数理解析研究所講究録
第 571 巻 1985 年 122-138

VALUATIONS ON MEROMORPHIC FUNCTIONS OF BOUNDED TYPE

Mitsuru Nakai

中井三留

Nagoya Institute of Technology

We are concerned with the following question: When is it true that any valuation on the field of single valued meromorphic functions of bounded type on a Riemann surface carrying nonconstant bounded holomorphic functions is a point valuation ? In this note we mention the following four results on this question which are natural growth of our personal communucations with Professor Frank Forelli at University of Wisconsin - Madison: 1. *The covering stability*, 2. *Stable surfaces*, 3. *Maximality and stability*, and 4. *Weak stability*. Proofs for these will in geral be omitted except one last spot.

We start by fixing terminology precisely. We denote by $M^\infty(W)$ the field of meromorphic functions of bounded type on an open Riemann surface W so that $M^\infty(W)$ is the quotient field of the algebra $H^\infty(W)$ of bounded holomorphic functuions on W. A *valuation* v on $M^\infty(W)$ is a group homomorphism of the multiplicative group $M^\infty(W)^* = M^\infty(W) - \{0\}$ onto the nonzero subgroup of the additive group $\mathbf{Z}$ of all integers such that

$$v(f + g) \geqq \min(v(f), v(g)) \quad (f, g \in M^{\infty}(W)^*)$$

where we make the convention that $v(0) = +\infty$.

We see that $v(M^{\infty}(W)^*) = \{me;\ m \in \mathbb{Z}\}$ where e is the minimum of the set of positive numbers in $v(M^{\infty}(W)^*)$. The valuation $\hat{v}$ defined by

$$\hat{v}(f) = e^{-1}v(f) \quad (f \in M^{\infty}(W)^*)$$

is referred to as the *normalization* of v. Two valuations v_1 and v_2 are said to be equivalent if $\hat{v}_1 = \hat{v}_2$.

Take a local parameter z at a point a in W with $z(a) = 0$. Let f be meromorphic at a and

$$f(z) = \sum_{\nu=k}^{\infty} c_{\nu} z^{\nu} \quad (c_k \neq 0)$$

be the Laurent expansion of f. The number k is uniquely determined by f and a and is usually denoted by $\partial_f(a)$ and called the *order* of f at a. We can easily check that

$$\partial.(a)\colon f \mapsto \partial_f(a)$$

is a valuation on $M^{\infty}(W)$ if $M^{\infty}(W) \neq \mathbb{C}$ (the complex number field), or what amounts to the same, if $H^{\infty}(W) \neq \mathbb{C}$. Any valuation v on $M^{\infty}(W)$ is said to be a *point valuation* on $M^{\infty}(W)$ at a if v is equivalent to the valuation $\partial.(a)$ for a point a in W.

For convenience we say that a Riemann surface W is H^{∞}-*stable*, or simply *stable*, if $H^{\infty}(W) \neq \mathbb{C}$ and every valuation on $M^{\infty}(W)$ is a point valuation. Then our problem is to clarify the stability of Riemann surfaces under consideration.

A valuation v on $M^{\infty}(W)$ is said to be *distinguished* if the following condition is satisfied: If $v(f) \geqq 0$ for an f in $M^{\infty}(W)$, then there exists a λ in $\mathbb{C}$ such that $v(f - \lambda) > 0$. Point valuations clearly satisfy this condition. Except the last section we do not a priori assume the distinguishedness for our valuations in this note which is one of important points to be stressed in our study. In this connection the following question is very important and is probably very difficult to resolve:

OPEN PROBLEM 1. *Is there any* W *such that* $M^{\infty}(W)$ *carries a nondistingushed valuation or is any valuation on any* $M^{\infty}(W)$ *automatically distinguished ?*

§ 1. THE COVERING STABILITY.

We say that a Riemann surface R or more precisely a triple (R, S, π) of Riemann surfaces R and S and an analytic mapping π of R into S is a *covering surface* of S. The surface S and π are referred to as the base surface and the covering map of the covering surface (R, S, π), respectively. We say that the covering surface (R, S, π) is *unbounded* if for any curve C on S with its initial point a in S and any α in $\pi^{-1}(a)$ there exists a curve Γ on R with α its initial point such that $\pi(\Gamma) = C$. Let a be in S and α in $\pi^{-1}(a)$. We can always find local parameters z and ζ about a and α respectively such that the local expression of the covering map $z = \pi(\zeta)$ takes the form $z = \zeta^m$. Here the positive integer m, the multiplicity of a, does not depend on the choice of local parameters z and ζ. If $m > 1$, then α is referred to as a branch point of order $m - 1$. For each a in S we let $\#(\pi^{-1}(a)) = \infty$ if $\pi^{-1}(a)$ is an infinite set and $\#(\pi^{-1}(a)) = n$ if the set $\pi^{-1}(a)$ consists of a finite n number of points where a branch point of order $m - 1$ is counted as m points. When (R, S, π) is unbounded, $\#(\pi^{-1}(a))$ is a constant $n \leqq \infty$ for every a in S. If $n < \infty$, then we say that (R, S, π) is n *sheeted* or more roughly *finitely sheeted* without referring to the specific sheet number n. Note that there may or may not be an infinite number of branch points in R. The following is the main and fundamental result in our study:

THEOREM 1. *The unbounded finite covering surface* R *is stable if and only if its base Riemann surface* S *is stable.*

Identifying $M^\infty(S)$ with $M^\infty(S)\circ\pi$ we can view that $M^\infty(R)$ is a field extension of $M^\infty(S)$. By the finiteness of R over S we can see by the standard symmetric function argument that $M^\infty(R)$ is a finite separable algebraic extension of $M^\infty(S)$. Using this we can prove the above theorem easily although we need a long series of elementary discussions. In the course we also use the fact that there is a valuation V on $M^{\infty}(R)$ for any given valuation v on $M^{\infty}(S)$ such that $V|M^{\infty}(S)$ is equivalent to v.

§ 2. STABLE SURFACES.

It is surprising that not many stable surfaces are known. Nonstable plane regions are in plenty. Let S_0 be the surface obtained from a plane region S with $H^{\infty}(S) \neq \mathbb{C}$ by removing a point a in S. Then the valuation v on $M^{\infty}(S_0)$ given by $\partial.(a)$ is seen not to be a point valuation on $M^{\infty}(S_0)$ and hence S_0 is not stable. If S is a suface of the Myrberg type, then S_0 can be stable (cf. Example 2 below). The following is , in essence, the only one known stable surface:

THEOREM 2. *The unit disk* Δ *is stable.*

Take any valuation v on $M^{\infty}(\Delta)$. One need to show that v is a point valuation. We can find a proof in an old paper [7] of Royden under the assumption that v is distinguished. Thus we need to show that v is automatically distinguished which is accomplished by using Blaschke products.

An m *sheeted disk* Δ_m is an m sheeted unbounded covering surface of the open unit disk Δ. We also call Δ_m a *finitely sheeted disk* without specifying the sheet number m. From Theorems 1 and 2 it follows the following

EXAMPLE 1. *Any finitely sheeted disk is stable.*

Plane regions bounded by finitely many mutually disjoint nondegenerate continuum are finitely sheeted disks by the Bieberbach-Grunsky theorem or more generally finite open Riemann surfaces are finitely sheeted disks by the Ahlfors theorem and therefore these are examples of stable surfaces.

Let Δ_2 be the 2 sheeted disk (Δ_2, Δ, π) with the sequence $\{x_n\}_1^\infty$ of projections x_n of branch points in Δ_2 lying over the positive real axis such that

$$1/2 < x_1 < x_2 < \dots < x_n < \dots < 1,$$

$\lim_n x_n = 1$, and

$$\sum_{n=1}^\infty (1 - x_n) = +\infty.$$

Let $\{\sigma_k\}_1^\infty$ be a sequence of disjoint closed disks contained in $\Delta \cap \{\mathrm{Re}\, z < 0\}$ converging to -1. We denote by E one of two connected pieces of $\pi^{-1}(\Delta \cap \{\mathrm{Re}\, z < 0\})$. Finally let

$$U = \Delta_2 - E \cap \pi^{-1}(\bigcup_{k=1}^\infty \sigma_n).$$

By the Myrberg type argument and the Blaschke theorem we have

$$H^\infty(U) = H^\infty(\Delta_2) = H^\infty(\Delta)\circ\pi.$$

Using this relation we can see the following

EXAMPLE 2. *The surface* U *is of infinite connectivity and of infinite genus and not representable as a finitely sheeted disk but stable.*

In these connections the most important and interesting question is the following

OPEN PROBLEM 2. *Is there any stable plane region of infinite connectivity ?*

§ 3. MAXIMALITY AND STABILITY.

A Riemann surface W' is an H^∞-extension of a Riemann surface W if $W \subset W'$ and $H^\infty(W')|W = H^\infty(W)$. A Riemann surface W is said to be H^∞-*maximal* when the following condition is satisfied: if there is an H^∞-extension W' of W, then $W' = W$. Because of Example 2 there can exist a stable surface which is not H^∞-maximal. However, for plane regions, the stability implies the H^∞-maximality. What happens to the converse ? It would be very nice if the converse is true but unfortunately we have the following

EXAMPLE 3. *There exists an H^∞-maximal bounded plane region which is not stable.*

We will seek such a region among (modified) Zalcman L-domains which are of the following form

$$X = \Delta_0 - \bigcup_{k=1}^{\infty} \overline{\Delta}(c_k, r_k).$$

Here Δ_0 is the punctured unit disk $0 < |z| < 1$ and

$$\overline{\Delta}(c_k, r_k) = \{|z - c_k| \leqq r_k\}$$

where $\{c_k\}_1^\infty$ and $\{r_k\}_1^\infty$ are zero sequences of positive real numbers such that

$$c_{k+1} + r_{k+1} < c_k - r_k < c_1 + r_1 < 1/2 \quad (k = 1,2,\ldots).$$

First we observe that any Zalcman L-domain X is H^∞-maximal. We do not know whether X can be stable or not but we will see that X can be unstable by a suitable choice of $\{r_k\}$ for any fixed centers $\{c_k\}$.

We need to consider the following auxiliary region

$$Y = \Delta_0(0, 1/2) - \bigcup_{k=1}^{\infty} \overline{\Delta}(c_k, \rho_k)$$

where $\Delta_0(0, 1/2)$ is the disk $0 < |z| < 1/2$ and $\{\rho_k\}$ is a zero sequence of positive real numbers such that

$$c_{k+1} + \rho_{k+1} < c_k - \rho_k < c_1 + \rho_1 < 1/2 \quad (k = 1,2,\ldots).$$

We choose $\{\rho_k\}$ convergent to zero enough rapidly so as to make the following condition valid:

[A] $z = 0$ *is an irregular boundary point of the region* Y *with respect to the Dirichlet problem.*

Now we choose $\{r_k\}$ in such a fashion that $0 < r_k < \rho_k$ for all $k = 1,2,\ldots$ and we further choose $\{r_k\}$ so small that

[B] $\sum_{k=1}^{\infty} r_k/\rho_k^n < +\infty \quad (n = 1,2,\ldots)$.

The choice of $r_k = \rho_k^k$ is an example. Clearly $\overline{Y} - \{0\} \subset X$. Because of the above condition [B] we can see the existence of the formal n^{th} derivative $f^{(n)}(0)$ of f at 0 given by

$$f^{(n)}(0) = \lim_{z \in \overline{Y}-\{0\}, z\to 0} f^{(n)}(z)$$
$$= (n!/2\pi i)\int_{\partial X} f(\zeta)\zeta^{-(n+1)}d\zeta.$$

Concerning these formal derivatives we have the *unicity theorem*: If $f^{(n)}(0) = 0$ for every $n = 1,2,\ldots$, then f is identically zero on X. This is derived by a potential theoretic argument by using the condition [A].

For each f in $H^{\infty}(X)$ we set

$$v(f) = \min\{n;\ f^{(n)}(0) \neq 0\}$$

if f is not identically zero, and set $v(0) = +\infty$. For any pair f_1 and f_2 of functions in $H^{\infty}(X)$ with f_2 not identically zero we set

$$v(f_1/f_2) = v(f_1) - v(f_2).$$

It is easy to check that the above value is certainly determined uniquely by the ratio, i.e. $f_1/f_2 = f_3/f_4$ implies $v(f_1/f_2) = v(f_3/f_4)$. Thus v can be defined on $M^{\infty}(X)$. It is simply a matter of checking formally to ascertain that v is a valuation on $M^{\infty}(X)$ which is not a point valuation on $M^{\infty}(X)$.

§ 4. WEAK STABILITY.

We say that a Riemann surface W is *weakly* H^∞*-stable*, or simply *weakly stable*, if $H^\infty(W) \neq \mathbb{C}$ and every distinguished valuation on $M^\infty(W)$ is a point valuation on $M^\infty(W)$. Hence the stability of W obviously implies the weak stability of W. The converse of this is closely related to Open problem 1 and we ask the following

OPEN PROBLEM 3. *Does the weak stability automatically imply the stability ?*

A boundary point ζ of a bounded plane region S is said to have an H^∞*-barrier* b_ζ on S if b_ζ is a nonzero member of $H^\infty(S)$ and $(z - \zeta)^{-n} b_\zeta(z)$ is bounded on S for every $n = 1,2,\ldots$. We also say that b_ζ is an H^∞-barrier at ζ on S. The importance of H^∞-barriers lies in the following fact: *If every boundary point of a bounded plane region* S *has an* H^∞*-barrier on* S, *then* S *is weakly stable.*

In view of the above fact it is important to determine when a boundary point ζ of S has an H^∞-barrier. Concerning this the first conclusion easily proved is the following

PROPOSITION 1. *If a boundary point* ζ *of a bounded plane region* S *has an* H^∞*-barrier on* S, *then* S *is regular with respect to the Dirichlet problem for* S.

It would be very nice if the converse of this is true but we do not have even the faintest idea at this moment to attack this question. Hence we mention the following

OPEN PROBLEM 4. *Does a boundary point* ζ *of a bounded plane region* S *which is regular with respect to the Dirichlet problem for* S *have an* H^{∞}*-barrier on* S *?*

The following result is very special and of auxiliary nature but it is certainly in the positive direction to the above question.

PROPOSITION 2. *Let* F *be a closed subset of the open unit disk* Δ *such that* $S = \Delta - F$ *is connected. Then any boundary point* ζ *of* S *lying on the unit circle has an* H^{∞}*-barrier on* S.

This technical result is now used to prove the following fact which is the main result in this section. At this point we stress that weak stability is a conformally invariant property. Thus, if we have a true statement that A implies the weak stability, then the statement that A' implies the weak stability is also true where A' is the conformal image of a property A.

THEOREM 3. *If any connected component of the boundary of a bounded plane region* R *is nondegenerate continuum, then the region* R *is weakly stable.*

The above theorem assures the existence of a weakly stable bounded plane region of infinite connectivity (cf. Open problem 2). The following is one such example:

$$R = \Delta - \bigcup_{k=1}^{\infty} \kappa_k$$

where $\{\kappa_k\}$ is a family of mutually disjoint closed disks κ_k in Δ converging only to the boundary point $z = 1$. Compare this with Zalcman L-domains. There are also weakly nonstable regions among L-domains (cf. Section 3).

As an illustration of (omitted) proofs in this note we give here a selfcontained complete proof of the above theorem 3 since it is relatively simple and elementary. In order to make the whole discussion self contained we also need to mention a proof of Proposition 2 which is, however, extremely simple.

PROOF OF PROPOSITION 2. Let $T = -\zeta^{-1}S + 1$ which is in the right halh plane $\mathrm{Re}\, w > 0$. The point $w = 0$ is in the boundary ∂T which is the image of ζ under the mapping

$$z \mapsto w = -\zeta^{-1}(z - \zeta)$$

from S onto T. Consider the branch of $\sqrt{w}$ in $\mathrm{Re}\, w > 0$ with $\sqrt{1} = 1$. Then it is easy to see that

$$b_0(w) = \exp(-1/\sqrt{w})$$

is an H^∞-barrier on T at $w = 0$. Thus ζ has an H^∞-barrier

$$b_\zeta(z) = b_0(-\zeta^{-I}(z - \zeta))$$

on S. $\square$

PROOF OF THEOREM 3. Let $\lambda \in \mathbb{C}$ and v be a valuation on $M^\infty(R)$. Since $nv(\lambda^{1/n}) = v(\lambda)$ shows that n divides $v(\lambda)$ for all $n = 1,2,\ldots$, we must conclude that $v(\lambda) = 0$, i.e.

$$v(\mathbb{C}^*) = \{0\}.$$

Next let $f \in H^\infty(R)$ and take a positive number c with $c \geqq \sup_R |f|$. Since $(f + c)^{1/n} \in H^\infty(R)$ and $nv((f + c)^{1/n}) = v(f + c)$, we must conclude , as above, that $v(f + c) = 0$. Thus $v(f) = v(f + c - c) \geqq \min(v(f + c), v(c)) = 0$, and we have shown that

$$v \text{ is nonnegative on } H^\infty(R).$$

Now fix an arbitrary distinguished valuation v on $M^\infty(R)$ and we will show that this v is a point valuation on $M^\infty(R)$. For the purpose we may replace v by its normalization and thus we may assume v as normalized from the very begining so that $v(M^\infty(R)^*) = \mathbb{Z}$.

Let I be the identity function: $I(z) = z$ identically. We assumed that R is a bounded plane region, and therefore, I is a member of $H^\infty(R)$. Hence $v(I) \geqq 0$. By the distinguishedness of v there exists a point a in $\mathbb{C}$ such that

$$v(I - a) > 0.$$

If an a' in $\mathbb{C}$ different from a also satisfies $v(I - a') > 0$, then we have the following contradiction:

$$\begin{aligned} 0 &= v(a - a') \\ &= v((I - a') - (I - a)) \\ &\geqq \min(v(I - a'), v(I - a)) > 0. \end{aligned}$$

Thus the point a in $\mathbb{C}$ with $v(I - a) > 0$ is uniquely determined by v. Such an a is said to be the *support* of v.

We claim that the support a of v satisfies $a \in \overline{R}$. In fact, if $a \notin \overline{R}$, then $(I - a)^{-1} \in H^{\infty}(R)$ and $v((I - a)^{-1}) \geqq 0$. However, since $v(I - a) > 0$, we have the following contradiction: $v((I - a)^{-1}) = - v(I - a) < 0$. Hence $a \in R$ or $a \in \partial R$.

We now assert that $a \in R$. Contrariwise assume that a is in ∂R. Let K be the component of ∂R containing the point a. By our assumption K is a nondegenerate continuum. Let ψ be a conformal mapping of the component of the complement of K containing R onto the unit disk Δ. Then we set

$$S = \psi(R) = \Delta - F$$

where ψ sends K to $\partial\Delta$ and F is a closed subset of Δ. Let

$$u(f) = v(f \circ \psi) \quad (f \in M^{\infty}(S)).$$

Then u is a distinguished valuation on $M^{\infty}(S)$. Let α be the support of u which is now known to be in $\overline{S}$.

We maintain that $\alpha \in \partial\Delta$. Suppose contrariwise that $\alpha \in \overline{\Delta - F} - \partial\Delta$. Then , since $\alpha \in \Delta$, there exists a point γ in $\psi^{-1}(\Delta)$ such that $\psi(\gamma) = \alpha$. Since $\psi \in H^{\infty}(R)$, we have

$$v(\psi - \psi(\gamma)) = u(I - \alpha) > 0.$$

Observe that $(\psi - \psi(\gamma))/(I - \gamma)$ is zero free on the simply connected region $\psi^{-1}(\Delta)$. Hence we can find an f_n in the subclass $H^{\infty}(\psi^{-1}(\Delta))$ of $H^{\infty}(R)$ such that

$$(\psi - \psi(\gamma))/(I - \gamma) = f_n^n$$

for each $n = 1,2,\ldots$. Then

$$v((\psi - \psi(\gamma))/(I - \gamma)) = nv(f_n)$$

for every $n = 1,2,\ldots$ and we can conclude, as before, that the left hand side of the above must be zero. Hence

$$v(I - \gamma) = v(\psi - \psi(\gamma)) > 0$$

and γ must be the support of v. By the uniqueness of the support of v we have $a = \gamma$ which must be in $\psi^{-1}(\Delta)$. This contradicts the fact that $a \in K$.

Since $\alpha \in \partial\Delta$, there exists, by Proposition 2, an H^{∞}-barrier b_{α} at α on S. Hence we have

$$(I - \alpha)^{-n} b_{\alpha} \in H^{\infty}(S) \quad (n = 1,2,\ldots)$$

and therefore

$$u((I - \alpha)^{-n} b_{\alpha}) \geqq 0$$

for every $n = 1,2,\ldots$ or equivalently

$$u(b_\alpha) \geqq nu(I - \alpha)$$

for every $n = 1,2,\ldots$. This is clearly a contradiction since $u(I - \alpha) > 0$. Thus we have shown that $a \in R$.

Take an arbitrary f in $H^\infty(R)$ and let $\partial_f(a) = n \geqq 0$. Set $g = f/(I - a)^n$. Note that $g(a) \neq 0$. Since $(g - g(a))/(I - a)$ belongs to $H^\infty(R)$, $v((g - g(a))/(I - a)) \geqq 0$ and thus

$$v(g - g(a)) \geqq v(I - a) > 0.$$

Since $v(g - g(a)) \neq v(g(a)) = 0$, we must have the equality instead of the inequality in

$$v(g) = v(g - g(a) + g(a)) \geqq \min(v(g - g(a)), v(g(a))) = 0$$

and thus we conclude that $v(g) = 0$, or equivalently

$$v(f) = nv(I - a) = v(I - a)\partial_f(a).$$

The relation can obviously be extended to $f \in M^\infty(R)$ and, since $v(M^\infty(R)^*) = \mathbb{Z}$, we must have $v(I - a) = 1$. Thus

$$v = \partial_\cdot(a)$$

on $M^\infty(R)$ and v is a point valuation on $M^\infty(R)$. $\square$

REFERENCES

[1] O. Endler: Valuation Theory, University Text, Springer, 1972.

[2] B. Fuglede: Sur les fonctions finement holomorphes, Ann. Inst. Fourier, Grenoble, 31 (1981), 57-88.

[3] M. Heins: Algebraic structure and conformal mapping, Trans. Amer. Math. Soc., 89 (1958), 267-276.

[4] M. Heins: Complex Function Theory, Academic Press, 1968.

[5] S. Lang: Algebra, Addison-Wesley, 1965.

[6] H. Royden: Rings of analytic and meromorphic functions, Trans. Amer. Math. Soc., 83 (1956), 269-276.

[7] H. Royden: Rings of meromorphic functions, Proc. Amer. Math. Soc., 9 (1958), 959-965.

[8] H. Royden: Algebras of bounded analytic functions on Riemann surfaces, Acta Math., 114 (1965), 113-142.

[9] O. Zariski and P. Samuel: Commutative Algebra, Vol. 2, Van Nostrand, 1960.

[10] O. Schilling: The Theory of Valuations, Mathematical Surveys, no 4, 1950.

[11] L. Zalcman: Bounded analytic functions on domains of infinite connectivity, Trans. Amer. Math. Soc., 144 (1969), 241-269.

Mitsuru Nakai
Department of Mathematics
Nagoya Institute of Technology
Gokiso, Showa, Nagoya 466, Japan