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Complexity in Basin Structures and Information Processing by the Transition among Attractors

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Abstract

Spatially extended dynamical systems are investigated in connection with the information processing problems. Coupled map lattices and cellular automata are used as simple models. Information theory for multi-attractor systems is constructed, where stability of attractors against noise, information storage by attractors, and connectivity among attractors are studied. Three quantities are introduced to characterize the complexity of basin structures and information processing among attractors. Numerical results for one-dimensional cellular automata are presented.

1. Introduction

Dynamical systems can be thought of an information processing machine. Studies in spatially extended systems which show turbulence have just been started[1)-10)]. Typical examples are (i) partial differntial equations, (ii) coupled map lattices, and (iii) cellular automata. Here the latter two models will be discussed, since they are more tractable in numerical simulations and are powerful for extracting essential features in nonlinear systems. Recently chaos in a low dimensional dynamical system has been investigated extensively.

When we try to make an intelligent system based on a dynamical system, following problems are important:

(a) Creation of information: As was beautifully shown by Rob Shaw[11], a dyanamical system with chaos can be thought of an information source.

(b) Storage of information: Memory is important both for the brain and the computer. If a dynamical system has a large number of attractors, information can be stored in each attractor. Hopfield[12] has considered a kind of cellular automata which has a large number of fixed patterns. The model is related with a spin glass model. In the term of a dynamical system, the important point is the existence of a large number of attractors. In the following sections, an information theory for a multi-attractor system is constructed, which characterizes the ability of the information storage in the system.

Stability of the storage is also important which is related with the problem of self-repair or retrieval[13]. We will consider the stability of each attractor against a noise and define the mutual information between attractors. Some examples of the calculation on the basin information will be shown in §3.

(c) Propagation of information: Information processing in real space is necessary for intelligent machines. Examples in dynamical systems can be seen in cellular automata or coupled map lattices. Studies in the CA with soliton-like excitations will be important for this reason[10]. The propagation of disturbances in CML is related with the Lyapunov spectra and vectors, which will be important for the future study[1][3].

(d) Evolution and adaptation: Adaptation is a marvellous aspect in cognitive systems in livings, such as the immune network and the brain. For these problems, a dynamical system with a closed set of variables is not adequate[14]. One candidate for a model for these systems is a dynamical system with a coupling (or parameter) variable in adaptation with an external noise or environment (order from a noise[15]). Detailed study of evoloving dynamical systems, however, is left for the future.

(e) Hierarchical structure: In computer systems and also in the brain, hierarchical structures must be of relevance[16]. A typical example for a tree-like structure can be seen in UNIX system developed by Bell Laboratory. One candidate for this structure is a spin glass model, where the ultrametric structure appears through a replica symmetry breaking[16][17]. Cellular automata or coupled map lattices may have this kind of ultrametric structure, which has to be elucidated in the future.

From the viewpoint of the creation and storage of information, a dynamical system can be classified into the following four types: (Here " a large number" means a quantity exponential to the system size, $(O(e^N))$, while " a small number" means a quantity less than some power of the system size $(o(N^a))$

(1) No creation and small strorage :A dynamical system with a small number of simple attractors.

(2) No creation but large storage: A dynamical system with a large number of simple attractors.

(3) Positive creation with small storage: A dynamical system with a small number of complex attractors.

(4) Positive creation with large storage: A dynamical system with a large number of complex attractors.

Here, simple attractor means a periodic one, while "complex" attractor means chaotic for usual dynamical systems with a continuous variable. For cellular automata, however, we have to change the definition, since the attractors are always periodic if the system size is finite. For CA with a finite size, the term "simple attractor" is used for the attractor with a short period $(O(1))$, while "complex" is used for the attractor with a long period (longer than $O(N)$).

The above classification, if applied to CA, seems to correspond to the classification by Stephen Wolfram[8][9], though the characterization may be slightly different. For CML, some examples for the above classification are (1)---homogeneous periodic solution, (2)---periodic solution with kinks, (3)---developed chaotic patterns, and (4)--- chaotic patterns with some kinks. We do not know whether the developed turbulence corresponds to either (3) or (4), since the number of attractors for Navier-Stokes equation remains unknown.

2. Information theory for Multi-attractor systems

(a) Complexity of basins:

We consider a system which has M attractors denoted by $\{a_i\}$ $(i=1,2,\cdots,M)$. First, we examine the volume of the basin of attraction for each attractor. The ratio of the volume in a given bounded phase space for the attractor a_i is denoted by $b_i(\sum_i b_i=1)$. Let us define the complexity for basins by

$$C_B=-\sum_i b_i \ln b_i,$$

which characterizes the information for the initial state necessary to predict the final state. If we are interested in the complex basin structure itself such as the fractal basin boundary[18] or the fractal basin structure[19], it will be of importance to define C_B in a small ball with radius ε and calculate how $C_B(\varepsilon)$ scales as ε goes to zero.

(b) Jumping among attractors by noise [20]

If a dynamical system has more than one attractors, a unique invariant measure cannot be attained. In a real physical situation, existence of a small noise is expected. A unique (or a small number of) measure is selected out by the inclusion of a noise to a system. Here we consider the case with a very weak noise. In that case, the system may be characterized as a process where the state stays at the original attractors (for most of the time) and a process where the state jumps out to some other attractors by the effect of a noise. That is, the dynamics is decomposed into

(residence at the original attractors without noise)
+

(transition among attractors by a noise)
Here, the time for the latter process is neglected compared with the time for the former.

Transition matrix between attractors is defined as follows: If a small noise is added for the attractor a_i, a jump from the attractor a_i to a_j occurs with some probability. The jumping process depends on the state of the dynamical system when the noise is applied and on the site at which a noise is applied and on the strength of the noise. The transition probability P_{ji} is defined as the ratio of the transition from a_i to a_j (the ratio is a measure calculated for all the states when a noise is applied and sites at which the noise is applied).

The probability that a system is in the attractor a_i for a weak noise case is given by

q_i= the i-th component for the eigenvector for P_{ji} corresponding to the eigenvalue 1.

If the eigenvalue 1 is degenerate with the multiplicity m_p, there are m_p invariant measures within the above weak-noise limit approximation.

In some cases, some attractor a_i is so weak that P_{ik} is zero for all k, as can be seen in the next section. If q_i is zero, the residence probability at the attractor a_i is zero.

(c) Complexity in the jumping process among attractors by noise

Once the probability meaure q_i is attained by a noise, we can define the complexity for the probability distribution for attractors by

$$C_M=-\sum_i q_i \ln q_i,$$

for a give invariant meaure. The meaning of the quantity C_M is as follows: After the transients have decayed out for a weak noise system, we make a measurement to determine at which attractor the system stays at that time. The information gain by the above measurement is C_M. The important differnce between C_B

and C_M is that the former information is concerning on the knowledge about the initial state, while the latter is related with the observation for the aged system with a noise.

Another important quantity is a dynamical information gain by noise. Let us assume that we knew that a system had initially been at the attractor a_i and have observed that the system is now at the attractor a_j after a noise was applied. How much information has been obtained through this observation? We can get the information about the noise, i.e., the time step when, and the site where, the noise is applied. The amount is given by $\ln(P_{ji}^{-1})$ bit. Thus the dynamical information gain per noise is given by

$$C_D=-\sum_{i,j} q_i P_{ji} \ln P_{ji},$$

since the ratio for the event $a_i \rightarrow a_j$ is $q_i P_{ji}$.

As is easily seen,

$$C_T=C_M-C_D$$

is non-negative. The quantity C_T corresponds to the mutual information [21] between attractors by noise.

If C_D is large, the information creation by noise is large. That is, the uncertainty about the attractor into which the system settles down after a noise is added is large. It can be also stated that if the mutual information is large ($C_D \ll C_M$), the information flow between attractors is large, i.e., the structure of the network of the transition among attractors is well organized.

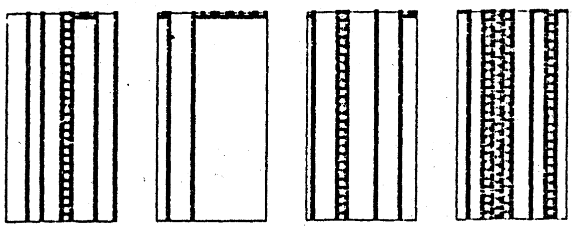
3. Complexity in CA with multi-attractors

As a simple example for the complexity theory for multi-attractor systems in §2, one-dimensional cellular automata are investigated. The models are (a) 2-state legal cellular automata with range 1[7] (b) 2-state totalistic cellular automata with range 2[8] (c) 2-state cellular automata with range 2 which have "soliton"-like excitations[10]. Here we use the rule number for model a) or the rule code for model b) intorduced by Stephen Wolfram[7)8)], to characterize the rule for CA. Some examples of the evolutions of CA are shown in Fig.1.

The method of calculations is as follows: (i) Take a one-dimensional cellular automaton with a size N ($8<N<23$) and simulate it for all initial configurations (i.e., 2^N possibilities). Here periodic boundary conditions are chosen. (ii) Enumerate all possible attractors (find $\{a_i\}$ and M ($i=1,2,\cdot$ M) and list up all the patterns. (iii) Calculate how many initial configurations are attracted into the attractor a_i. The number of such initial configurations divided by 2^N gives b_i, from which the basin complexity C_B is calculated. (iv) Take an attractor a_i whose period is denoted as t_i. We change a value of one lattice site for a_i. There are $N\times t_i$ possibilities for this flip-flop. We simulate the CA starting from the configurations obtained by all these possible flip-flops and check which attractor the state is attracted into. The number of such configurations which are attracted into a_j divided by $N\times t_i$ gives p_{ji}. The eigenvector for p_{ji} corresponding to the eigenvalue 1 gives q_i. From p_{ji} and q_i, measure complexity C_M and dynamical complexity C_D are calculated. Here instead of obtaining all possible eigenvectors, we choose an initial vector $(b_1,b_2,b_3,\cdots\cdots,b_M)^T$ and multiply the matrix $\{p_{ji}\}$ many times and how the vector is settled down and obtain $\{q_1,q_2,\cdots\cdots,q_M\}^T$. This corresponds to the selection of one measure closest to the distribution proportional to the volumes of basins, if the measure is not unique (i.e., nonergodic). For a finite one-dimensional CA, such a nonergodic case seems to be rare.

For the classification of attractors, the configurations which coincide by the spatial translation are regarded as the same attractor. For example, the patterns 11000001, 11100000 and 00111000 are regarded as the same.

The results for various CA are summarized as follows



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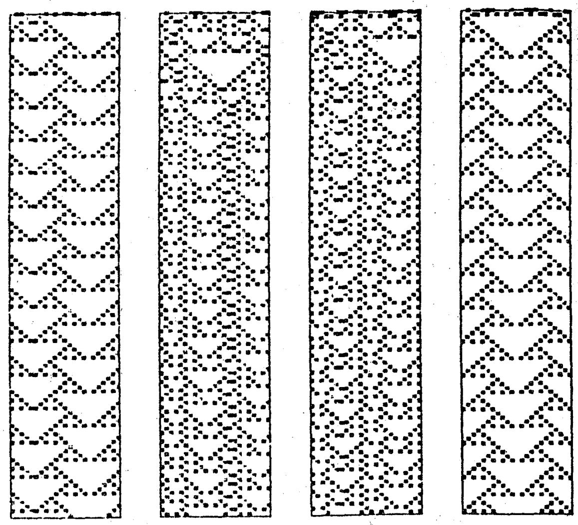

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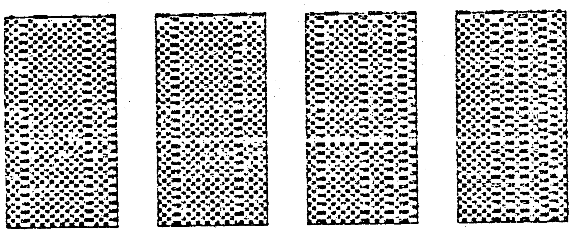

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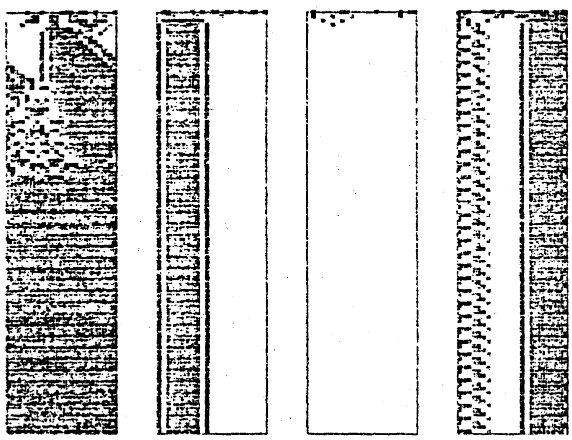


d)

Fig.1

Some patterns of evolution of one-dimensional CA.

a)Rule 108 (range=1; a)) N=23, time step=50
b)Rule 50 (range=1;a)) N=23, time step=50
c)Rule 146 (range=1;a)) N=23, time step=120
d)Rule 52 (range=2;totalistic;b)) N=23, time step=120

4 initial configurations are chosen, from which the states are attracted into cycles.

corresponding to the classes in §1.

(1) class 1:

Number of attractors M remains small (about 1~4) even if N is increased up to 19. As N goes to large, C_B, C_M, and C_D rapidly go to zero.

(2) class 2:

As is expected, the number of attractors increase as exp(const.xN). The basin complexity and measure complexity increase as axN+const., where a is some constant which seems to take the same values both for C_B and C_M, though the additive const. is different between the two. The dynamical complexity C_D increases as bxN+const., where b is smaller than a. (see Fig.2)

The class 2 behavior is understood by the superposition of local oscillators. If a local oscillator has a period t and spatial range r, the number of attractors is roughly given by $(t+1)^{N/r}$, since there are (t+1) possibilities in each r region (put the oscillator or not and put it with which phase of the oscillation). This argument is easily extended to the case where there are more than one types of local oscillators. The linear increase of complexity is explained in the same way.

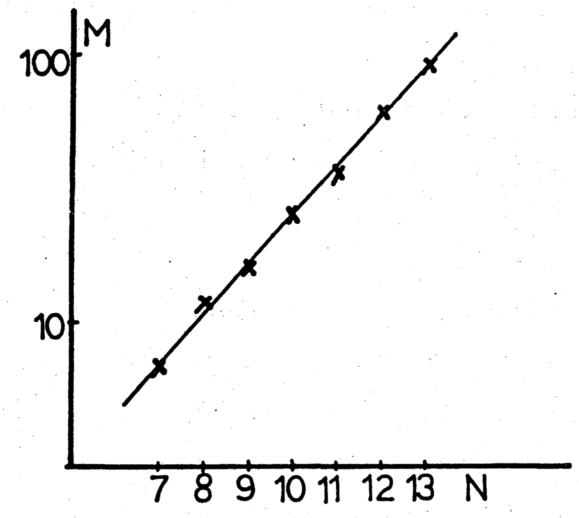
If a local oscillator exists as a kink in a zigzag structure (see Fig. 1b), there appears a difference by whether N is odd or not (see Fig. 3) .

(3) class 3:

The class 3 behavior of CA is characterized by triangles with various sizes. Number of attrcators and complexities as a function of system size are shown in Fig.4, with some patterns of typical attractors. Though the behavior is very complicated, following points are common in class 3 CA.

(i) Number of attractors change irregularly as the system size N. The increase by the size is at most bounded by some power of system size N.

(ii) The attractors which have a large region of basins are the



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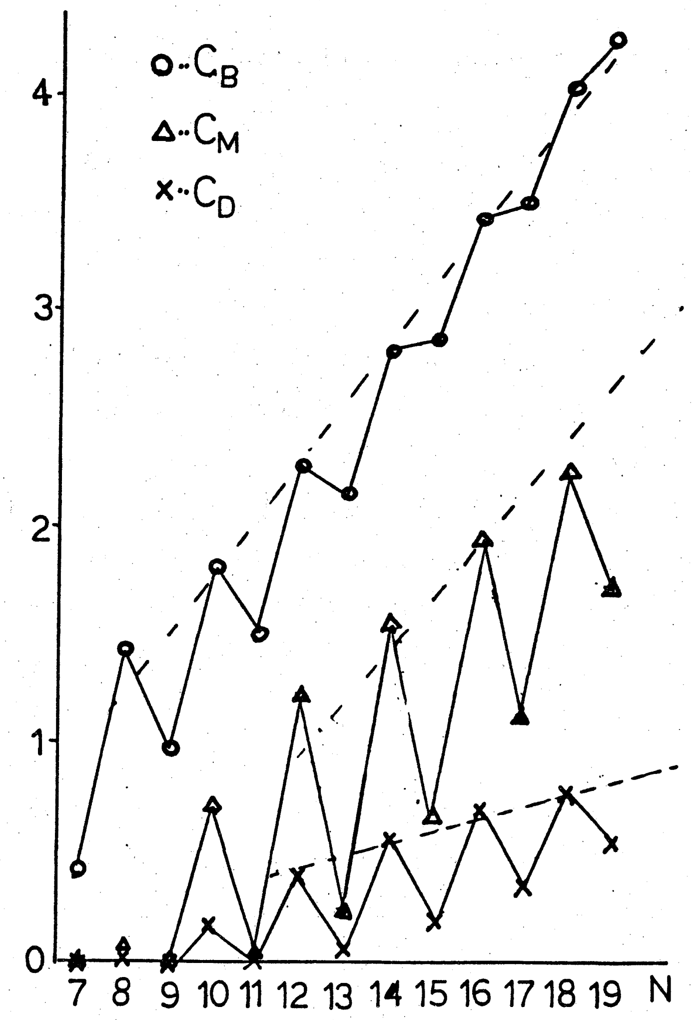
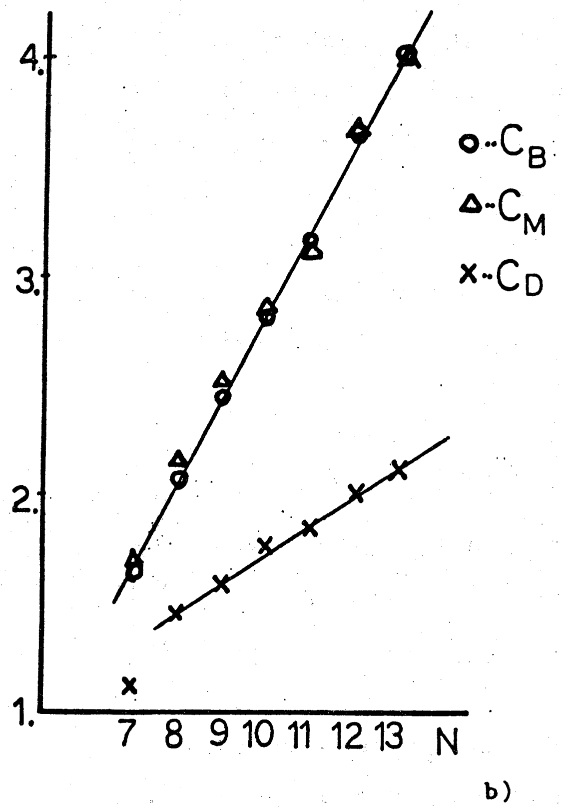


Fig.3

Three complexities as a function of system size. for rule 50 (range=1;a))



b)

Fig.2

a) Number of attractors M vs. system size N for rule 108 (a)).

b) Three complexities as a function of system size. for the same rule as Fig.2a).

ones which have a triangle structure and the one in which all the sites take zero (000···000). As N is increased the basin for the latter regions decrease irregularly. Among the attractors with triangle structures, the attractor with a larger size of triangles has a larger size of basin of attractions.

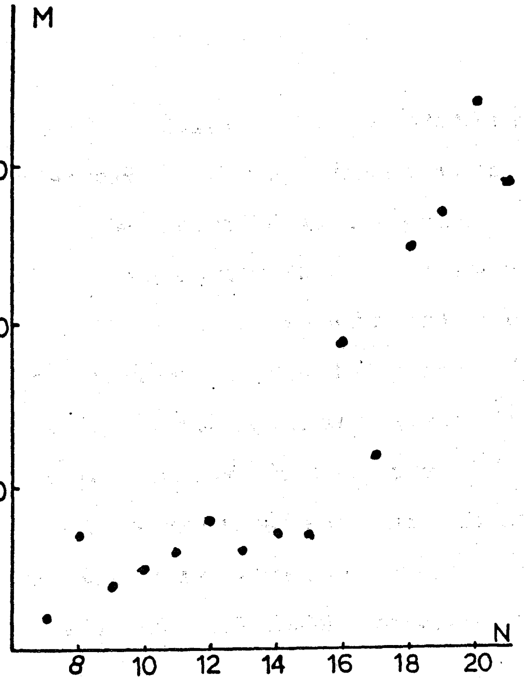
(iii) The complexities also change irregularly as the system size. They seem to increase slowly as the system size. Generally speaking, C_D is not so small compared with C_M. That is, the mutual information C_M-C_D is small compared with the cases which belong to other classes. Thus, the connectivity among attractors by a small noise is random compared with CA for the other classes.

(vi) The irregular behavior as a size change seems to depend on some number theoretic properties of the size and rules. For example, there occurs singular behavior around at $N=2^k-i$ (i=0,or 1,or -1 which depends on the rule). For the rule 146 with range 1 (model a)), the rapid decrease of the number of attractors occurs at N=14 and the basin of "all zero" attractor has 99% ratio at N=15 ($=2^4-1$). The complexities C_M and C_D take small values (or vanish) around at $N=2^k$.

(4) class 4:

The class 4 behavior for CA characterized by Stephen Wolfram is long-time transients and the existence of local oscillators and the sensitive dependence of patterns on the initial configurations. The characteristic features for the basin structure for the class 4 systems may be summarized as follows:

(i) The number of attractors increase essentially exp(const.xN) though the increase is rather irregular (see Fig. 5). The pattern of attractor which has a large region of basins change as size, though "all 0" or "all 1" has a large basin of attractions in many rules. As N is increased, attractors of essentially new type appear successively, which is a typical difference from other classes.



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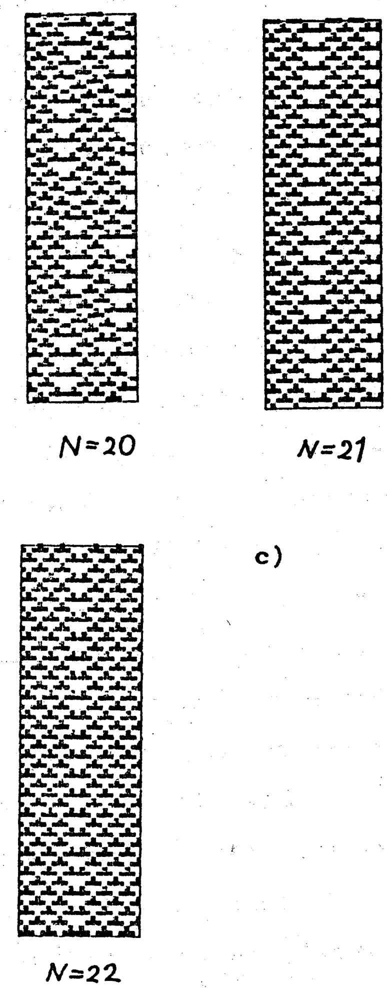

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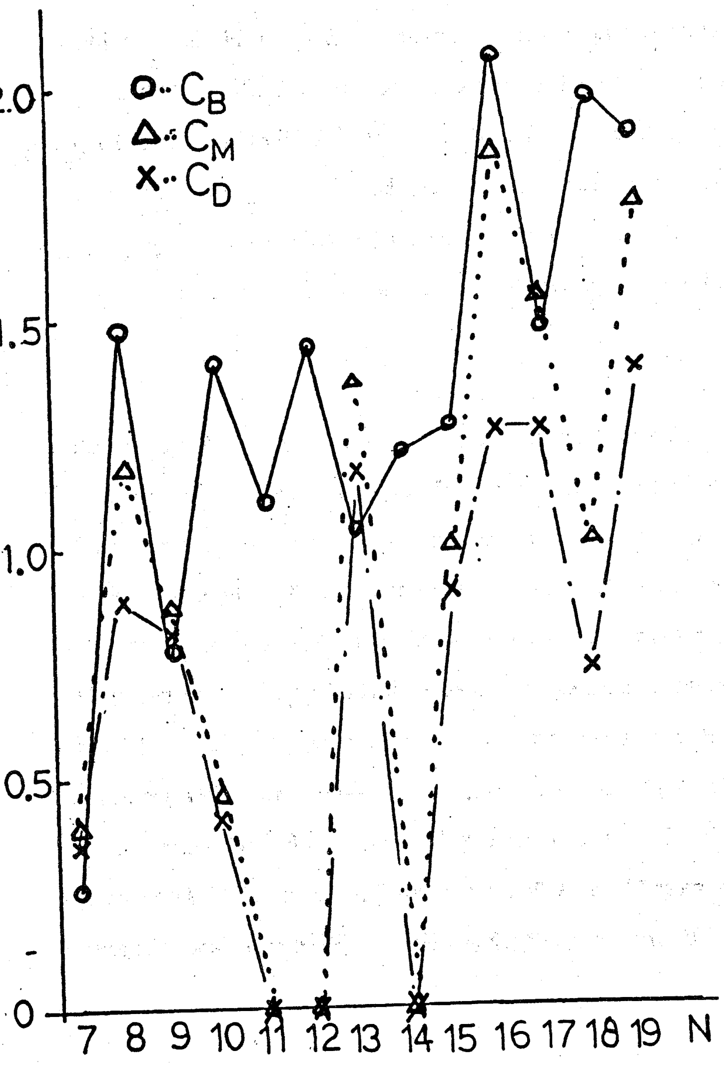


b)

Fig.4

a) Number of attractors M vs. system size N for rule 54(range=1;a)).

b) Three complexities as a function of system size. for the same rule as Fig.4a).

c) A pattern for the attractor whioh has the largest basin of attractions for N=23.

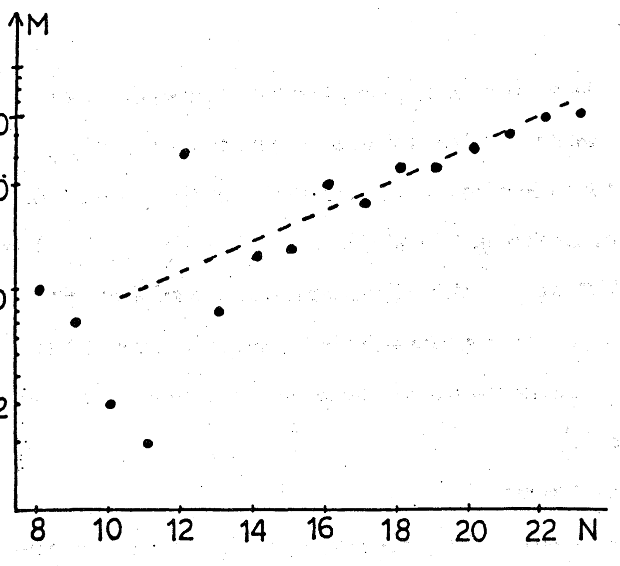
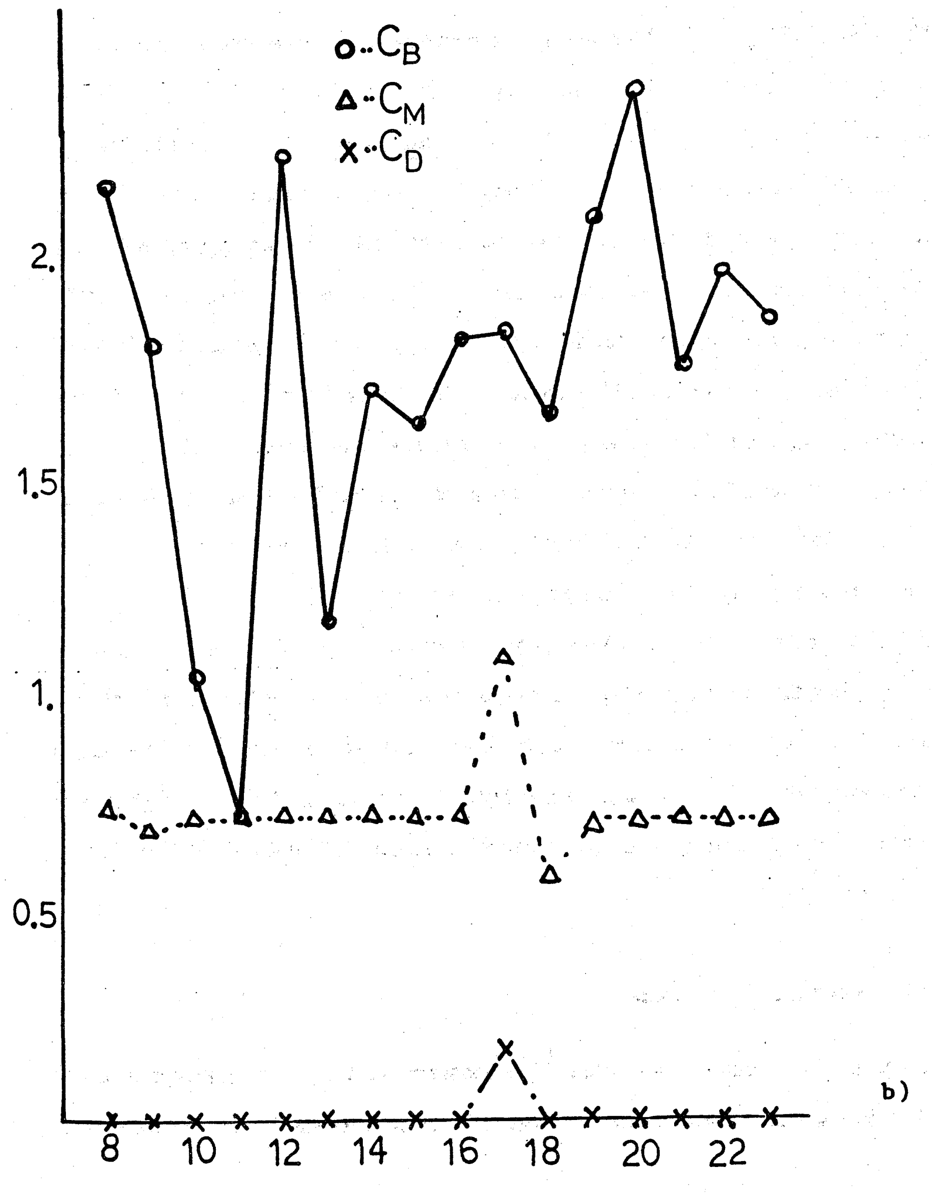


Fig.5

a) Number of attractors M vs. system size N for rule 52(range=2;totalistic;b)).

b) Three complexities as a function of system size. for the same rule as Fig.5a).



(ii) The basin complexity C_B takes a comparatively large value, which changes irregularly as size. The measure complexity C_M is much smaller than C_B, since the probability measure (by a noise) for "all 0" or "all 1" is much larger than the ratio for the basin of attractions to such states. The dynamical complexity is much smaller, which means that the mutual information is rather large. In other words, the transition between attractors by noise is regularly structurized.

(5) CA with soliton-like excitations

Quite recently, Aizawa et al.[10] have investigated a class of cellular automata which allows 00101100 to move right or left (i.e., 01011000 or 00010110) after one step. There are 2^{13} possible rules of this type for the legal CA with 2-states and range=2 which permit this type of soliton-like solutions. Simulations for all these rules have been performed. New interesting behavior which do not belong to the above types of classification is soliton-like behavior. For some rules, the dynamics of system is governed only by the soliton-like excitations (1011) and their collisions. If the "solitons" pass through each other by collisions, the system shows a kind of integrable behavior. For this class, the basins for the state of superpositions of "solitons" go larger as the system size is increased. The important difference between this type of behavior and the usual integrable systems studied in soliton theory is that our system is integrable only after the transients have decayed out. Thus, our system may be termed as "integrable system on an attractor". In a variety of dissipative systems which show soliton-like behaviors, the above notion will be of importance.

4. Discussion and Future Problems

In the present paper we have discussed the storage of information and information processing in stochastic cellular automata. We have studied the complexity of networks among attractors connected by a noise. Detailed account for the complexity in CA will be reported elsewhere[22].

If some adaptive process is included to change the rule or the strength of noise (susceptibility against noise) adaptively to choose a large network, the formation process of networks may be discussed from the above model.

Another important process is the information propagation in real space in CA or CMLs. If one lattice is perturbed from the outside, three possible patterns are possible: (i) The disturbance is localized in some limited space even if the dynamics shows a turbulent behavior. (ii) The disturbance propagates with some velocity. For some CMLs, the propagation is rather smooth, while it has a large fluctuation in some CMLs. (iii) The propagation of disturbance is rather irregular. The disturbance is localized for a long time and is transmitted quite rapidly after a long waiting time. The phenomena can be termed as "tunneling". The above three patterns can be understood from the viewpoint of the structures of Lyapunov vectors (extended or localized). The relation of the information propagation for these models with the Lyapunov vectors has to be clarified in the future[3].

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