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TOWARD THE CLASSIFICATION OF THE PATTERNS GENERATED BY ONE-DIMENSIONAL CELL AUTOMATA

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ABSTRACT

The basic patterns in the cellular automata have been phenomenologically classified into four types;

Type I vanishing patterns
II localized patterns
III chaotically propagating patterns
IV long-lived irregular patterns

A new theoretical method to predict these types is proposed by using the elementary cellular atutomata. Furthermore, it is discussed that the above mentioned phenomenological classification is not sufficient, and that the several elementary excitations are playing important roles in the process of pattern formation. For an example, a special group of the cellular automata, which permits a certain solitary wave solution, is studied and the variety of the pattern formation will be discussed together with the collision processes among elementary excitations.

1. Introduction

Automaton systems defined on the cellular structure have been studied extensively from the various kinds of interest in natural science[4~6]; morphogenesis and self-production process, activity of the central nervous systems, artificial intelligence, and mathematical language theory, etc. The cellular automaton (C.A.) is proposed as a simplified model of the complex huge systems, but the global behaviors of C.A. has not been elucidated still now except for some simple cases.[3,8] The difficulty in understanding the global characters of C.A. seems to mainly come from the essential discreteness of the system; space and time as well as state are discrete. As the results, the differentiability and the metric structure of C.A. is still ambiguous. The ordinary techniques developed in the smooth systems such as the perturbational approaches are not applied successfully.

One of the important problems in C.A. is to understand the space-time patterns. Recently, numerical studies have made clear the variety of patterns,[1,2] and the statistical properties have been analyzed by using the ergodic-theoretical concepts; entropies, Lyapunov exponents, and dimensions, etc. However, these approaches are too rough to estimate the detail behavior of each individual system as well as to predict the interrelation among several rules which belong to a lineal family.

In §3, we will study a special 1-dimentional family which admits a certain solitary wave solution. The elementary excitations, such as soliton, breather and kink, play important roles for the growth of complex patterns. The transient behavior of each elementary excitation gives us a significant information about the individual C.A. system. If the global behaviors of C.A. are decomposed into the collision processes among these elementary excitations, one can expect to classify the patterns in a unified manner. As the detailed analysis will be reported in another paper,[7] only a few examples will be explained in this paper.

Another purpose of this paper is to propose a new algebraic method to predict the global behaviors of C.A. systems. Some theoretical approaches have been proposed, but only a few systems with additive rules have been analysed so far. It has been desired to develope a new systematic method to attack the non-additive cases. Our approach is based on the irreducible decomposition of the local rule. Each local rule is uniquely decomposed into the sum of the symmetric and asymmetric fundamental functions. As the results, the rules with a special symmetric function are classified into a common family. In this paper, only the basic idea of our method will be explained by using the simple C.A. systems with nearest neighbor interactions. The further application to the more complex systems will be reported in the forthcoming paper.[9]

Let us consider the following systems,

$$S_i^{t+1} = F(S_{i-1}^t,\ S_i^t,\ S_{i+1}^t) \quad , \tag{1-1}$$

where S_i^t is the 1 or 0 state on the lattice site i at time t, and F is the mapping function describing the time evolution. Denoting the function R of Z as follows,

$$R(Z) = S_{i-1}^t \cdot Z^2 + S_i^t \cdot Z^1 + S_{i+1}^t \cdot Z^0 \ , \tag{1-2}$$

the mapping F is the two-valued function on the real number R. Under the conditions that F is symmetric, $F(R(Z)) = F(Z^2 R(Z^{-1}))$, and $F(0) = 0$, the number of the admissible rule is $2^5 = 32$. These admissible rules are denoted by $F(\overline{R}(2))$ in the same way as ref.1, i.e.,

$$\begin{aligned}\overline{R}(2) = (&0,\ 4,\ 18,\ 22,\ 32,\ 36,\ 50,\ 54,\ 72,\ 76,\ 90,\ 94,\\ &104,\ 108,\ 122,\ 126,\ 128,\ 132,\ 146,\ 150,\ 160,\ 164,\\ &178,\ 182,\ 200,\ 204,\ 218,\ 222,\ 232,\ 236,\ 250,\ 254).\end{aligned} \tag{1-3}$$

In these 32 rules, there are four identical cases derived from the mirror image relation. Two systems F and F' are called to be mirror image with each other, if they are satisfying the next relation,

$$F(X, Y, Z) = F'(X^*, Y^*, Z^*) + 1 \quad (\text{mod.}2) \qquad (1\text{-}4)$$

where $A^* = A + 1$ (mod.2). It is clear that the space time patterns created by F and F' are identical when the value of each site is exchanged as $1 \to 0$ and $0 \to 1$. Denoting the mirror image relation by $F \sim F'$, it is easy to see that,

$$F(164) \sim F(218) \quad , \quad F(132) \sim F(222)$$

$$F(160) \sim F(250) \quad , \quad F(128) \sim F(254) \quad . \qquad (1\text{-}5)$$

Therefore, the number of independent admissible rules is reduced to 28. The global classification of these 28 rules are given in the next section.

2. Algebraic structure of elementary cellular automata

The general theoretical approach to C.A. systems has not yet been succeeded except for the linear or additive systems. The purpose of this section is to provide a new algebraic method to study the global properties of C.A. systems. The method proposed here is easily applicable to the more complex C.A. systems. The fundamental idea is the irreducible decomposition of the local rule F.

Theorem

If F(X, Y, Z) is a two-valued function of X,Y, and Z, then the function is uniquely expressed by the following symmetric functions (f_0, f_1, and f_2) and an asymmteric function p,

$$f_0 = X + Y + Z$$

$$f_1 = XY + YZ + ZX$$

$$f_2 = XYZ \qquad (2\text{-}1)$$

$$p = (X - Y)(Y - Z)(Z - X) \quad .$$

In the cellular automata given by eq.(1-1), these fundamental functions are changed as follows,

$$f_0 = S_{i-1} + S_i + S_{i+1}$$

$$f_1 = S_{i-1}\cdot S_i + S_i\cdot S_{i+1} + S_{i+1}\cdot S_{i-1}$$

$$f_2 = S_{i-1}\cdot S_i\cdot S_{i+1} \qquad (2\text{-}2)$$

$$p = (S_i - S_{i-1})(S_i - S_{i+1}) \quad (\mathrm{mod.}2) \quad .$$

In fact, the arbitrary function F can be constructed by the sum of these fundamental functions.

$$F(0) = 0$$
$$F(4) = p + f_1 p$$
$$F(18) = f_0 + f_2 + p + f_1 p$$
$$F(22) = f_0 + f_2$$

$$
\begin{aligned}
F(32) &= f_1 + p \\
F(36) &= p \\
F(50) &= f_0 + f_2 + p \\
F(54) &= f_0 + f_2 + f_1p \\
F(72) &= f_1 + f_2 + f_1p \\
F(76) &= f_1 + f_2 + p \\
F(90) &= f_0 + f_1 + p \\
F(94) &= f_0 + f_1 + f_1p \\
F(104) &= f_1 + f_2 \\
F(108) &= f_1 + f_2 + p + f_1p \\
F(122) &= f_0 + f_1 + p + f_1p \\
F(126) &= f_0 + f_1 \\
F(128) &= f_2 \\
F(132) &= f_2 + p + f_1p \\
F(146) &= f_0 + p + f_1p \\
F(150) &= f_0 \\
F(160) &= f_2 + f_1p \\
F(164) &= f_2 + p \\
F(178) &= f_0 + p \\
F(182) &= f_0 + f_1p \\
F(200) &= f_1 + f_1p \\
F(204) &= f_1 + p \\
F(218) &= f_0 + f_1 + f_2 + p \sim F(164) \\
F(222) &= f_0 + f_1 + f_2 + f_1p \sim F(132) \\
F(232) &= f_1 \\
F(236) &= f_1 + p + f_1p \\
F(250) &= f_0 + f_1 + f_2 + p + f_1p \sim F(160) \\
F(254) &= f_0 + f_1 + f_2 \sim F(128)
\end{aligned}
\tag{2-3}
$$

In other words, the whole admissible functions of F are an additive groups of $\{0, f_0, f_1, f_2, p, f_1p\}$.

Among the elements of this group, only one special element $\{f_0\}$ can generate the propagating wave in C.A. systems. Therefore, the additive sub-group $\{0, f_1, f_2, p, f_1p\}$ can not create the propagating patterns such as type III,

$$F \in \{0, f_1, f_2, p, f_1p\} \rightarrow \text{Type I or II} \tag{2-4}$$

Furthermore, the formation of the localized patterns (Type II) is determined only by the element $\{p\}$, and as the results the Type I and II are separated as,

$$F \in \{0,\ f_1,\ f_2,\ f_1p\} \rightarrow \text{Type I}$$

$$F \in p + \{0,\ f_1,\ f_2,\ f_1p\} \rightarrow \text{Type II} \quad . \tag{2-5}$$

The Type III patterns are realized in the rules with $\{f_0\}$. Indeed, the triangular-like patterns can propagate in the system with $\{f_0\}$ when the collision of two triangular patterns does not occur. However, there are three singular rules, $f_0 + f_1 + f_1p$, $f_0 + f_2 + p$, and $f_0 + p$, which reveal the triangular patterns only transiently. When many triangular patterns are excited in the system, these three rules can not creat the Type III patterns persistently, but are settled down in the Type II patterns after a long transient time. In the work done by Wolfram, it is stated that the Type IV patterns do not exist in the nearest neighbor interaction case, but our classification implies that the above mentioned singular rules are considered to be the prototypes of the Type IV patterns. The reason why the propagating nature in these 3 singular rules is abolished in a finite time can be explained by tracing the non-propagating properties of the mirror image function of $\{f_0\}$. These points will be discussed in another paper.

3. Cellular automata with a certain solitary wave

The classification of the global patterns in C.A. is not completely finished, but the more precise classification is necessary. For example, in the Type IV case the propagating solitary wave patterns are often created, and the global patterns are strongly correlated to the existence of such soliton modes. In this section, we will discuss some typical behaviors generated in the C.A. systems with a certain soliton solution.

Let us consider the following systems with the interactions among the five neighboring sites,

$$S_i^{t+1} = F(S_{i\pm2}^t,\ S_{i\pm1}^t,\ S_i^t) \quad , \tag{3-1}$$

and assume that the local rule F admit the simple soliton solution $\cdots\ 0\overrightarrow{1011}0\ \cdots$ or $\cdots\ 0\overrightarrow{1101}0\ \cdots$. The mapping F is the two-valued function defined on the 20 coordinates under the conditions that the rule is symmetric. The number of the admissible rule is 2^{13} if the equilibrium condition F(0, 0, 0) = 0 is satisfied. As the systematic survey will be published in another paper, here only a few cases listed in Table I will be discussed.

<u>Example 1</u>

The collision process of two solitons becomes,

$$\begin{aligned} &\cdots\ 0\overrightarrow{1011}00\overleftarrow{1101}0\ \cdots \quad \rightarrow \quad \cdots\ 0\ \cdots \\ &\cdots\ 0\overrightarrow{1011}000\overleftarrow{1101}0\ \cdots \quad \rightarrow \quad \cdots\ 0\overleftarrow{1101}0\ \cdots\ 0\overrightarrow{1011}0\ \cdots\ . \end{aligned} \tag{3-2}$$

From the random initial condition of $\{S_i^0\}$, some solitons are created, but the number of soliton is decreased by the even phase collision.

<u>Example 2</u>

$$\begin{aligned} &\cdots\ 0\overrightarrow{1101}00\overleftarrow{1011}0\ \cdots \quad \rightarrow \quad \cdots\ 0\overleftarrow{1011}0\ \cdots\ 0\overrightarrow{1101}0\ \cdots \\ &\cdots\ 0\overrightarrow{1101}000\overleftarrow{1011}0\ \cdots \quad \rightarrow \quad \cdots\ 0\widetilde{101}0\ \cdots \quad \text{(Breather)} \end{aligned} \tag{3-3}$$

Table I Examples of soliton system

(*-coordinates determine a pure soliton)

coordinates $(S_{i-2},S_{i-1},S_i,S_{i+1},S_{i+2})^t$		mapping $F = S_i{}^{t+1}$				
		Ex.1	Ex.2	Ex.3	Ex.4	Ex.5
0,0,0,0,0	*	0	0	0	0	0
1,0,0,0,0	*	0	0	0	0	0
0,1,0,0,0	*	0	1	1	0	1
0,0,1,0,0		0	1	1	0	0
1,1,0,0,0	*	1	0	0	1	0
0,1,1,0,0	*	1	0	0	1	0
1,0,0,0,1		1	0	0	1	1
1,0,1,0,0	*	0	0	0	0	0
1,0,0,1,0		0	1	1	0	1
0,1,0,1,0		0	0	1	0	0
1,1,0,1,0	*	1	1	1	1	1
1,1,0,0,1		1	0	0	1	0 or 1
0,1,1,1,0		0	1	1	0 or 1	1
1,0,1,1,0	*	0	1	1	0	1
1,0,1,0,1		0	1	0	0	0 or 1
1,1,1,0,0		0	0	0	1	1
1,1,0,1,1		1	0	0	1	1
1,1,1,0,1		1	1	1	1	1
1,1,1,1,0		0	0	0	0	0
1,1,1,1,1		1	0	0	1	0
Soliton		$\overrightarrow{1011}$	$\overrightarrow{1101}$	$\overrightarrow{1101}$	$\overrightarrow{1011}$	$\overrightarrow{1101}$
Breather		×	o	×	×	o or ×
Kink		×	×	o	×	×
Nucleus		×	×	×	×	o or ×

In this case, there is a typical localized pattern with period 4;
$\cdot\cdot$ 01010 $\cdot\cdot \to \cdot\cdot$ 0100010 $\cdot\cdot \to \cdot\cdot$ 011101110 $\cdot\cdot \to \cdot\cdot$ 0110110 $\cdot\cdot \to \cdot\cdot$ 01010$\cdot\cdot$.
This kind of localized pattern is called breather for short. The collisions between soliton and the breather becomes,

$$
\begin{aligned}
&\cdot\cdot\ 0\overrightarrow{1101}00000\widetilde{1010}\ \cdot\cdot &&\to\ \cdot\cdot\ 0\widetilde{1010}\ \cdot\cdot\ , \\
&\cdot\cdot\ 0\overrightarrow{1101}0000\widetilde{1010}\ \cdot\cdot &&\to\ \cdot\cdot\ 0\overleftarrow{1011}0\ \cdot\cdot\cdot\ 0\overrightarrow{1101}0\ \cdot\cdot\ , \\
&\cdot\cdot\ 0\overrightarrow{1101}000\widetilde{1010}\ \cdot\cdot &&\to\ \cdot\cdot\ 0\overleftarrow{1011}0\ \cdot\cdot\ , \\
&\cdot\cdot\ 0\overrightarrow{1101}00\widetilde{1010}\ \cdot\cdot &&\to\ \cdot\cdot\ 0\overleftarrow{1011}0\ \cdot\cdot\cdot\ 0\overrightarrow{1101}0\ \cdot\cdot\ .
\end{aligned}
\tag{3-4}
$$

After long period, the breathers dominate the final stage.

Example 3

$$
\begin{aligned}
&\cdot\cdot\ 0\overrightarrow{1101}00\overleftarrow{1011}0\ \cdot\cdot &&\to\ \cdot\cdot\ 0\overleftarrow{1011}0\ \cdot\cdot\cdot\ 0\overrightarrow{1101}0\ \cdot\cdot \\
&\cdot\cdot\ 0\overrightarrow{1101}000\overleftarrow{1011}0\ \cdot\cdot &&\to\ \cdot\cdot\ 0\overleftrightarrow{10101}0\ \cdot\cdot\quad \text{(kink)}
\end{aligned}
\tag{3-5}
$$

Only two coordinates are different from the case of example 2, but the final stage is dominated by the kink or some solitons. The collisions between soliton and kink becomes,

$$
\begin{aligned}
&\cdot\cdot\ 0\overrightarrow{1101}00\overleftarrow{1010}1 &&\to\ \cdot\cdot\ 0\overleftarrow{1011}0\ \cdot\cdot\cdot\ 0\overrightarrow{11010}101 \\
&\cdot\cdot\ 0\overrightarrow{1101}000\overleftarrow{10101} &&\to\ \cdot\cdot\ 0\overrightarrow{110101}
\end{aligned}
\tag{3-6}
$$

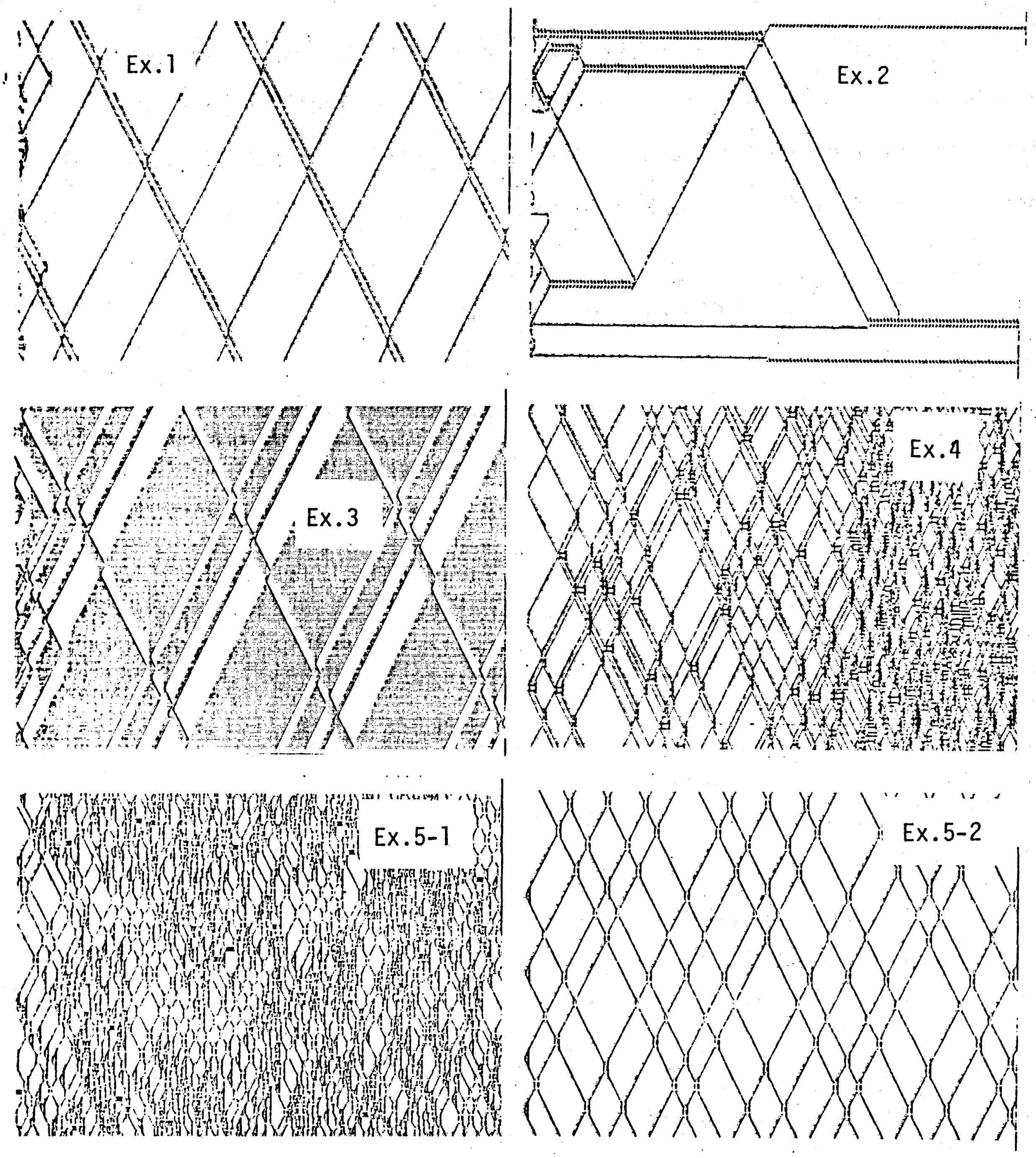

Fig. 1 Time evolution of each soliton system
The vertical axis is site and the horizontal axis is time.
In the case of Ex.5, the dilute gases of soliton behaves quasi-periodically (Ex.5-2), but the soliton-turbulence is realized in the dense gases (Ex.5-1).

Example 4

$$\cdot\cdot\ 0\overrightarrow{1011}00\overleftarrow{1101}0\ \cdot\cdot \quad \rightarrow \quad \cdot\cdot\ 0\overleftarrow{1101}00\overleftarrow{1101}0\ \cdots\ 0\overrightarrow{1011}00\overrightarrow{1011}0\ \cdot\cdot$$

$$\cdot\cdot\ 0\overrightarrow{1011}000\overleftarrow{1101}0\ \cdot\cdot \quad \rightarrow \quad \cdot\cdot\ 0\overleftarrow{1101}0\ \cdots\ 0\overrightarrow{1011}0\ \cdot\cdot$$

The number of soliton is increased by the even phase collision, and finally the soliton-turbulence state is realized.

Example 5

Two solitons are always conserved after the odd and even phase collisions, but the n-soliton collisions ($n \geqq 3$) induce the multiplication of soliton. As the results, the soliton-turbulence state is realized after a long time. In this case, there may exist the breather pattern and the nucleus which emit the soliton-chain persistently. However, they are very unstable and easily destroyed by the collision with solitons.

The results of the simulation are illustrated in Figure 1, where the state s = 1 are shown by the black spot in the white region (s = 0). The system size is 200, and the periodic boundary condition is adopted.

In this section, we have not discussed the collision processes of many solitons. These problems will be reported elesewhere together with the statistical analysis of the soliton-turbulence.

4. Discussions

The variety of the global patterns created in C.A. systems are not exhausted at all by the 4 typical patterns obtained from the phenomenological classification. As discussed in §3, it is possible to exist many elementary excitations in the system, and the complex patterns can be created by the collisions of those excitations (soliton, breather, and kink). Furthermore, there often exist some compound excitations;

(A) Giant breather which emits the solitons periodically,

(B) Giant soliton which emits the simple breathers periodically.

Our studies shown in §3 are the first step necessary for elucidating the roles of the elementary excitations.

The research of the complex behaviors in C.A. is an interesting statistical mechanical subject. The global properties of C.A. must be understood theoretically from each deterministic rule. The basic idea proposed in §2 can be applied straightforwardly to the more complex C.A. systems; for example the totalistic rules treated in ref. 2 have been successfully analysed by the same method.

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