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# ANOTHER CONSTRUCTION OF COUNTEREXAMPLES TO COLEMAN'S CONJECTURE

岡　伸篤

Nobuatsu Oka

Department of Mathematics
Faculty of Science
Nagoya University
Chikusa-ku Nagoya
464 (JAPAN)

## ABSTRACT

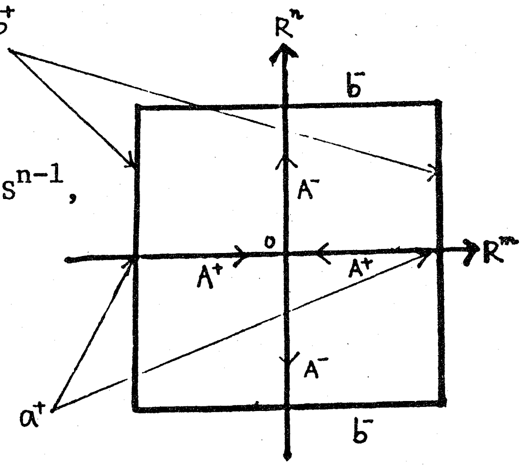
We construct another counterexample which is different from Pixton's and, using this example, we construct countably many distinct counterexamples on $R^m \times R^2$ $(m \geqq 3)$.

## §1. INTRODUCTION

Suppose that $\psi$ is a flow on $R^{m+n}$ that satisfies the following conditions:

(1) The origin $\{0\}$ is an isolated invariant set for $\psi$ which has the isolating block $B = D^m \times D^n$ (cf. F. W. Wilson [6]).

Figure 1




(2) The structure of B is

$$b^+ = \partial D^m \times D^n,$$

$$b^- = D^m \times \partial D^n,$$

$$\tau = b^+ \cap b^- = \partial D^m \times \partial D^n = S^{m-1} \times S^{n-1},$$

$$A^+ = D^m \times \{0\},$$

$$A^- = \{0\} \times D^n,$$

$$a^+ = \partial D^m \times \{0\} = b^+ \cap A^+,$$

and

$a^- = \{0\}\times\partial D^n = b^- \cap A^-$.

(See Figure 1)

$B^+$ is the ingressing set $\{p\in\partial B|p.(-\varepsilon,0)\cap B = \phi$ for some $\varepsilon>0\}$, $b^-$ is the egressing set $\{p\in\partial B|p.(0,\varepsilon)\cap B = \phi$ for some $\varepsilon>0\}$ $A^+$ is the stable manifold $\{p\in B|p.[0,\infty)\subset B\}$ of $\{0\}$ in B and $A^-$ is the unstable manifold $\{p\in B|p.(-\infty,0]\subset B\}$ of $\{0\}$ in B. Here p.I means $\{\psi(p,t)|t\in I\}$ for any interval I.

C. Coleman [1] raised a conjecture at $5^{th}$ International Conference on Nonlinear Oscillations (Kiev 1970). The following is the reformulated one in terms of the isolating blocks by Wilson [6].

Coleman's conjecture: If the flow $\psi$ satisfies the conditions (1) and (2), then it is locally topologically equivalent to the standard hyperbolic example $\psi_{mn}$ generated by the differential equations

$$\dot{x} = -x, \quad \dot{y} = y \qquad (x,y) \in R^m\times R^n.$$

We say that $\psi$ is locally topologically equivalent to $\psi_{mn}$ at $\{0\}$ if there is a homeomorphism $\chi:U \to \chi(U) \subset R^{m+n}$ on some neighbourhood U of $\{0\}$ that takes each orbit segment of $\psi$ in U onto an orbit segment of $\psi_{mn}$ in $\chi(U)$ and preserves the natural orientation of the orbits. Note that, in this conjecture, U is the isolating block B of $\psi$ and $\chi(U)$ is the isolating block B of $\psi_{mn}$.

In 1980, D. A. Neumann [2] constructed the first counterexample in the case that both $A^+$ and $A^-$ are two dimensional, i.e., he showed that there is a flow $\psi$ on $R^2\times R^2$ that is not topologically equivalent at $\{0\}$ to $\psi_{22}$. Furthermore, he constructed uncountably many distinct counterexamples (cf. [3]). R. B. Walker [5] also gave the counterexample in the case that either $A^+$ or $A^-$ is two dimensional. On the other hand, Wilson [8] showed that the conjecture is true when $A^+$ or $A^-$ is one dimensional.

In 1981, D. Pixton [4] constructed a counterexample in the general case $R^m \times R^N$ $(m \geq 2,\ n \geq 2)$.

In this paper we construct another counterexample which is different from Pixton's and, using this example, we construct countably many distinct counterexamples on $R^m \times R^2$ $(m \geq 3)$.

## §2 NOTATIONS AND PROPOSITIONS

To detect countably many distinct counterexamples we use the fluctuations which were used by Walker when he constructed countably many counterexamples in the case that $A^+$ and $A^-$ are two dimensional (cf. Walker [5]).

Let $\psi$ be a flow on the isolating block $B_\psi$ satisfying (1) and (2) and let $h_\psi : b^+ - a^+ \to b^- - a^-$ be a map which maps each point $p \in b^+ - a^+$ to the first point in which the semiorbit $p.[0,\infty)$ meets $b^-$. This map $h_\psi$ is called the Poincaré map of the flow $\psi$. [The Poincaré map of the standard flow $\psi_{mn}$ is denoted by $h_{mn}$.]

Let $\chi : B \to B$ be a homeomorphism on B giving the topological equivalence between the flow $\psi$ and the standard flow $\psi_{mn}$. Then $\chi$ induces

$$\chi|_{b^+} : (b^+, a^+) \to (b^+, a^+)$$

and

$$\chi|_{b^-} : (b^-, a^-) \to (b^-, a^-).$$

In general, if the flows $\psi$ and $\psi$ on $R^{m+n}$ are topologically equivalent, then the Poincaré maps $h_\psi$ and $h_\psi$ satisfy

$$(*) \qquad (\chi|_{b^-}) \circ h_\psi = h_\psi \circ \chi|_{b^+ - a^+} .$$

where the homeomorphism $\chi$ maps the isolated block $B_\psi$ to $B_\varphi$, i.e., $h_\psi$ and $h_\varphi$ are topologically equivalent.

Since $b^+ - a^+ = \partial D^m \times (D^n - \{0\}) \cong S^{m-1} \times S^{n-1} \times (0.1]$ and $b^- - a^- = (D^m - \{0\}) \times \partial D^n \cong S^{m-1} \times S^{n-1} \times (0,1]$ have $(\mu,\nu,r)$-coordinates, the Poincaré map of the standard flow is the identity map with respect to these coordinates.

Let $\tau_r$ be $S^{m-1} \times S^{n-1} \times \{r\} \subset S^{m-1} \times S^{n-1} \times (0,1]$ and $\tau^+$ (resp. $\tau^-$) be $\tau_r$ in $b^+ - a^+$ (reep . $b^- - a^-$). Let

$$L^+(\mu) = \{\mu\} \times D^N \subset b^+,$$

$$L^+(S^{m-2}) = S^{m-2} \times D^n \subset b^+ \quad [\, S^{m-2} \text{ is the equator of } S^{m-1}],$$

$$L^+(\nu) = \partial D^m \times \{\nu\} \times (0,1] \subset b^+,$$

$$l_r^+(\mu) = L^+(\mu) \cap \tau_r^+,$$

$$l_r^-(\mu) = L^-(\mu) \cap \tau_r^-,$$

and

$$l_r^+(S^{m-2}) = L^+(S^{m-2}) \cap \tau_r^+.$$

In the same way, let

$$L^-(\nu) = D^m \times \{\nu\} \subset b^-,$$

and

$$l_r^-(\nu) = L^-(\nu) \cap \tau_r^-.$$

Let $d_\mu$ (resp. $d_\nu$) be the metric on $S^{m-1}$ (resp. $S^{n-1}$) $\subset R^m \times R^n \cong R^{m+n}$. Then $d_\mu$ (resp. $d_\nu$) induces a pseudo-metric $d_\mu$ (resp. $d_\nu$) on $b^+$ and $b^-$, i.e., $d_\mu((\mu',\nu',r'),(\mu'',\nu'',r'')) = d_\mu(\mu',\mu'')$ [resp. $d_\nu((\mu',\nu',r'),(\mu'',\nu'',r'')) = d_\nu(\nu',\nu'')$] then each $L^+(\mu_0)$ has a $\varepsilon$-neighbourhood $N_\varepsilon^\mu(L^+(\mu_0)) = \cup\{L^+(\mu) \mid d_\mu(\mu,\mu_0)<\varepsilon\}$ and $L^+(S^{m-2})$ has a $\varepsilon$-neighbourhood $N_\varepsilon^\mu(L^+(S^{m-2})) = \cup\{L^+(\mu) \mid d_\mu(S^{m-2},\mu)<\varepsilon\}$. Here after, the flow $\psi$ on $R^m \times R^n$ ($m \geq 3$ or $n \geq 3$) is assumed to satisfy (1) and (2).

If the flow $\psi$ has the isolated block B with the m-dimensional stable manifold $A^+$ and the n-dimensional unstable manifold $A^-$, then we call $\psi$ <u>m×n type flow</u>. Let $\psi$ and $\varphi$ be m×n type flows and suppose the homeomorhism $\chi$:B → B maps the orbits of $\psi$ to the orbits of $\varphi$. In other words, suppose the Poincaré maps $h\psi$ and $h\varphi$ satisfy (*). Then we have the following propositions.

<u>Proposition</u> 1 (Walker [5]). For any $\varepsilon>0$, there exists an $r_\varepsilon$ (>0) such that for any r ($0<r<r_\varepsilon$) the following inclusions hold:

(i) $(\chi|_{b^+-a^-})\ l_r^+(\mu) \subset N_\varepsilon^\mu(L^+(\chi|_{a^+}(\mu)))$,

(ii) $(\chi|_{b^--a^-})\ l_r^-(\nu) \subset N_\varepsilon^\nu(L^-(\chi|_{a^-}(\nu)))$.

<u>Proposition 2</u>. For any tubular neighbourhood $N(L^+((\chi|_{a^+})(S^{m-2})))$ of $L^+((\chi|_{a^+})(S^{m-2}))$ in $b^+$, there exists an $r_N$ (>0) such that for any r ($<r_N$) the following inclusions hold:

(iii) $(\chi|_{b^+-a^+})\ l_r^+(S^{m-2}) \subset N(L^+((\chi|_{a^+})(S^{m-2})))$,

(iv) $(\chi|_{b^--a^-})\circ h_\psi l_r^+(S^{m-2}) \subset h_\psi(N(L^+((\chi|_{a^+})(S^{m-2}))))$.

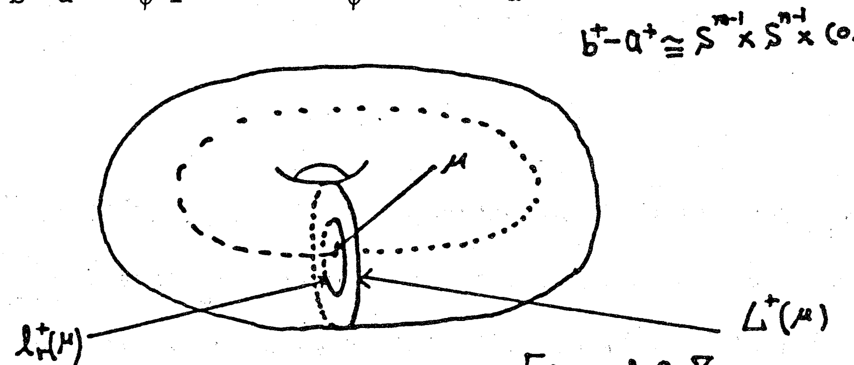



Figure 2

Next theorem is very effective to show the existence of a flow $\psi$ which corresponds with a given Poincaré map.

Theorem 3 (Wilson). Let $P: S^{m-1}\times(0,\varepsilon]\times S^{n-1} \to S^{m-1}\times(0,\varepsilon]\times S^{n-1}$ be a $C^r$-diffeomorphism which is a strongly $C^r$-isotopic to the identity relative to $\tau_\varepsilon = S^{m-1}\times\{\varepsilon\}\times S^{n-1}$ $(r>0)$. Then, there is a $C^r$-flow $\psi$ on B which coincides with the standard type flow $\psi_{mn}$ in a neighbourhood of $A^+ \cup A^- \cup \tau-\{0\}$ and which has the property that $h_\psi|_{S^{m-1}\times(0,1]\times S^{n-1}} = P$.

Now we proceed to define fluctuations. In this paragraph, let every flow be assumed to be m×2 type ($m \geqq 3$). Let $\Gamma_r: I \to \tau_r^-$ be a closed arc and consider $\hat{\nu}\circ\Gamma_r: I \to R$ where $\hat{\nu}: b^- \to R$ is a lift of the circular coordinate function $\nu: b^- \to S^1$.

For given $\Delta$ $(0<\Delta<\pi)$, let $FL_\nu(\Gamma_r;\Delta)$ be the cardinal number of $\nu$-fluctuations and let $\{s_0, s_1, s_2, \ldots, s_n, \ldots\} \subset I$ be a sequence of fluctuation points which are defined as follows:

Put $s_0 = 0$ (the origin of $\hat{\nu}$ coordinate) and

$s_1 = \mathrm{Min}\{s\in I \,|\, |\hat{\nu}\circ\Gamma_r(s) - \hat{\nu}\circ\Gamma_r(s_0)| = \Delta\}$ if it exists; otherwise define $FL_\nu(\Gamma_r;\Delta) = 0$. If $s_1$ exists then define $\sigma = \mathrm{sign}\,(\hat{\nu}\circ\Gamma_r(s_1) - \hat{\nu}\circ\Gamma_r(s_0))$. If $s_{i-1}$ exists for $i>1$ then define

$s_i = \mathrm{Min}\{s > s_{i-1} \,|\, \hat{\nu}\circ\Gamma_r(s) - \hat{\nu}\circ\Gamma_r(\bar{s}) = (-1)^{i-1}\sigma\Delta\}$ for some $\bar{s}$ $(s_{i-1} \leqq \bar{s} \leqq s)\}$

if it exists; otherwise define $FL_\nu(\Gamma_r;\Delta) = i-1$. Now we have the following lemma.

Lemma 4 (Walker). For all $\Delta$ $(0<\Delta<\pi)$, there exists an $r_\Delta$ $(>0)$ such that for all $r$ $(0<r<r_\Delta)$ and closed are $\Gamma_r: I \to \tau_r^-$ the following inequality holds:

$$FL_\nu(\Gamma_r;\Delta) \leqq FL_\nu((\chi|_{b^- - a^-})\circ\Gamma_r; c(\Delta)/2)$$

where

$c(\Delta) = \mathrm{Min}\{\Delta,\Delta'\}$ $(\Delta' = \mathrm{Min}\{d_\nu((\chi|_{a^-})(\nu),(\chi|_{a^-})(\nu+\Delta)) \mid \nu\in a^-\})$ and $d_\nu$ is the metric on $a^-$.

Remark. Since $a^-$ is compact, there exists nonzero lower bound of $d_\nu((\chi|_{a^-})(\nu),(\chi|_{a^-})(\nu+\Delta))$.

## §3 CONSTRUCTION OF THE COUNTEREXAMPLE

Since the isolating block of m×n type ($m\geqq 3$, $n\geqq 3$) is homeomorphic to $D^m\times D^n$, $b^- - a^-$ is divided into

$(D^m-\{0\})\times\partial D^n \cong S^{m-1}\times(0,1]\times S^{n-1} \cong S^{m-1}\times(0,1]\times(I\times S^{n-2}\cup D_+^{n-1}\cup D_-^{n-1})$.

Changing the order of the product $S^{m-1}\times(0,1]\times I\times S^{n-2}$, let us consider the coordinate $(\mu,\nu_1,\nu_2,r)$ on $S^{m-1}\times I\times S^{n-2}\times(0,1]$.

Let $\bar{I}_1$, $\hat{\bar{I}}_1$, $\bar{I}_2$ and $\hat{\bar{I}}_2$ be m-dimensional annuli homeomorphic to $S^{m-2}\times I$. Let the $\nu$-coordinates of $\bar{I}_1$, $\hat{\bar{I}}_1$, $\bar{I}_3$ and $\hat{\bar{I}}_3$ be c, the $\nu_1$-coordinate of $\bar{I}_2$ and $\hat{\bar{I}}_2$ be d and put e as in Figure 4. (Also see Figure 3.) Let $\Sigma$ be a $C^\infty$-isotopically deformed $S^{m-2}\times I$ in $S^{m-1}\times I$ as in Figure 3, and $I_i$ (resp. $\hat{I}_i$) and $\Sigma$ (i=1,2,3) be $(\nu_2,r)$-saturation of $\bar{I}_i$ (resp. $\hat{\bar{I}}_i$ and $\bar{\Sigma}$), i.e., $\Sigma=\{(\mu,\nu_1,{}_2,r)\mid(\mu,\nu_1)\in\bar{\Sigma}\}$ and $I_i=\{(\mu,\nu_1,\nu_2,r)\mid(\mu,\nu)\in\bar{I}_i\}$.

Firstly, for sufficiently small r (>0), we construct the diffeomorphism $h_\psi$ which is $C^\infty$-isotopic to the identity and satisfies $h_\psi(l_r^+(S^{m-2})) = \Sigma\cap\tau_r^-$. Here, $S^{m-2}$ denotes the equator of $a^+\simeq S^{m-1}$ and the isotopy preserves $(\nu_2,r)$-coordinate and fixes $S^{m-1}\times(0,1]\times D_+^{n-1}$ and $S^{m-1}\times(0,1]\times D_-^{n-1}$.

Since the isotopy fixes the small neighbourhood of the boundary $S^{m-1}\times I$ in Figure 3, the $\mu$-coordinates of two endpoints of the interval I coincide. Hence, by taking account of the construction of $C^\infty$-isotopy and Theorem 3, it will be shown that for given $h_\psi$ there exists an m×n-type flow ψ with Poincaré map $h_\psi$.

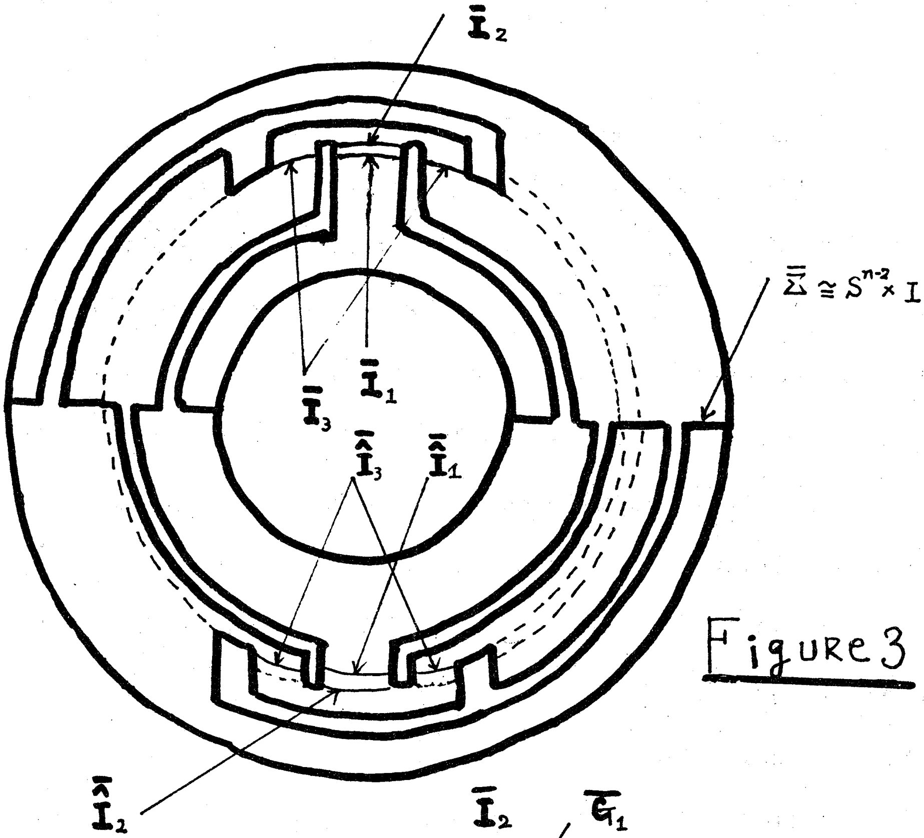



Figure 3

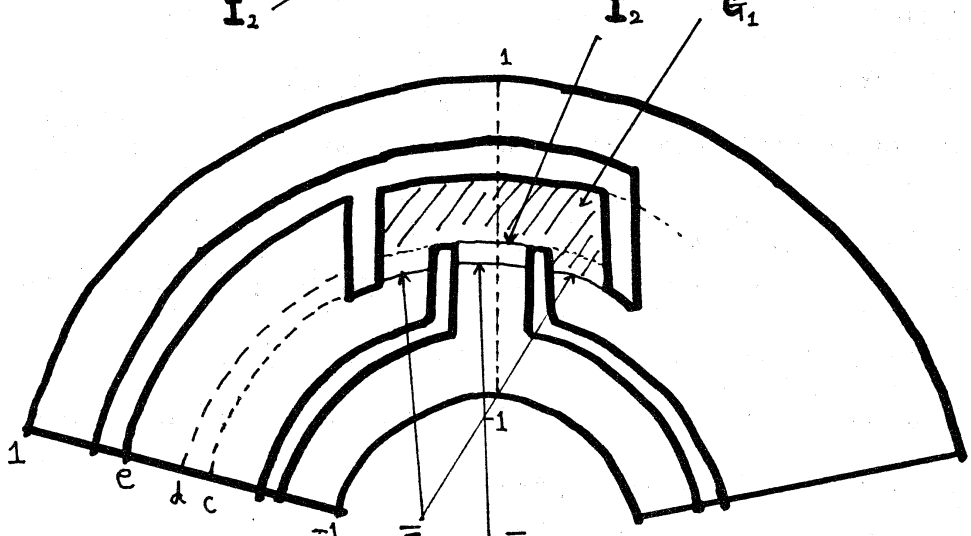



Figure 4

## §4 PROOF THAT $\psi$ AND $\psi_{mn}$ ARE NOT TOPOLOGICALLY EQUIVALENT

Suppose that the flows $\psi$ and $\psi_{mn}$ ($m \geqq 3$, $n \geqq 3$) are topologically conjugate, i.e.,

(1) $$(\chi|_{b^-}) \circ h_\psi = h_{mn} \circ (\chi|_{b^- - a^-})$$

and let the topological imbedding $\gamma_{r'}: a^- \hookrightarrow b^- - a^-$ be defined by $\gamma_{r'}(\nu_1,\nu_2) = (\chi|_{b^- - a^-})^{-1}(\hat{\mu}_0,\nu_1,\nu_2,r')$, here $\hat{\mu}_0$ is an arbitrary element of $\overline{\partial N(L^+((\chi|_{a^+})(S^{m-2})))} \cap a^+$ and $r'$ is chosen smaller than $r'_\varepsilon$ of which existence is assured in Proposition 1 where $\chi|_{b^-}$ is replaced by $(\chi|_{b^-})^{-1}$. Furthermore, choose $r''$ sufficiently small so that $(\chi|_{b^- - a^-})(S^{m-1}\times S^{n-1}\times(0,r'')) \subset S^{m-1}\times S^{n-1}\times(0,r_\varepsilon)$, here $r_\varepsilon$ is determined according to $\varepsilon$ and $\chi|_{b^-}$ in Proposition 1, and choose $r'$ to be smaller than $\mathrm{Min}\{r'_\varepsilon, r''\}$. Then $\Gamma_{r'} = \gamma_{r'}(a^-)$ does not intersect $\Sigma$ by Proposition 2 (iV) since the condition (1) holds.

In fact, the inclusion

$$h_\psi(1^+_r(S^{m-2})) \subset (\chi|_{b^- - a^-}) \circ h_{mn}(N((\chi|_{a^+})(S^{m-2})))$$

holds for any $r$ smaller than $r_0 = \mathrm{Min}\{r_\varepsilon, r_N\}$. On the other hand, by the definition of $\mu$-coordinate of $\Gamma_{r'}$, the inclusion

$$\Gamma_{r'} \subset (\chi|_{b^- - a^-}) \circ h_{mn}(\overline{\partial N((\chi|_{a^+})(S^{m-2}))}) \qquad (r' < r_0)$$

holds. Hence it follows from the inclusion $\Gamma_{r'} \subset S^{m-1}\times S^{n-1}\times(0,r_\varepsilon]$ that $\Gamma_{r'}$ does not intersect $\Sigma$ for any $r'$ smaller than $r_\varepsilon$.

Let $G_1$ (resp. $G_2$) be the $(\nu_2,r)$-saturation of the shaded region $\bar{G}_1$ (resp. $\bar{G}_2$) in Figure 3, i.e., for example,

$$G_1 = \{(\mu,\nu_1,\nu_2,r) \in S^{m-1}\times I\times S^{n-1}\times(0,1] \mid (\mu,\nu_1) \in \bar{G}_1\},$$

and let

(2) $$H^j_{r'} = \gamma^{-1}_{r'}(G_j) \qquad (j=1,2).$$

Then $\pi(G_1 \cup G_2)$ is included in $\{(\nu_1,\nu_2)\,|\,c<\nu_1<e\}$ $(\subset I\times S^{n-2} \subset a^-)$ where $\pi: b^- - a^- \to a^-$ is the natural projection defined by $\pi(\mu,\nu,r)=(\nu)$.

Let $i_{r'}: a^- \hookrightarrow S^{m-1}\times S^{n-1}\times(0,1]$ be a topological imbedding defined by $i_{r'}(\nu_1,\nu_2)=(\mu_0,\nu_1,\nu_2,r')$, then it follows that

(3) $\quad \gamma_{r'} = (\chi|_{b^- - a^-})^{-1} \circ i_{r'}$ and

(4) $\quad d_\nu((\chi|_{a^-})^{-1}(H^j_{r'}),\ \pi\ (\chi|_{b^- - a^-})^{-1} \circ i_{r'}(H^j_{r'})) < \varepsilon$

(j=1,2).

Hence we have

(5) $\quad (\chi|_{a^-})^{-1}(H^j_{r'}) \subset \{(\nu_1,\nu_2)\,|\,c-\varepsilon<\nu_1<e+\varepsilon\}$ $\qquad$ (j=1,2).

Note that it follows from (2) and (3) that

$$\pi\circ(\chi|_{b^- - a^-})^{-1} \circ i_{r'}(H^j_{r'}) = \pi\circ\gamma_{r'}(H^j_{r'}) = \pi(G_j) \qquad (j=1,2).$$

Though $\Gamma_{r'} \cap I_i \neq \phi$ or $\Gamma_{r'} \cap I_i \neq \phi$ (i=1,2,3) hold as in Figure 4, these conditions do not occur simultaneously since $S^{m-2}\times S^{n-1}\times(0,1]$ divides $S^{m-1}\times S^{n-1}\times(0,1]$ into two components. Hence let us assume the case $\Gamma_{r'}\cap I_i \neq \phi$ (i=1,2,3). (See Figure 6.)

Since $\partial H^i_{r'} = \gamma_{r'}^{-1}(\partial G_i) = \gamma_{r'}^{-1}(I_1 \cup I_3)$, it follows that $d_\nu((\chi|_{a^-})^{-1}(\partial H^1_{r'}),\ \pi\circ(\chi|_{b^- - a^-})^{-1} \circ i_{r'}\circ\gamma_{r'}^{-1}(I_1\cup I_3)) < \varepsilon$ for $r'(<r'_\varepsilon)$.

Noting that both $I_1$ and $I_3$ are included in $\{(\mu,\nu_1,\nu_2,r)\,|\,\nu_1=c\}$ $(\subset b^- - a^-)$ we have

(6) $\quad \partial(\chi|_{a^-})^{-1}(H^1_{r'}) \subset \{(\nu_1,\nu_2)\ |\ c-\varepsilon<\nu_1<c+\varepsilon\}(\subset a^-)$.

Now we will show

(7) $\quad (\chi|_{a^-})^{-1}(H^1_{r'}) \subset \{(\nu_1,\nu_2)\,|\,c-\varepsilon<\nu_1<c+\varepsilon\}$.

Let C be the region $I\times S^{n-2}\times\{0\} \cap \{(\nu_1,\nu_2)\,|\,\nu_1 \geqq c+\varepsilon\} \subset a^-$, then C is connected. Suppose that $(\chi|_{a^-})^{-1}(H^1_{r'}) \cap C \neq \phi$, hence $(\chi|_{a^-})^{-1}(H^1_{r'})$ includes C. This contradicts (5). Thus $(\chi|_{a^-})^{-1}(H_{r'}) \cap C = \phi$, and (7) follows from $\Gamma_{r'} \cap I_2 = \phi$ and

$d_\nu((\chi|_{a^-})^{-1}\circ\gamma^{-1}(I_2), \pi\circ(\chi|_{b^- -a^-})^{-1}\circ i_{r'}\, \gamma^{-1}(I_2)) < \varepsilon$,
where $(\chi|_{a^-})^{-1}\circ \gamma_{r'}^{-1}(I_2) \subset (\chi|_{a^-})^{-1}(H^1_{r'})$ and
$\pi\circ(\chi|_{b^- -a^-})^{-1}\circ i_{r'}\circ\gamma_{r'}^{-1}(I_2) = \pi(I_2) = d$, hence it follows that
$(\chi|_{a^-})^{-1}(H^1_r)$ must meet $\{(\nu_1,\nu_2)|d-\varepsilon<\nu_1<d+\varepsilon\}$. This also leads to the contradiction and hence $\psi$ and $\psi_{mn}$ are not topologically equivalent.

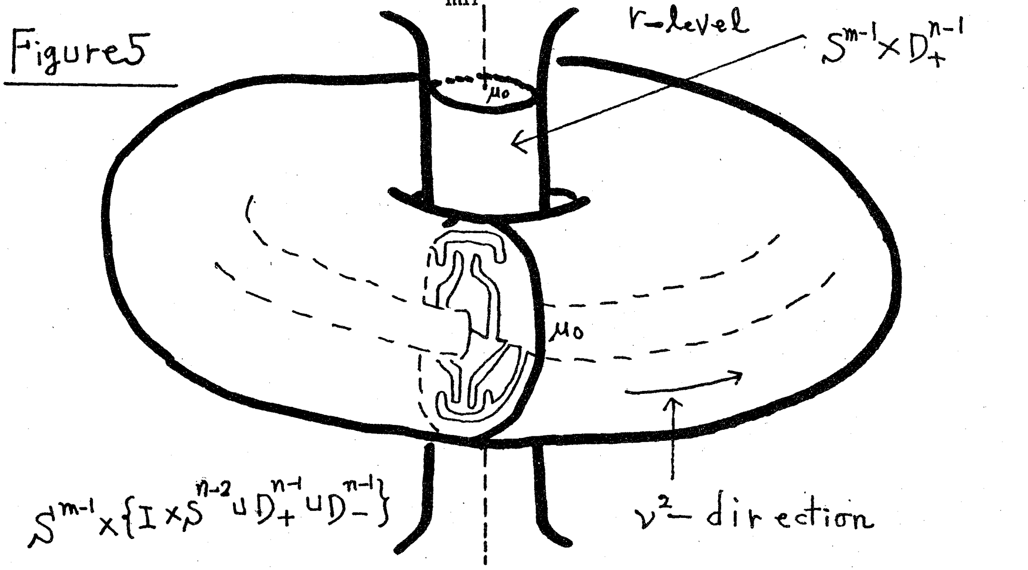



Figure5

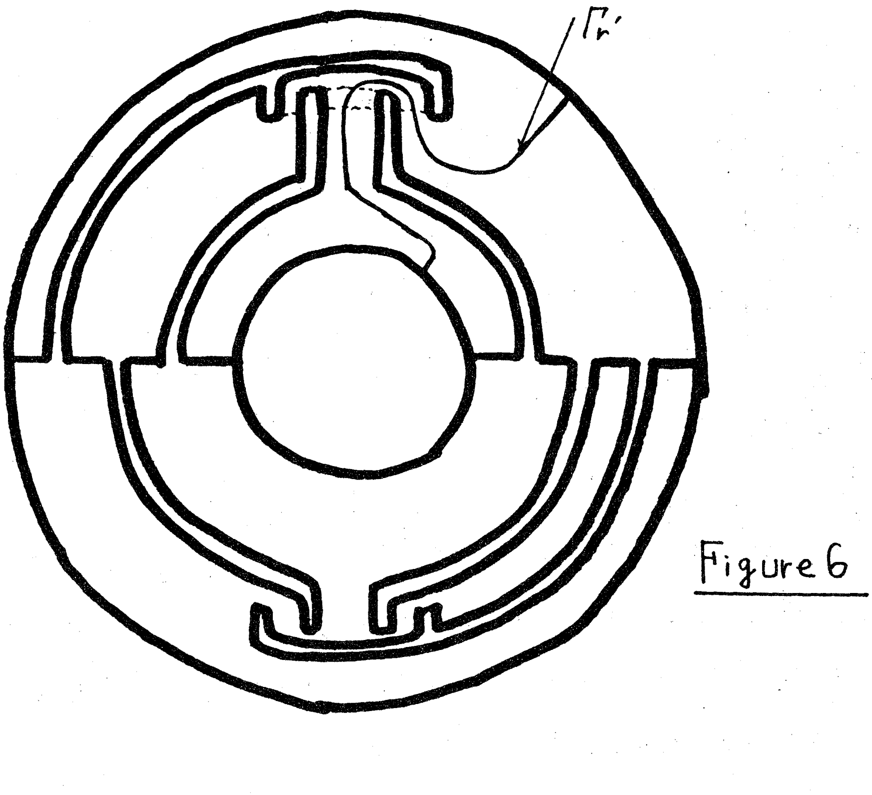



Figure 6

## §5 CONSTRUCTION OF THE MULTIPLE EXAMPLE

In this section, all flows are assumed to be of $m\times 2$ type.

Thus, $b^+\cong \partial D^m\times D^2 \cong S^{m-1}\times D^2,$

$b^-\cong D^m\times\partial D^2 \cong D^m\times S^1,$

$a^+\cong S^{m-1}\times\{0\},$

$a^-\cong \{0\}\times S^1$

and

$$b^+- a^+ \cong b^- - a^- \cong S^{m-1}\times S^1\times(0,1].$$

Divide $S^{m-1}\times S^1$ into $S^{m-1}\times I^+$ and $S^{m-1}\times I^-$ $(I^+\cup I^- = S^1)$ and deform $S^{m-2}\times I^+$ into $\Sigma^+$ as in Figure 8 by a diffeomorphism on $S^{m-1}\times I^+$ which is isotopic to the identity relative to $S^{m-1}\times[-1,-1+\varepsilon_1]$ and $S^{m-1}\times[1-\varepsilon_2,1]$, where I denotes $[-1,1]$ and both $\varepsilon_1$ and $\varepsilon_2$ are sufficiently small positive numbers. On the other hand, $S^{m-1}\times I^-$ remain fixed during the deformation and we call this a standard cassette. Glue this standard cassette and $S^{m-1}\times I^+$ at each boundaries by the identity map id: $S^{m-1} \to S^{m-1}$, i.e., glue

$$S^{m-1}\times\{-1\} \subset S^{m-1}\times I^+ \text{ to } S^{m-1}\times\{+1\} \subset S^{m-1}\times I^-$$

and

$$S^{m-1}\times\{+1\} \subset S^{m-1}\times I^+ \text{ to } S^{m-1}\times\{-1\} \subset S^{m-1}\times I^-.$$

Then we obtain a deformed $S^{m-2}\times S^1$ in $S^{m-1}\times S^1$, in other words, we obtain a $C^\infty$-diffeomorphism which is isotopic to the identity and it deforms $S^{m-2}\times I^+ \subset S^{m-1}\times I^+ \subset S^{m-1}\times S^1$ to $\Sigma^+$ as in Figure 8. Using this diffeomorphism, we can construct an r-preserving diffeomorphism on $S^{m-1}\times S^1\times(0,\varepsilon]$ which is isotopic to the identity relative to $S^{m-1}\times S^1\times\{\varepsilon\}$ for some small $\varepsilon(>0)$ [see D.A. Neumann [2]] and is extendable to a $C^\infty$-diffeomorphism h on $S^{m-1}\times S^1\times(0,1]$ such that $h|_{S^{m-1}\times S^1\times[\varepsilon,1]}$ is the identity map. By Theorem 4, there is a flow $\psi_h$ which has h as its Poincaré map.

Now, for sufficiently small r, we can assume that $S^{m-2}\times I^{+}\times\{r\}$ $(\subset b^{-} - a^{-})$ is $\Sigma^{+}$ as in Figure 8. Denote the deformed $S^{m-2}\times S^{1}$ in this r-level by $\Sigma_r$ and let $\Sigma$ be an r-saturation of $\Sigma_r$, then, for small r>0, h satisfies the equality $h(l_r^{+}(S^{m-2})) = \Sigma \cap \tau_r$ and, in particular, $h(l_r^{+}(S^{m-2})) \cap S^{m-1}\times I^{+}\times\{r\} = \Sigma^{+}$. Let $S^{m-1}\times I_j$ (j= 1,2,3,...,k) be k-copies of $S^{m-1}\times I$ which have $\Sigma^{+}$ as the deformed $S^{m-2}\times I$ as in Figure 8, and glue $S^{m-1}\times I_1$, $S^{m-1}\times I_2$, ... and $S^{m-1}\times I_k$ one after another at each boundaries in the same way as stated above. Lastly glue a standard cassette.

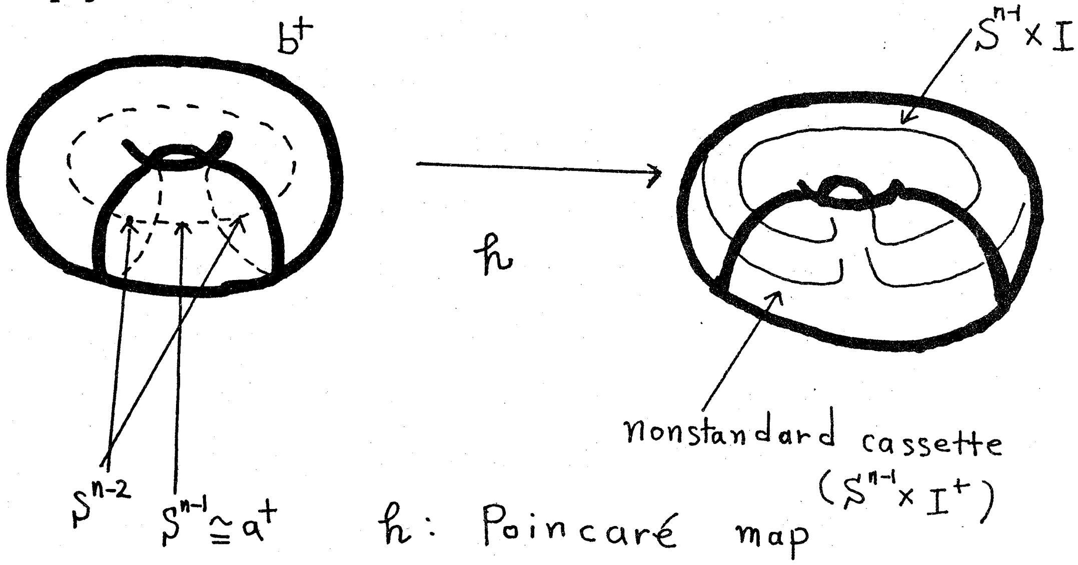



Figure 7

§6 PROOF OF MAIN RESULTS

In this section, we will show that the difference of numbers of nonstandard cassettes leads to countably many nonconjugate flows. (See Figure 10 and 11.) Consider the annulus $S^{m-1}\times I$ obtained by glueing one nonstandard cassette and one standard cassette (see Figure 7). As mentioned in the last section, we get a Poincharé map $h_{\psi_1}$ such that

$h_{\psi_1}(l_r^+(S^{m-2})) \cap I\times S^{m-1}\times\{r\}$ is the annulus $S^{m-2}\times I$ in Figure 7 for small r. By Wilson [7] and Neumann [1], there is a flow $\psi_1$ of which Poincaré map is equal to $h_{\psi_1}$. Also consider the annulus $S^{m-1}\times I$ obtained by glueing two nonstandard casettes and one standard casette, and construct a Poincaré map $h_{\psi_2}$ such that $h_{\psi_2}(l_r^+(S^{m-2})) \cap I\times S^{m-1}\times\{r\}$ is the annulus $S^{m-2}\times I$ in Figure 9 and a flow $\psi_2$ of which Poincaré map is equal to $h_{\psi_2}$.

Suppose that $\psi_1$ and $\psi_2$ are topologically conjugate, then the equality $\chi\circ h_{\psi_2} = h_{\psi_1}\circ(\chi|_{b^+-a^+})$ holds. Hence we have

$$h_{\psi_2}\, l_r^+(S^{m-2}) \subset (\chi|_{b^--a^-})^{-1}\circ h_{\psi_1}(N\,(L^+((\chi|_{a^+})(S^{m-2}))))$$

by Proposition 2 (iv). Define, as in section 4, a differentiable imbedding $\alpha_r^\mu : a^- \hookrightarrow \tau_r$ by $\alpha_r^\mu(\mu,\nu) = (g_\mu(\mu,\nu), g_\nu(\mu,\nu), r)$ for any $\mu_0 \in S^{m-1} - \overline{N\,(L^+((\chi|_{a^+})(S^{m-2})))}$ and small positive number r.

Here we suppose that both $\alpha_r^\mu(a^-) \cap \partial I^+\times S^{m-1}\times\{r\}$ and $\alpha_r^\mu(a^-) \cap \partial I^-\times S^{m-1}\times\{r\}$ have the same $\mu$-coordinate $\mu_0$ and $g_\mu$ (resp. $g_\nu$) is the $\mu$ (resp. $\nu$)-coordinate function when $S^{m-1}\times S^1\times\{r\}$ is divided into $S^{m-1}\times I^+\times\{r\}$ and $S^{m-1}\times I^-\times\{r\}$. (See Figure 11.)
Since $S^{m-2}$ is differentiably imbedded in $S^{m-1}$, we can choose an arc $\alpha$ so that its fluctuation is less than three and $\alpha \cap h_{\psi_1}(N\,(L^+((\chi|_{a^+})(S^{m-2})))) = \phi$ for some small positive number $\varepsilon$.
In particular, choose $\alpha_r^\mu$ as such an $\alpha$ and apply Lemma 4 to $(\chi|_{b^--a^-})^{-1}\circ\alpha_r^\mu = \Gamma_r$, then it follows that

$$FL_\nu(\Gamma;\Delta) \leqq FL_\nu((\chi|_{b^--a^-})\,\Gamma_r;\, c(\Delta)/2) \leqq 3,$$

where $\Delta = |\tilde{\nu}(I_1)-\tilde{\nu}(I_2)|$. (See Figure 11.) This contradicts $FL_\nu(\Gamma;\Delta) \geqq 5$ (see Figure 12), and we conclude that $\psi_1$ and $\psi_2$ are not topologically conjugate.

Moreover, we can show inductively that two $B_{m2}$ flows ($m \geq 3$) are not orbit conjugate if the numbers of nonstandard cassettes are different from each other.

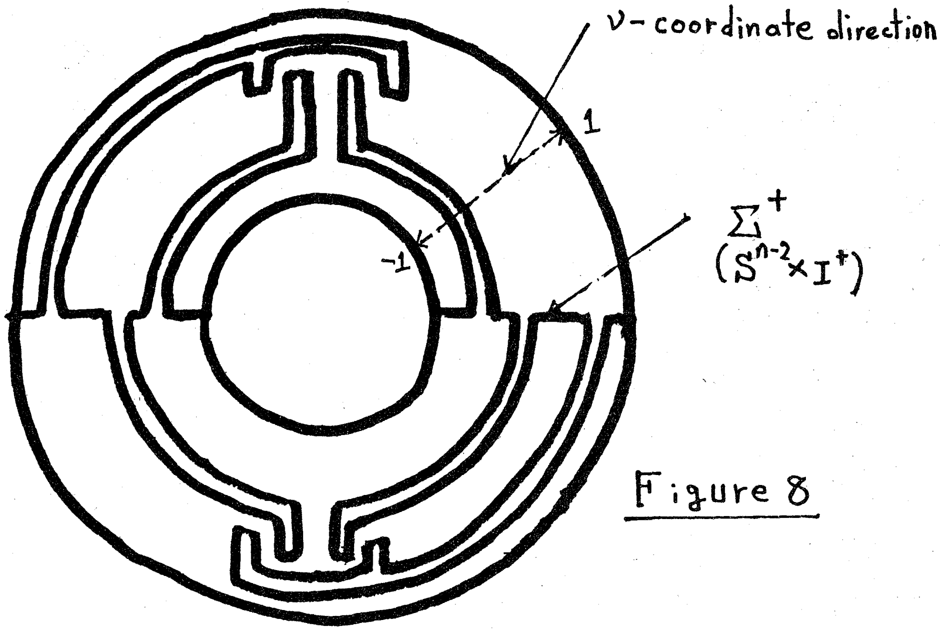



Figure 8

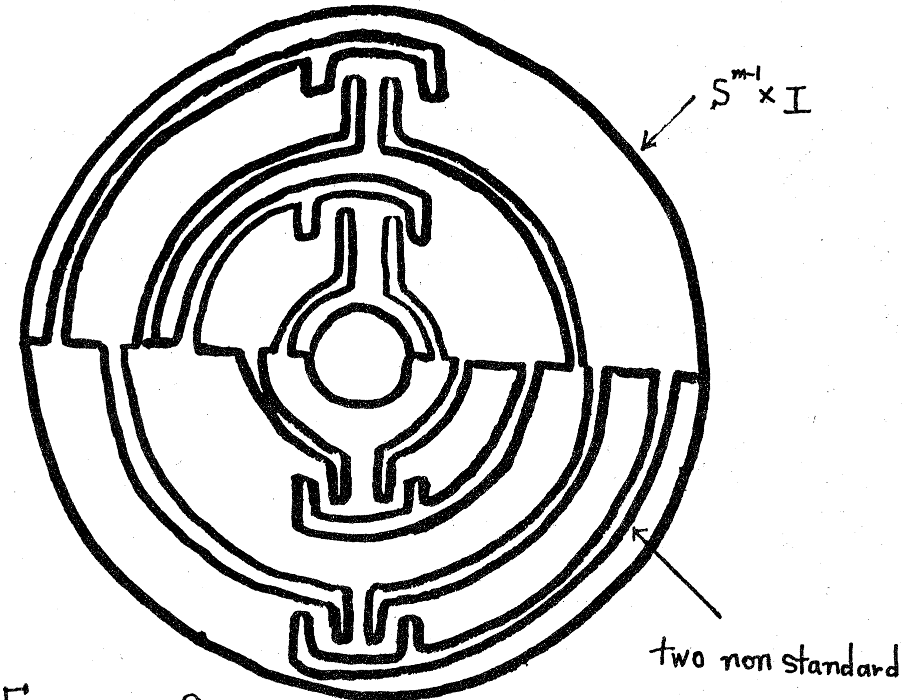



Figure 9

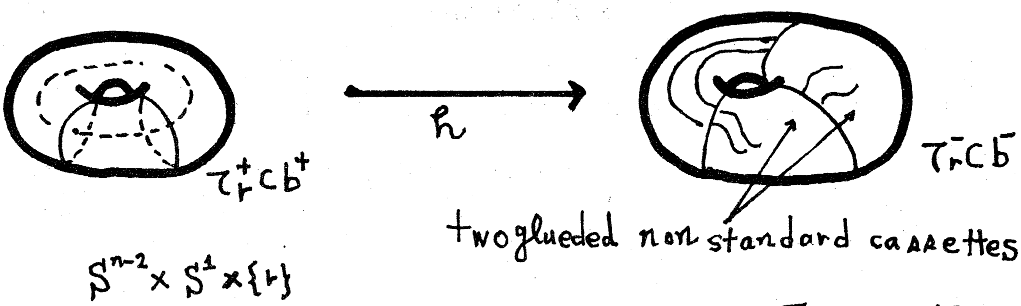



Figure 10

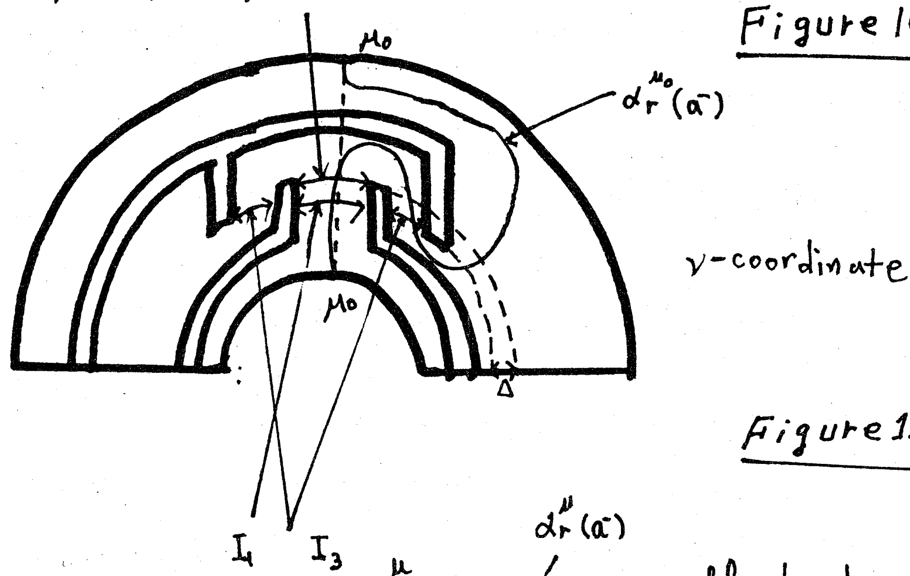



Figure 11

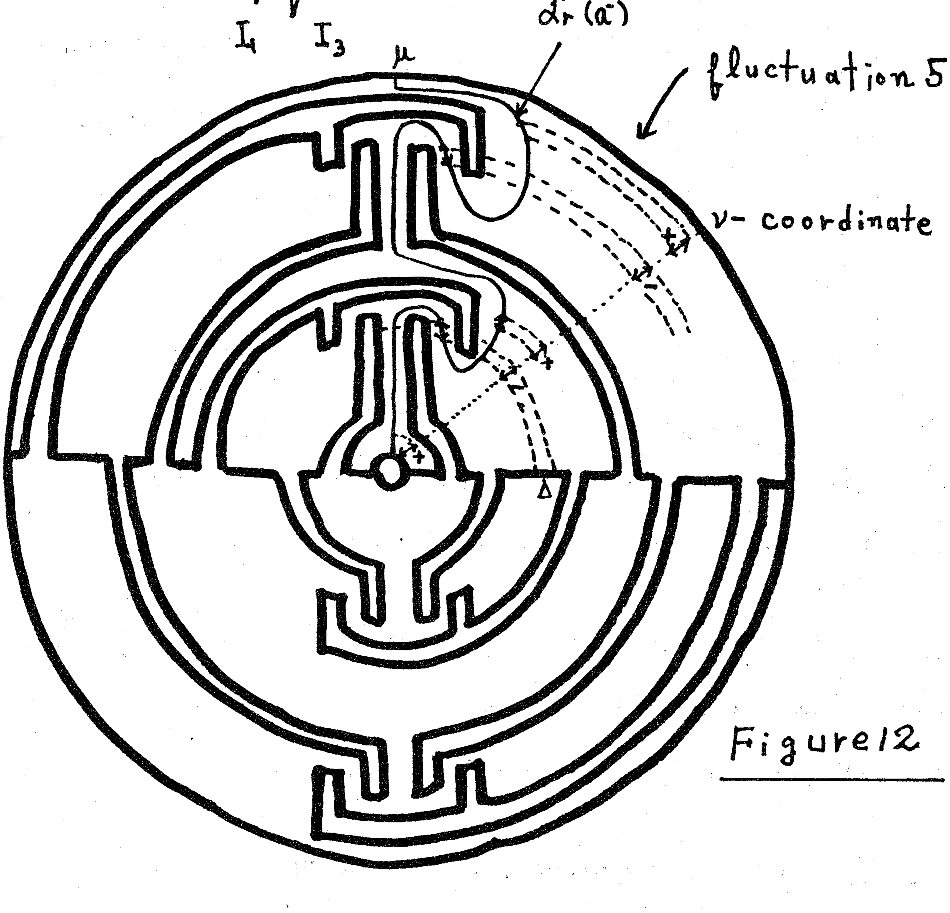



Figure 12

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