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Matrices which are knot module matrices

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Everyone knows the Alexander matrix of a classical knot group is equivalent to its conjugate transpose. However the question of how close is the connection between this 'hermitian symmetry' and the property of being the Alexander matrix of a classical knot group seems to have received little attention prior to the author's paper [P1], where we gave explicit answers to this question on both the module and presentation matrix level (see [P1, Remark 2, and Theorem 4]). In fact, the connection is so close that we thought it would be worth while to give an explicit example of a hermitian matrix which does not occur as the Alexander matrix of a classical knot group; this was the motivation for the present paper.

Our results rest on a generalization of theorem 4 in

[P1] which gives a characterization of those Z<t>-matrices which present the middle dimensional homology module of a simple odd dimensional knot (Theorem A). Among the applications given is an example of a hermitian matrix M which does <u>not</u> present such a 'knot module' (Corollary 3).

Preliminaries

Let $k \subset S^{\nu+2}$ be a tame ν-dimensional knot in the $\nu+2$-dimensional sphere $S^{\nu+2}$. The covering translation group $Aut(\tilde{X})$ of the universal abelian cover $\tilde{X}$ is infinite cyclic, $Aut(\tilde{X}) = \langle t \rangle$, and the action of t can be used to give the homology groups $H_i(\tilde{X})$ a Z<t>-module structure. Let $\mathcal{S}_\nu = \{ H_{[\nu+1/2]}(\tilde{X}) : \tilde{X}$ is the universal abelian cover of a simple ν-dimensional knot complement$\}$.
Then

<u>Theorem 1</u> Suppose ν_1 and ν_2 are odd numbers and $\nu_1 = \nu_2 \bmod 4$. Then $\mathcal{S}_{\nu_1} = \mathcal{S}_{\nu_2}$.

Proof (Kearton[K]) Let ν be an odd number, and $H_{(\nu+1/2)}(\tilde{X})$ be any element of $\mathcal{S}_\nu$. Choose $\varepsilon=\pm1$ so that $\varepsilon=\nu \bmod 4$. Then there exists a square Z-matrix V with $\det(V-\varepsilon V')=\pm1$ such that

$$H_{(\nu+1/2)}(\tilde{X}) = \mathcal{M}_{(tV-\varepsilon V')}.$$

Here $\mathcal{M}_{(tV-\varepsilon V')}$ is the Z<t>-module with presentation matrix $tV-\varepsilon V'$.

Definition A matrix such as V above is called a Seifert matrix. When V is a Seifert matrix, $\mathcal{M}_{(tV-\varepsilon V')}$ is called a (type ε) knot module, and any matrix N with $\mathcal{M}_N$ Z<t>-isomorphic to $\mathcal{M}_{(tV-\varepsilon V')}$ is called a (type ε) knot module matrix.

2. The main theorem

Let N be an (n×n) Z<t>-matrix which satisfies

$$\Delta(t) = \det N = c_0+c_1t+\ldots+c_dt^d, \quad c_0c_d\neq 0, \text{ and } \Delta(1)=\pm1.$$

Let $R=Z[1/c_0,1/c_d]$ denote the smallest subring of the rational numbers Q in which c_0 and c_d are units, $\bar{N}'$ be the conjugate transpose of N, and $\Lambda = Z[t,t^{-1},(1-t)^{-1}]$ be the ring of integral polynomials in t, t^{-1}, and $(1-t)^{-1}$.

Our main theorem is

<u>Theorem A</u> N presents a (type ε)-knot module if and only if there is an $(n+e)\times(n+e)$ Λ-unimodular matrix C such that

$$\begin{pmatrix} N & 0 \\ 0 & I \end{pmatrix} C = \varepsilon\left(\overline{\begin{pmatrix} N & 0 \\ 0 & I \end{pmatrix} C}\right) .$$

Here I is the $(e\times e)$ identity matrix, and

$$e = \begin{cases} d, & \text{if } \varepsilon = 1, \text{ or } \varepsilon = -1 \text{ and } n \text{ is even} \\ d+1, & \text{if } \varepsilon = -1, \text{ and } n \text{ is odd.} \end{cases}$$

<u>Proof.</u> The case $\varepsilon = 1$ is just Theorem 4 of Pizer[P1]. Moreover, generalizing this theorem to the case $\varepsilon = -1$ requires only slight alterations to the method given in [P1], so we omit a proof. (Note that replacing 'skew-symmetric t-isometry' by 'symmetric t-isometry' in Theorem 3 and (4.1) of [P1] gives a characterization of (type -1) knot modules.)

<u>Example 1</u> ([P1]) $\varepsilon = 1$, $N = tV-V'$, V a $d\times d$ Seifert matrix. Put $z = (1-t)^{-1}$. Then direct calculation shows $\bar{z} = -tz = (1-z)$. In particular the identity $\bar{z} = -tz$ shows

$$\begin{pmatrix} (tV-V') & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} zI & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} -\bar{z}+\bar{z}\bar{t}V' & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} \bar{z}I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} (\overline{tV-V'})' & 0 \\ 0 & I \end{pmatrix}$$

Hence $C = \begin{pmatrix} zI & 0 \\ 0 & I \end{pmatrix}$, the block sum of zI and I, satisfies the condition of Theorem A.

Example 2 $\varepsilon = -1$, $N = tV+V'$, V a $d \times d$ Seifert matrix. As Trotter[T1, p.178] notes, d is even.

Let $E = \oplus_{d/2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the block sum of d/2 copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Then $$\begin{pmatrix} tV+V' & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} zI & 0 \\ 0 & E \end{pmatrix} = -\begin{pmatrix} \bar{z}V+\bar{z}\bar{t}V' & 0 \\ 0 & E' \end{pmatrix}$$

$$= -\begin{pmatrix} \bar{z}I & 0 \\ 0 & E' \end{pmatrix} \begin{pmatrix} (\overline{tV+V'})' & 0 \\ 0 & I \end{pmatrix}$$

Hence $C = \begin{pmatrix} zI & 0 \\ 0 & E \end{pmatrix}$ satisfies the condition of Theorem A.

The following corollary follows easily from Theorem A.

Corollary 1 Suppose X is a square (n×n) $Z\langle t\rangle$-matrix such that $\det X(1) = \pm 1$. For any natural number k, if $X = (-t)^k \bar{X}'$, then X is a (type 1) knot module matrix, while if $X = t^{(2k-1)}\bar{X}'$, and n is even, then X is a (type -1) knot module matrix.

Proof If $X = (-t)^k \bar{X}'$, take $C = \begin{pmatrix} z^k I & 0 \\ 0 & I \end{pmatrix}$ and apply Theorem A.

When $X = t^{(2k-1)}\overline{X}'$, and n is even, take $C = \begin{pmatrix} z^{2k-1}I & 0 \\ 0 & E \end{pmatrix}$, and apply Theorem A.

<u>Remark</u> That $X = \overline{X'}$ implies X is a (type 1) knot module matrix was proved by Rolfsen[R2] using surgical methods.

We shall have need of the following results. Let $[N]_+$ denote the Λ-equivalence class of N under the equivalence relation of Fox[F,p.199]. Trotter[T2,T3] (and indepentently, the author) have shown

<u>Lemma 1</u> There is a finite procedure for transforming N via Λ-elementary transformations into a matrix of the form zI-B, where B is an R-unimodular d×d Z-matrix.

That the Z<t>-module presented by N satisfies the conditions required for Trotter's algorithm to be valid follows from $\Delta(1)=\pm 1$; see Crowell[C, Theorem 1.3], and [T1, p.179].

<u>Theorem B</u> $[N]_+ = [zI-B]_+$ presents a (type ε) knot module if and only if there is a d×d' Z-matrix U such that

1. $(zI-B)U = \varepsilon(\overline{(zI-B)U})'$,
2. U is R-unimodular, and $U^{-1}B^k(I-B)^k$ is a Z-matrix, for some k.

Proof See [P1, Theorem 4]. When $\varepsilon = 1$, 1 is established in the remarks after equation(6) and equation(4), and 2 is just assertion (4.3). The case $\varepsilon = -1$ is left to the reader.

We can now establish

Corollary 2. There are type 1 knot modules which are not type -1 knot modules, and vice versa.

Proof The proof is by example.

Let $N_1 = \begin{pmatrix} t_2+t+1+\bar{t} & 4t-1 \\ 4-t & 1+t \end{pmatrix}$. $N_1 = t\bar{N}_1$ and the Alexander polynomial $\Delta(t) = \det N_1$ of N_1 is $\Delta(t) = t^4+6t^3-15t^2+6t+1$. Note that $\Delta(1) = -1$. Hence by Corollary 1, N_1 is a (type -1) knot module matrix. We claim, however, that N_1 is not a (type 1) knot module matrix. (Note that the second elementary ideal is $(5,1+t)_{Z\langle t\rangle}$, whence N_1 is not cyclic, that is, equivalent to $\Delta(t)$.) Direct calculation shows $N_1 \sim tI_4-A_1$, where I_4 is the 4×4 identity matrix, and

$$A_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 15 & 4 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & -5 & -1 \end{pmatrix}.$$

Writing $B_1 = (I_4-A_1)^{-1}$, $[tI_4-A_1]_+ = [zI-B_1]_+$. Hence if N_1 presents a (type 1) knot module, by Theorem B there is a Z-unimodular skew-symmetric U such that $B_1U = U(I-B_1')$. ($c_0=c_4=1$, so R=Z) But it is easy to see that $B_1U = U(I-B_1')$ if and only if $A_1U = U(A_1')^{-1}$. We are thus lead to considering the equality

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 15 & 4 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & -5 & -1 \end{pmatrix} \begin{pmatrix} 0 & p & q & r \\ -p & 0 & s & t \\ -q & -s & 0 & u \\ -r & -t & -u & 0 \end{pmatrix} = \begin{pmatrix} 0 & p & q & r \\ -p & 0 & s & t \\ -q & -s & 0 & u \\ -r & -t & -u & 0 \end{pmatrix} \begin{pmatrix} -5 & 0 & -1 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & -1 & 0 & -1 \end{pmatrix}$$

The general solution is

$$S = \begin{pmatrix} 0 & p & p+4u & u \\ -p & 0 & p+3u & -5p-16u \\ -p-4u & -p-3u & 0 & u \\ -u & 5p+16u & -u & 0 \end{pmatrix}$$

which has determinant $\det S = [(5p+16u)(p+4u)+u(2p+3u)]^2$. Hence $\det S$ is a unit in Z if and only if $(5p+16u)(p+4u)+u(2p+3u) = 5p^2+38pu+67u^2 = \pm 1$. Making the change of variables $p = 4p'+u'$; $u=-p'$, we see the quadratic form [5,38,67] is equivalent to [-5,2,5]. But [5,38,67] has discriminant $38^2-20.67 = 104$, and all reduced forms with this discriminant occur in the two chains [1,10,-1],[-1,10,1] and [5,2,-5],[-5,8,2],[2,8,-5],[-5,2,5], [5,8,-2],[-2,8,5]. (see the algorithm in Dickson[D,p.103]). But Theorem 86 of [D] states that the absolute value of the lower bound for numbers represented by $5p^2+38pu+67u^2$ for integers p and u, not both zero, is the lower bound of the $|a_i|$, where $[a_i,b_i,a_{i+1}]$ constitute the chain of reduced forms equivalent to $[5,38,67]$. The lower bound is therefore equal to two. Thus $5p^2+38pu+67u^2 = \pm 1$ has no integral solutions, a contradiction to the Z-unimodularity of U. Hence N_1 does <u>not</u> present a type 1 knot module matrix.

It can be shown that any square Z<t>-matrix N with determinant $\Delta(t) = ct^2+(1-2c)t+c$ is a type 1 knot module matrix. (see Pizer[P2]) On the other hand, the Alexander matrix of the trefoil knot, $N_2 = t^2-t+1$, is <u>not</u> a type -1 knot module matrix. Indeed, $(t^2-t+1) \sim \begin{pmatrix} t & 1 \\ -1 & t-1 \end{pmatrix} = tI_2-A_2$, where $A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. By the above, N_2 presents a type -1 knot module if and only if there is a Z-unimodular <u>symmetric</u> U such that $A_2U = U(A_2')^{-1}$. We are thus led to considering the equality

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

which has the general solution $U = \begin{pmatrix} -2c & c \\ c & -2c \end{pmatrix}$. However

$3c^2 = \det U = \pm 1$ has no solutions in integers. Hence N_2 does not present a type -1 knot module.

<u>Corollary 3</u> $[N] = [\overline{N}']$ does <u>not</u> imply N is a knot module matrix.

<u>Proof</u> Let N be the matrix

$$N = \begin{pmatrix} tI_4-A_1 & 0 \\ 0 & tI_2-A_2 \end{pmatrix}$$

where A_1 and A_2 are the matrices defined above. tI_4-A_1 is a type -1, but not a type 1, knot module matrix. tI_2-A_2 is a type 1, but not a type -1, knot module matrix. Because each block is equivalent to its conjugate transpose , $[N] = [\overline{N}']$. . Writing $B_i = (I-A_i)^{-1}$, i=1,2, we see $N \sim M = \begin{pmatrix} zI_4-B_1 & 0 \\ 0 & zI_2-B_2 \end{pmatrix}$. Now suppose that N is a type ε knot module matrix. Then by Theorem B, there is a 6×6 Z-unimodular matrix U such that $MU = \varepsilon\overline{U}'\overline{M}'$. Partition U into a 4×4 matrix U_1, 4×2 matrix U_2, 2×4 matrix U_3, and 2×2 matrix U_4, $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$. Then $MU = \varepsilon\overline{U}'\overline{M}'$

implies

$$(zI_4-B_1)U_1 = \varepsilon\overline{U}_1'(\overline{zI_4-B_1})' \qquad \ldots(1)$$
$$(zI_2-B_2)U_4 = \varepsilon\overline{U}_4'(\overline{zI_2-B_2})' \qquad \ldots(2)$$
$$(zI_4-B_1)U_2 = \varepsilon\overline{U}_3'(\overline{zI_2-B_2})' \qquad \ldots(3)$$

Equating the coefficients of z in (3) shows $U_2 = -\varepsilon U_3'$.

Equating the constant terms in (3) thus implies

$$B_1U_2 = U_2(I-B_2') \qquad \text{...(4)}$$

Equation (4) is a linear equation in the elements of U_2, and direct calculation shows the only solution is $U_2 = 0$. Hence $U_3 = 0$, and $U = \begin{pmatrix} U_1 & 0 \\ 0 & U_4 \end{pmatrix}$, the block sum of U_1 and U_4. But U is Z-unimodular, hence U_1 and U_4 must be Z-unimodular. Equations (1) and (2) together with Theorem B therefore imply zI_4-B_1 and zI_2-B_2 are both type $\mathcal{E}$ knot module matrices, a contradiction.

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