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Witt group and nilpotent group of odd order

by

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Abstract. The equivariant Witt group $W_0(D,G)$ of a finite nilpotent group G over a Dedekind domain D is studied. We introduce a Morita correspondence on the set of orthogonal representations. We determined the structure of $W_0(D,G)$ of a finite nipotent group G of odd order.

§0. Introduction. Let D be a Dedekind domain with the quotient field K and Λ be a D-order in the f.d. semisimple K-algebra A. Throughout this paper, we always assume that Λ has an anti-involution (-) with $\overline{a+b}=\bar{a}+\bar{b}$, $\overline{ab}=\bar{b}\bar{a}$, $\bar{\alpha}=\alpha$ for all $a,b\in\Lambda$, $\alpha\in D$. We extend this involution into A, naturally.
An orthogonal representation of Λ is a pair (V,b) of the Λ-lattice V and the Λ-invariant, D-valued, nonsingular symmetric bilinear form b on V, where "Λ-invariant" means $b(\alpha v,w)=b(v,\bar{\alpha}w)$ for $\alpha\in\Lambda$, $v,w\in V$. An orthogonal representation (V,b) is metabolic when there is a Λ-invariant sublattice N in V such that $N^{+}=\{v\in V:b(v,n)=0$ for all $n\in N\}$ is equal to N.
The equivariant Witt group $W_0(D,\Lambda)$ is the Grothendieck group on the isometry classes of orthogonal representations of Λ modulo

the subgroup generated by metabolic forms. Most interesting case is that where Λ is the integral group ring DG and the involution is given by the inverse of G. In this case, this group is the same as $GW(G,D)$ in [3] and $GW_0(D,G)$ in [2].

At first, we will investigate the structure of $W_0(D,G)$ of a finite nilpotent group G. We consider the Lenstra's formula [5] on Witt groups as we have done on the Grothendieck group $G_0(DG)$ in [6]. Let Y be the set of all isomorphism classes of irreducible K-characters θ of G. For the irreducible KG-mdoule V corresponding to θ, let $\bar{\theta}$ be the irreducible K-character of G corresponding to $V^*=\mathrm{Hom}_K(V,K)$. Let e_θ be the central primitive idempotent of KG (or KG/G_p if $p=ch(K)$ is not prime to the order $|G|$ of G, where G_p denotes a Sylow p-subgroup of G) corresponding to θ. Set $D_\theta=D(\frac{1}{\deg\theta})$. We have an analogue of Theorem 1 in [6].

Theorem A. Let G be a finite nilpotent group. Then we have the following isomorphism: $W_0(D,DG) \cong \oplus_{\theta=\bar{\theta}\in Y} W_0(D_\theta,D_\theta Ge_\theta)$.

We will next introduce a Morita correspondence and we get the following:

Corollary. Let G be a finite nilpotent group of odd order and assume that K is an algebraic number field. We then have the isomorphism: $W_0(D,DG) \cong \oplus_{\chi\sim_K\bar{\chi}\in T} \mathcal{H}_0(D\langle\chi\rangle)$, where T is the set of representatives for the K-conjugacy classes of irreducible complex-characters of G and $D\langle\chi\rangle=D(\frac{1}{\deg\chi},\chi(g):g\in G)$.

§1. Notations and Lenstra's formula. We will adopt the notations form [1] and [6]. All modules in this paper are finitely generated left modules, unless otherwise specified. Let G be a finite nilpotent group and write $G=\Pi_p G_p$ as the direct product of its Sylow p-subgroups G_p. For a set S of primes, set $G_S = \Pi_{p\in S} G_p$ and θ_S denotes an irreducible constituent of $\theta_{|G_S}$. Since $\theta_{|G_S}$ is homogeneous, θ_S is defined uniquely. Then, the canonical homomorphisms $G \longrightarrow G_S \longrightarrow G$ induce, by restriction, an exact functor N_S. Namely, for an DG-module M, $N_S M$ is the D-module M on which G_p acts as given for $p\in S$ and trivially for $p\notin S$. Let (M,b) be an orthogonal representation of DG, then $(N_S M,b)$ is an orthogonal representation of DG. We set $N_S(M,b)=(N_S M,b)$. We have to note that this functor is compatible with the Witt class, that is, for sub DG-module N of M, the orthogonal complement of N in (M,b) is equal to that of N in $N_S(M,b)$.

At first, we construct the following diagram:

$$(1.1)\quad \begin{array}{ccccc} W_0(D,DG) & \longrightarrow & W_0(K,KG) & \xrightarrow{\partial} & W_0(K/D,DG) \\ & & \downarrow \phi_K & & \downarrow \oplus\phi_\beta \\ \oplus_{\theta\in\hat{Y}} W_0(D_\theta,D_\theta Ge_\theta) & \rightarrow & \oplus_{\theta\in\hat{Y}} W_0(K,KGe_\theta) & \xrightarrow{\oplus\partial_\theta} & \oplus_{\theta\in\hat{Y}} W_0(K/D\ ,D_\theta Ge_\theta) \end{array}$$

where $\hat{Y}=\{\theta\in Y : \bar{\theta}=\theta\}$. Then we will show that this deagram is commutative and that ϕ_K and ϕ_β are all isomorphisms. If so, we have the desired isomorphism by the well known result [1],

$$W_0(D,DG) \cong \operatorname{Ker}\partial \cong \operatorname{Ker} \oplus\partial_\theta \cong \oplus_{\theta\in\hat{Y}} W_0(D_\theta,D_\theta Ge_\theta).$$

(1.2). Definition of ϕ_*. Let (M,b) be an orthogonal representation of KG. Since every orthogonal representation of

KG decomposes into the orthogonal sum of homogeneous components and metabolic forms, we assume that M is a $KG(e_\theta+e_{\bar\theta})$-module and we put $\phi_K(M,b)=\Sigma_{S\subset\pi(\theta)}(N_S(M,b)) \in \oplus_{\chi\in Y} W_0(K,KG(e_\chi+e_{\bar\chi}))$ If $\bar\theta\neq\theta$, then M and $\phi_K(M,b)$ are metabolic. Therefore, extending it linearly, we have the homomorphism:
$\phi_K:W_0(K,KG) \longrightarrow \oplus_{\theta\in\hat{Y}} W_0(K,KGe_\theta)$. On the other hand, let β be a prime deal of D and set $F=D/\beta$ and $p=ch(F)$. Then since every orthogonal representation of FG is equivalent to some orthogonal representation of FG on which G_p acts trivially. We know that this isomorphism: $W_0(F,FG)\cong W_0(F,FG/G_p)$ is given by $N_{\pi(G)-\{p\}}$. Then every orthogonal representation of FG/G_p decomposes into the orthogonal sume of β-torsion orthogonal representations of DGe_θ with $\theta\in\hat{Y}$, $p\notin\pi(\theta)$ and hyperbolic forms of $DG(e_\theta\oplus e_{\bar\theta})$ with $\theta\notin\hat{Y}$. Let (M,b) be an orthogonal represenation of FG and assume that M is a DGe_θ-module with $p\notin\pi(\theta)$. Now we put $\phi_\beta((M,b)) = \Sigma_{S\subset\pi(\theta)} N_S(M,b) \in \oplus_{\chi\in\hat{Y}} W_0(F,FGe_\chi)$ and extending linealy it,we have the homomorphism:

$$\phi_\beta:\ W_0(F,FG) \xrightarrow{N_{\pi(G)-\{p\}}} W_0(F,FG/G_p) \longrightarrow \oplus_{\chi\in Y,p\notin\pi(\chi)} W_0(F,FGe_\chi).$$

Lemma 1.3. ϕ_* are the isomorphisms.

Proof. We will give the inverse maps. Let (H,h) be an orhtogonal representation of KGe_θ (or FGe_θ with $p\notin\pi(\theta)$) and we put $\phi'_K(H,h)_\theta$ (or $\phi'_\beta(H,h)_\theta$) $= \Sigma_{S\subset\pi(\theta)}(-1)^{|\pi(\theta)-S|} N_S(H,h)$, where $|\pi(\theta)-S|$ denotes the number of elements in $\pi(\theta)-S$. We can easily check $\phi'\phi=1$ and $\phi\phi'=1$.

We next show that the diagram (1.1) is commutative. Let

(V,b) be an orthogonal representation of KG and assume that V is a KGe_θ-module. Let I be a full DG-lattice in V with $b(I,I)\subset D$ and set $J=\{v\in V: b(v,I)\subset D\}$. Then $\partial(V,b)$ is given by $(J/I,\tilde{b})$, where $\tilde{b}$ is b+D/D. Decomposing into the β-torsion parts, we have $\partial(V,b)=\oplus_\beta (J_\beta/I_\beta,\tilde{b}_\beta)$, where $(\ )_\beta$ denotes the β-localization, that is, $J_\beta=D_\beta\otimes J$. Therefore, let $p_\beta=ch(D/\beta)$ and we have

$$(\oplus\phi_\gamma)\partial(V,b)=\Sigma_\beta\phi_\beta(J_\beta/I_\beta,\tilde{b}_\beta)=\Sigma_\beta\Sigma_{S\subset\pi(\theta)-\{p_\beta\}} N_S(J_\beta/I_\beta,\tilde{b}_\beta).$$

On the other hand, $\phi_K(V,b)=\Sigma_{S\subset\pi(\theta)} N_S(V,b)$ and

$$(\oplus\partial_\chi)\phi_K(V,b)=\Sigma_{S\subset\pi(\theta)}\partial_{\theta_S}(N_S(V,b))=\Sigma_{S\subset\pi(\theta)}((D_{\theta_S}\otimes J)/(D_{\theta_S}\otimes I),\tilde{b}_S)$$
$$=\Sigma_{S\subset\pi(\theta)}\Sigma_{p_\beta\notin\pi(\theta_S)}(J_\beta/I_\beta,\tilde{b}_\beta)=(\Sigma\phi_\beta)\partial(V,b).$$

This completes the proof of Theorem A.

§2. Morita correspondence. Let Δ be a maximal D-order in the semisimple K-algebra A. Let M_Δ be a progenerator of Δ and set $\Lambda=End_\Delta(M)$. Then M can be considered as a (Λ,Δ)-bimodule and $M^*=Hom\,(M,\Delta)$ is a (Δ,Λ)-bimodule. As well known, the Morita correspondence states that the tensoring $M\otimes_\Delta$ gives an equivalence functor of the category of Δ-left modules to the category of Λ-left modules and the tensoring $M^*\otimes_\Lambda$ gives the inverse functor. The purpose in this section is to construct a Morita correspondence from the set of orthogonal representations of Δ onto the set of orthogonal representations of Λ. Since ${}_\Lambda M_\Delta$ is a progenerator, there are isomorphisms:

μ: $M\otimes_\Delta M^* \longrightarrow \Lambda$ given by $\mu(m\otimes f)m_1=m(fm_1)$, and

τ: $M^*\otimes_\Lambda M \longrightarrow \Delta$ given by $\tau(f\otimes m)=fm$, for $m,m_1\in M$, $f\in M^*$.

We assume that Δ and Λ have anti-involutions, respectively. We will use the same symbol (-) to denote them.

Now we assume that there is an isomorphsm $b:M \longrightarrow M^*$ satisfying the following four conditions.

1) $b(rm) = b(m)\bar{r}$ for $r\in\Lambda$, $m\in M$,

2) $b(rs) = \bar{s}b(m)$ for $s\in\Delta$,

3) $\tau(b(n)\otimes_{\Lambda}m) = \overline{\tau(b(m)\otimes_{\Lambda}n)}$ for $n,m\in M$, and

4) $\mu(m\otimes_{\Delta}b(n)) = \overline{\mu(n\otimes_{\Delta}b(m))}$.

We will show that under the conditions (C), the tensoring $M\otimes_{\Delta}$ gives an equivalence functor of category of orthogonal representations of Δ to that of Λ. It will be easily proved that this functor sends the set of metabolic forms of Δ onto the set of metabolic forms of Λ. Therefore, we have the following:

Theorem B. Under the above conditions, we have;

$W_0(D,\Delta) \cong W_0(D,\Lambda)$ and $WH_0(D,\Delta) \cong WH_0(D,\Lambda)$, where $WH_0(D,\Delta)$ is the Grothendieck group on the isometry classes of orthogonal representations of Δ modulo the subgroup generated by hyperbolic forms.

We start the proof of Theorem B. Let (H,h) be an orthogonal representation of Δ. Let $\phi(h)$ be the bilinear form on $M\otimes_{\Delta}N$ defined by $\phi(h)(m_1\otimes_{\Delta}n_1,\ m_2\otimes_{\Delta}n_2) = h(n_1,\tau(b(m_1)\otimes_{\Lambda}m_2)n_2)$.

Lemma 2.1. The above definition is well defined and $\phi(h)$ is a Λ-invariant, symmetric bilinear form.

Proof. For $\alpha,\beta\in\Delta$ and $m,u\in M$, $n,v\in H$, we have

$\phi(h)(m\alpha\otimes n,u\beta\otimes v)=h(n,\tau(b(m\alpha)\otimes u\beta)v)=h(n,\alpha\tau(b(m)\otimes u)\beta v)$

$=h(\alpha n,\tau(b(m)\otimes u)\beta v)=\phi(h)(m\otimes\alpha n,u\otimes\beta v)$. And for $r\in\Lambda$, we get $\phi(h)(r(m\otimes n),u\otimes v)=h(n,\tau(b(rm)\otimes u)v)=h(n,\tau(b(m)\otimes\bar{r}u)v)$ $=\phi(h)(m\otimes n,\bar{r}(u\otimes v))$. Moreover, we obtain $\phi(h)(m\otimes n,u\otimes v)=h(n,\tau(b(m)\otimes u)v)=h(\tau(b(m)\otimes u)v,n)=h(v,\overline{\tau(b(m)\otimes u)}n)$ $=h(v,\tau(b(u)\otimes m)n)=\phi(h)(u\otimes v,m\otimes n)$.

Therefore, the mapping $\Phi[(H,h)]=(M\otimes_\Delta N,\ \phi(h))$ sends the set of isometry classes of orthogonal representations of Δ (containing singular forms) into the set of isometry classes of orthogonal representations of Λ (containing singular forms). Similarly, we can define the mapping $\Psi[(S,s)]=(M^*\otimes_\Lambda S,\ \phi'(s))$, where $\phi'(s)(f_1\otimes t_1,f_2\otimes t_2)=s(t_1,\mu(b^{-1}(f_1)\otimes f_2)t_2)$ for $f_1,f_2\in M^*$, $t_1,t_2\in S$ and (S,s) is an orthogonal representation of Λ.

Lemma 2.2. $\Phi\Psi=1$ and $\Psi\Phi=1$.

Proof. Let (H,h) be an orthogonal representation of Δ. Then we have $(\phi\phi')(h)(f\otimes m\otimes n,g\otimes u\otimes v)=\phi(h)(m\otimes n,\mu(b^{-1}(f)\otimes g)(u\otimes v))$ $=h(n,\tau(b(m)\otimes_\Lambda(\mu(b^{-1}(f)\otimes_\Delta g)u))v)=h(n,\tau(b(m)\otimes_\Lambda b^{-1}(f)(gu))v)$ $=h(n,\tau(b(m)\otimes b^{-1}(f))(gu)v)=h(\tau(b(m)\otimes b^{-1}(f))n,(gu)v)$ $=h(\tau(b(b^{-1}(f))\otimes m)n,guv)=h(fmn,guv)$, for $f,g\in M^*$, $m,u\in M$, and $n,v\in H$. Identifying H and $M^*{}_\Lambda\otimes M\otimes_\Delta H$, we have $(\phi\phi')(h)=h$ and so $(\Phi\Psi)=1$. Similarly, we get $(\Psi\Phi)=1$.

By Lemma 2.2, we see that if (H,h) is nonsingular, then $\Phi(H,h)$ is also nonsingular, which proves Theorem B. This completes the proof of Theorem B.

We now start the proof of Corollary. Let θ be a faithful irreducible KG-character with $\bar{\theta}=\theta$ and T be the simple component of KG corresponding to θ. Then it follows from Feit [Theorem14.4 and 14.5] that T is the full matrix algebra $M_n(K(\chi))$ over the field $k(\chi)=K(\chi(g):g\in G)$ and there is a representation $\zeta:G \longrightarrow T$ satisfying $\zeta(g^{-1})={}^t\overline{\zeta(g)}$ and $\zeta(D_\theta G)=M_n(D\langle\chi\rangle)$, where χ is an irreducible complex character of G whose K-conjugacy class is θ and t denotes the transpose and (-) denotes the complex conjugate. Taking M as a row vector $nD\langle\chi\rangle=D\langle\chi\rangle\oplus\dots\oplus D\langle\chi\rangle$ and M^* as a column vector, we can apply the Morita correspondence on $M_n(D\langle\chi\rangle)$ and $D\langle\chi\rangle$ and we get the isomorphism:

$W_0(D_\theta,Ge_\theta) \cong W_0(D_\chi,D\langle\chi\rangle)$. Since the relative different of $D\langle\chi\rangle/D_\theta$ is a unit, we have that the trace $tr_{D\langle\chi\rangle/D_\theta}$ gives the isomorphism: $\mathcal{K}_0(D\langle\chi\rangle) \cong W_0(D_\chi,D\langle\chi\rangle)$.
This completes the proof of Corollary.

Reference.

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