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GRÖBNER BASIS OF IDEAL OF CONVERGENT POWER SERIES

小林英恒　古川昭夫　佐々木建昭
H. Kobayashi*), A. Furukawa**), T. Sasaki***)

*) Department of Mathematics, Nihon University
Kanda, Chiyoda-ku, Tokyo 101, Japan

**) Department of Mathematics, Tokyo Metropolitan University
Fukazawa, Setagaya-ku, Tokyo 158, Japan

***) The Institute of Physical and Chemical Research
Wako-shi, Saitama 351-01, Japan

Abstract

This paper develops a theory of Gröbner basis of ideal of convergent power series. The basis is constructed by calculating "S-power series" successively, where the S-power series is an analogue of S-polynomial and constructed so as to cancel the head terms of initial polynomials of two power series. By using the finite generation property of monoideal, it is proved that a finite number of successive constructions of S-power series provide us a Gröbner basis of power series ideal. Furthermore, a construction procedure of the Gröbner basis is discussed from the viewpoint of constructive algebra.

§1. Introduction

In discussing polynomial ideals and related problems, the Gröbner bases are very useful ideal bases [1]. The Gröbner bases allow us to solve the following problems within reasonable computation steps [2]: determine whether or not a given polynomial is an element of a given ideal, calculate the intersection of two polynomial ideals, simplify a polynomial with polynomial side-relations, solve a system of algebraic eqautions with/without parameters, calculate the polynomial solutions of a linear equation with polynomial coefficients, and so on.

A construction method of Gröbner basis for polynomials in $K[x_1,\dots,x_n]$, with K a number field, was discovered by Buchberger in 1965 [1]. Lauer [3] extended the Buchberger's method to include the polynomials with coefficients in the ring of integers. However, as far as the authors know, no attempt was made to construct a Gröbner basis of an ideal in a ring of power series. In this paper, we construct a Gröbner basis on a ring of convergent power series $K\{x_1,\dots,x_n\}$ and discuss some properties of it.

As we will see below, our construction is an almost straightforward extension of the method for polynomials, but we used some results of the theory of monoideal to prove the finite generation property of the Gröbner basis for power series. We follow to Hironaka [4] to use monoideals in discussing ideals of infinite power series. Because the theory of monoideal is essential in our extension, §2 is devoted to survey this theory briefly. With the notions of monoideal, the theory of Gröbner basis can be formulated simply. Hence, in §3, we reformulate the conventional Gröbner basis theory from the viewpoint of monoideal. The development of a theory of Gröbner basis of power series ideal is done in §4 and §5, and the constructivity of a Gröbner basis for infinite power series ideal is discussed in §6.

§2. Monoideal

Let Z_0 be the set of non-negative integers, and Z_0^n the Cartesian product of Z_0 with n a positive integer. An element A of Z_0^n is written as $(\alpha_1,\dots,\alpha_n)$ and we define $|A| = \alpha_1 + \cdots + \alpha_n$.

Definition I-1 [monoideal]. A subset I_M of Z_0^n is a monoideal if

$$I_M + Z_0^n = I_M. \quad \square$$

Figure 1 illustrates a monoideal in Z_0^2, where all the lattice points inside the shaded area constitute the monoideal and the lattice points (1,3) and (2,2) are generators of the monoideal.

|| Fig.1 ||

Proposition I-1. A monoideal I_M is finitely generated. That is, there exist a finite number of elements $A_1, \ldots, A_s$ in I_M satisfying

$$I_M = \bigcup_{i=1}^{s} (A_i + Z_0^n). \quad \square$$

Proof. We use an induction on n. When $n = 1$, it is obvious that I_M is generated by a single element $A_1 = (\alpha_1)$, $\alpha_1 = \min\{|A| \mid A \in I_M\}$.

Next, assuming that every monoideal in Z_0^n, $n < k$, is finitely generated, we consider the case of $n = k$. Let

$$\hat{I}_M = \{(\alpha_1,\ldots,\alpha_{k-1}) \mid (\alpha_1,\ldots,\alpha_{k-1},\alpha_k) \in I_M\},$$

then $\hat{I}_M$ is obviously a monoideal in Z_0^{k-1}. Hence, by induction assumtion, there exist a finite number of generators $\hat{A}_1, \ldots, \hat{A}_\ell$ such that $\hat{I}_M = \bigcup_{i=1}^{\ell} (\hat{A}_i + Z_0^{k-1})$. For $i=1,\ldots,\ell$, let $\hat{A}_i = (\alpha_{i1},\ldots,\alpha_{i,k-1})$ and $\alpha_{ik} = \min\{\alpha_k \mid (\alpha_{i1},\ldots,\alpha_{i,k-1},\alpha_k) \in I_M\}$. Denoting $(\alpha_{i1},\ldots,\alpha_{i,k-1},\alpha_{ik})$ by A_i, we decompose I_M as $I_M = \bigcup_{i=1}^{\ell} (A_i + Z_0^k) + I'_M$, with $I'_M \cap [\bigcup_{i=1}^{\ell} (A_i + Z_0^k)] = \phi$. Then, each element $(\alpha_1,\ldots,\alpha_k)$ of I'_M satisfies $\alpha_k < \bar{\alpha}_k = \max\{\alpha_{1k},\ldots,\alpha_{\ell k}\}$. For each α in $\{0,\ldots,\bar{\alpha}_k-1\}$, let $\hat{I}_{M,\alpha} = \{(\alpha_1,\ldots,\alpha_{k-1}) \mid (\alpha_1,\ldots,\alpha_{k-1},\alpha) \in I'_M\} + Z_0^{k-1}$, then $\hat{I}_{M,\alpha}$ is a monoideal in Z_0^{k-1} and it is finitely generated by induction assumption. Therefore, I_M is finitely generated. $\square$

Corollary to Prop. I-1. Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_s \subseteq \cdots$ is an increasing sequence of monoideals, then there exists an integer T such that

$$I_T = I_{T+1} = \cdots. \quad \square$$

Proof. Let $I_\infty = \bigcup_{i=1}^{\infty} I_i$, then we can see that I_∞ is a monoideal. Prop. I-1 assures that there are a finite number of elements $A_1, \ldots, A_t$ of I_∞ such that $I_\infty = \bigcup_{i=1}^{t} (A_i + Z_0^n)$. For each i, $1 \leq i \leq t$, there is a number $J(i)$ such that $A_i \in I_{J(i)}$.

Let $T = \max\{J(1),\ldots,J(t)\}$, then $A_i \in I_T$ for $i=1,\ldots,t$. So $I_\infty = \bigcup_{i=1}^{t}(A_i + Z_0^n) \subseteq I_T$. Hence, we see $I_\infty \subseteq I_T \subseteq I_{T+1} \subseteq \cdots \subseteq I_\infty$. □

Let $K[x_1,\ldots,x_n]$ be a ring of polynomials in n variables $x_1, \ldots, x_n$ with coefficients in a number field K. We abbreviate $K[x_1,\ldots,x_n]$ to $K[x]$. Let $f_1, \ldots, f_r$ be elements of $K[x]$, and I the ideal $(f_1,\ldots,f_r)$ in $K[x]$ generated by $f_1, \ldots, f_r$. We express f in $K[x]$ as $f = \sum_A a_A x^A$, where $A = (\alpha_1,\ldots,\alpha_n)$, $a_A \in K$, and x^A is an abbreviation of $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$. We call $\alpha_1 + \cdots + \alpha_n$ the degree of the term x^A, i.e., $\deg(x^A) = |A|$.

Definition I-2 [lexicographic order $\gg$ in Z_0^n].

For any elements $A = (\alpha_1,\ldots,\alpha_n)$ and $B = (\beta_1,\ldots,\beta_n)$ of Z_0^n, we define $A \gg B$ iff there is an integer i, $1 \leq i \leq n$, such that $\alpha_j = \beta_j$ for all j, $1 \leq j < i$, and $\alpha_i > \beta_i$. □

Note. The following theory is valid if we use another definition of order so far as Z_0^n becomes a well-ordered Abelian semigroup by the order.

Definition I-3 [exponent set, leading exponent, head term].

Exponent set of f, leading exponent of f, and head term of f, which are abbreviated to exs(f), lex(f), and ht(f), respectively, are defined as follows:

$$\mathrm{exs}(f) = \{A \in Z_0^n \mid a_A \neq 0 \text{ in } f = \sum a_A x^A\},$$

$$\mathrm{lex}(f) \in \mathrm{exs}(f),\ \mathrm{lex}(f) \gg \text{any other element of } \mathrm{exs}(f),$$

$$\mathrm{ht}(f) = \text{a term } a_A x^A \text{ of } f, \text{ where } A = \mathrm{lex}(f).\ \square$$

Definition I-4. The lex(I), with I a polynomial ideal, is a subset of Z_0^n defined by

$$\mathrm{lex}(I) = \{\mathrm{lex}(f) \mid f \neq 0,\ f \in I\}.\ \square$$

Proposition I-2. The set $E = \mathrm{lex}(I)$ is a monoideal. □

Proof. The relation $E \subseteq E + Z_0^n$ is trivial because $(0,\ldots,0) \in Z_0^n$, so we have only to show $E + Z_0^n \subseteq E$. Let $A + B$ be any element of $E + Z_0^n$ such that $A \in E$ and $B \in Z_0^n$. By definition, there exists a polynomial f in I such that $\mathrm{lex}(f) = A$. Since $\mathrm{lex}(x^B f) = A + B$ and I is an ideal, we have $x^B f \in I$. That is, $A + B \in E$. □

§3. Gröbner basis of polynomial ideal —— formulation based on monoideal ——

The essential operations in the Gröbner basis theory are monomial reduction and monomial cancellation. A monomial x^A is fully characterized by its exponent A, and we have seen that an exponent set is well formulated by a monoideal. Therefore, we naturally expect that the theory of Gröbner basis can be formulated simply from the veiwpoint of monoideal.

Definition I-5 [reducibility].

Let $F = \{f_1,\ldots,f_r\}$ be a subset of K[x], and put $E = \bigcup_{i=1}^{r}[\mathrm{lex}(f_i) + Z_0^n]$. A polynomial h in K[x] is called reducible with respect to F if $\mathrm{exs}(h) \cap E \neq \phi$ and h is called irreducible w.r.t. F if $\mathrm{exs}(h) \cap E = \phi$. □

Definition I-6 [reduction].

With the notations in Def. I-5, let $h' \in K[x]$. The h' is called a reduction of h w.r.t. F and written as $h \xrightarrow[F]{} h'$ if one of the followings holds:

(a) $h' = h$ when h is irreducible w.r.t. F,

(b) $h' = h - c \cdot x^A f_k$ when $\mathrm{exs}(h) \cap [\mathrm{lex}(f_k)+Z_0^n] \neq \phi$,

where c and A are determined as follows: let $\mathrm{ht}(f_k) = a_{A_k} x^{A_k}$, hence h contains a term proportional to x^{A_k} and let the term be $b_{A+A_k} x^{A+A_k}$ then $c = b_{A+A_k}/a_{A_k}$. □

Definition I-7 [normal form].

Suppose h in K[x] is reduced successively as $h \xrightarrow[F]{} h' \xrightarrow[F]{} \cdots \xrightarrow[F]{} \tilde{h}$, and if $\tilde{h}$ is irreducible w.r.t. F then $\tilde{h}$ is called a normal form of h w.r.t. F. We denote the above reduction sequence by $h \xrightarrow[F]{*} \tilde{h}$. □

Proposition I-3. Let $F = \{f_1,\ldots,f_r\}$ be a subset of K[x]. Given a polynomial h in K[x], we can reduce h to a normal form $\tilde{h}$ w.r.t. F by a finite sequence of reductions. □

Proof. See [1], or refer to a proof for the power series case in §4. □

Definition I-8 [Gröbner basis].

Let $I = (f_1,\ldots,f_r)$ be an ideal in K[x]. A subset $G = \{g_1,\ldots,g_s\}$ of K[x] is called a Gröbner basis of I if the following conditions are satisfied:

(1) $(g_1,\ldots,g_s) = I$,

(2) for any element f of I, $f \xrightarrow[G]{*} 0$. □

Definition I-9 [S-polynomial].

Let f and g be polynomials in K[x], and put ht(f) = $a_A x^A$ and ht(g) = $b_B x^B$. Let u and v be monomials satisfying LCM(x^A,x^B) = $u \cdot x^A$ = $v \cdot x^B$, where LCM is the least common multiple. Then, S-polynomial of f and g, to be abbreviated to Sp(f,g), is defined by

$$Sp(f,g) = u \cdot f - (a_A / b_B) v \cdot g. \quad \square$$

Proposition I-4. Let I be an ideal in K[x]. Let the monoideal E = lex(I) be generated by A_1, ..., A_s, i.e., E = $\bigcup_{i=1}^{s} (A_i + Z_0^n)$. If G = $\{g_1, \ldots, g_s\}$ is a subset of I satisfying lex(g_i) = A_i for every i in {1,...,s}, then

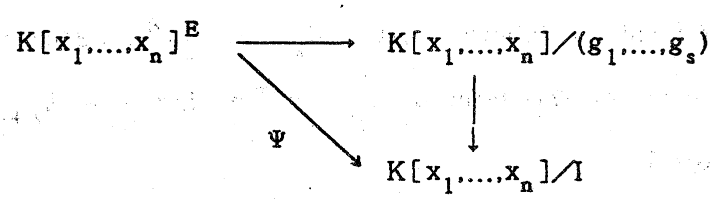
(a) $(g_1, \ldots, g_s) = I$,

(b) for any f in K[x], f is an element of I iff $f \xrightarrow[G]{} 0$. $\square$

Proof. Prop. I-3 assures that there exist polynomials h_1, ..., h_s such that

$$f = \sum h_i g_i + \tilde{f}, \quad exs(\tilde{f}) \cap E = \phi.$$

Proof of (a). Let $K[x]^E = \{h \in K[x] \mid exs(h) \cap E = \phi\}$, then we have a commutative diagram:



We can see that the mapping Ψ is surjective. On the other hand, (b) means that the Ψ is injective. So, proof of (a) reduces to proof of (b).

Proof of (b). Suppose f $\in$ I and $\tilde{f} \neq 0$, then lex($\tilde{f}$) $\in$ E because $\tilde{f} \in$ I. This contradicts to the assumption that $\tilde{f}$ is irreducible. Conversely, if $\tilde{f}$ = 0 then it is obvious that f $\in$ I. $\square$

Corollary to Prop. I-4. With the notations in Prop. I-4, let G = $\{g_1, \ldots, g_s\}$ be a Gröbner basis of I, and h a polynomial in K[x]. Let $\tilde{h}_1$ and $\tilde{h}_2$ be normal forms of h w.r.t. G, then $\tilde{h}_1 = \tilde{h}_2$. $\square$

Proof. By definition of normal form, we have exs($\tilde{h}_i$) E = ϕ, i=1,2. On the other hand, lex($\tilde{h}_1 - \tilde{h}_2$) $\in$ E because $\tilde{h}_1 - \tilde{h}_2 \in$ I. Hence, if $\tilde{h}_1 - \tilde{h}_2 \neq 0$, then we have a contradiction. $\square$

So far, we did not mention how to construct a Gröbner basis. The construction of the basis is most important in the theory of Gröbner basis, and it was given by Buchberger [1].

Procedure BUCHBERGER

input: an ideal $I = (f_1,\dots,f_r)$ in $K[x]$.

output: a Gröbner basis $G = \{g_1,\dots,g_s\}$ of I.

```
    G := {g_1:=f_1 ,..., g_r:=f_r};
    P := {(g_i,g_j) | g_i,g_j ε G, i ≠ j};
    while P ≠ φ do begin
        p_ij := a pair (g_i,g_j) in P;
        P := P - {p_ij};
(*)     g~ := a normal form of Sp(g_i,g_j) w.r.t. G;
        if g~ ≠ 0 then begin
            P := P ∪ {(g,g~) | g ε G};
            G := G ∪ {g~};
            end;
        end.
```

In the above procedure, the size of G is increasing one by one, so we denote the G explicitly by G_0, G_1, G_2, ..., where $G_0 = F$ and $G_i = G_{i-1} \cup \{\tilde{g}_i\}$, $i \geq 1$, with $\tilde{g}_i$ the i-th generated polynomial. Writing $\tilde{g}_i = g_{r+i}$, so $G_i = \{g_1,\dots,g_{r+i}\}$, we put $E_i = \bigcup_{j=1}^{r+i} [\mathrm{lex}(g_j) + Z_0^n]$.

Proposition I-5. The above construction procedure terminates. That is, there exists a positive integer T such that for any pair (f_i,f_j) in G_T, we have $Sp(f_i,f_j) \xrightarrow[G_T]{} 0$. □

Proof. Since $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$ is an increasing sequence of monoideals, Corollary to Prop. I-1 assures that there is an integer T such that $E_T = E_{T+1} = \cdots$. Suppose there is a pair (g_i,g_j) in G_T satisfying $Sp(g_i,g_j) \xrightarrow[G_T]{} \tilde{g}$, $\tilde{g} \neq 0$, that is $\mathrm{exs}(\tilde{g}) \cap E_T = \phi$. Then, by the construction, there is an integer t, $t > T$, such that $\tilde{g} \in G_t$, hence $\mathrm{lex}(\tilde{g}) \in E_t$. On the other hand, $E_T = E_t$ by definition of E_T. This is a contradiction. □

Theorem 1 [Buchberger].

Let $I = (g_1,\ldots,g_s)$ be an ideal in $K[x]$ and G the set $\{g_1,\ldots,g_s\}$. If

$$Sp(g_i,g_j) \xrightarrow[G]{} 0 \quad \text{for any pair } (g_i,g_j),\ i \neq j,\ 1 \leq i,j \leq s,$$

then G is a Gröbner basis of I. □

Proof. See [1], or refer to a proof for the power series case in §4. □

§4. Gröbner basis for truncated power series

Let $C\{z_1,\ldots,z_n\}$ be a ring of convergent power series with coefficients in the complex number field C. We abbreviate $C\{z_1,\ldots,z_n\}$ to $C\{z\}$. Let $f_1, \ldots, f_r$ be elements of $C\{z\}$, and I the ideal $(f_1,\ldots,f_r)$ in $C\{z\}$ generated by $f_1, \ldots, f_r$. We express f in $C\{z\}$ as $f = \sum_A a_A z^A$, where $A = (\alpha_1,\ldots,\alpha_n)$, $a_A \in C$, and z^A is an abbreviation of $z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n}$.

Before defining a Gröbner basis for power series, we consider in this section a power series ideal modulo $(z_1,\ldots,z_n)^{M+1}$. We call a Gröbner basis for this truncated power series an M-Gröbner basis. In this and the next sections, we omit several short proofs which are analogous to those given in §3.

Definition II-1 [order $\triangleright$ in Z_0^n].

For any element A and B of Z_0^n, we define $A \triangleright B$ iff either $|A| < |B|$ or $A \gg B$ when $|A| = |B|$. □

Notes. We can formulate the following theory by using another definition of $\triangleright$. For example, if we define $\triangleright$ by using a weight function for variables, we obtain the theory in a quite general form. The important point in such a definition is that the lower degree terms are of higher order, and this is essential for power series.

Definition II-2 [initial polynomial].

The initial polynomial of f, which is abbreviated to in(f), is the sum of the lowest degree terms of f:

$$\mathrm{in}(f) = \sum_{|A|=\mathrm{lowest}} a_A z^A. \ \square$$

Definition II-3 [order, leading exponent, head term].

Order of f, leading exponent of f, and head term of f, which are abbreviated to ord(f), lex(f), and ht(f), respectively, are defined as follows:

$$\mathrm{ord}(f) = \deg(\mathrm{in}(f)),$$

$$\mathrm{lex}(f) = \mathrm{lex}(\mathrm{in}(f)),$$

$$\mathrm{ht}(f) = \mathrm{ht}(\mathrm{in}(f)),$$

where the lex and ht in the right hand side are defined by Def. I-3. □

Definition II-4 [M-equality of power series].

Two power series f and g in $C\{z\}$ are equal within degree M, and denoted by $f =_M g$, iff $\mathrm{ord}(f-g) > M$. □

Definition II-5 [M-reducibility of power series].

Let $F = \{f_1,\ldots,f_r\}$ be a subset of $C\{z\}$, and put $E = \bigcup_{i=1}^{r} [\mathrm{lex}(f_i) + Z_0^n]$. Let a power series h in $C\{z\}$ be decomposed as $h = h_d + h_{d+1} + \cdots$, where h_i is the sum of all terms of degree i of h. The h is called reducible within degree M (abbreviated to M-reducible) with respect to F if $\mathrm{exs}(h_i) \cap E \neq \phi$ for some $i \leq M$, and h is called M-irreducible w.r.t. F if $\mathrm{exs}(h_i) \cap E = \phi$ for all $i \leq M$. □

Definition II-6 [M-reduction of power series].

With the notations in Def. II-5, let $h' \in C\{z\}$. The h' is called an M-reduction of h w.r.t. F and written as $h \xrightarrow[F,M]{} h'$ if one of the followings holds:

(a) $h' = h$ when h is M-irreducible w.r.t. F,

(b) $h' = h - c \cdot x^A f_k$ when $\mathrm{exs}(h_i) \cap [\mathrm{lex}(f_k)+Z_0^n] \neq \phi$ for some $i \leq M$,

where c and A are determined as follows: let $\mathrm{ht}(f_k) = a_{A_k} x^{A_k}$, hence h contains a term proportional to x^{A_k} and let the term be $b_{A+A_k} x^{A+A_k}$, $|A+A_k| \leq M$, then $c = b_{A+A_k}/a_{A_k}$. □

Definition II-7 [M-normal form of power series].

Suppose h in $C\{z\}$ is reduced successively as $h \xrightarrow[F,M]{} h' \xrightarrow[F,M]{} \cdots \xrightarrow[F,M]{} \tilde{h}$, and if $\tilde{h}$ is M-irreducible w.r.t. F then $\tilde{h}$ is called an M-normal form of h w.r.t. F. We denote the reduction of h to its normal form $\tilde{h}$ by $h \xrightarrow[F,M]{} \tilde{h}$. In particular, we write $h \xrightarrow[F,M]{} 0$ if $\tilde{h} =_M 0$. □

Proposition II-1. Let $F = \{f_1,\ldots,f_r\}$ be a subset of $C\{z\}$ and M a positive integer. Given a power series h in $C\{z\}$, we can reduce h to an M-normal form $\tilde{h}$ w.r.t. F by a finite sequence of reductions. □

<u>Proof.</u> In this proof, we denote the sum of all the k-th degree terms of h by h_k, so $h = h_d + h_{d+1} + \cdots$. If $d > M$ then h is already M-irreducible w.r.t. F, so we assume $d \leq M$. We put $\mathrm{lex}(f_i) = A_i$, $i=1,\ldots,r$, and $E = \bigcup_{i=1}^{r}(A_i + Z_0^n)$.

<u>Step 1.</u> We show, by the transfinite induction w.r.t. $\mathrm{lex}(h_d)$, that there is a finite sequence of M-reductions

$$h_d \xrightarrow[F,M]{} \cdots \xrightarrow[F,M]{} \tilde{h}_d + h'_{d+1} + h'_{d+2} + \cdots,$$

where $\tilde{h}_d$ is either 0 or M-irreducible w.r.t. F.

Step 1-1. When $\mathrm{lex}(h_d) = (0,\ldots,0)$, or $d = 0$.

If $\mathrm{ord}(f_i) > 0$ for all i then h_0 is obviously M-irreducible w.r.t. F, otherwise there is a reduction such that $h_0 \xrightarrow[F,M]{} 0 + (\text{terms of degree} \geq 1)$. Hence, the claim in Step 1 is right in this case.

Step 1-2. Assuming that the claim in Step 1 is right for any h'_d such that $\mathrm{lex}(h'_d) \lhd A$, we show the claim is right for h_d with $\mathrm{lex}(h_d) = A$. If $\mathrm{exs}(h_d) \cap E = \phi$ then there is nothing to prove, so we assume $\mathrm{exs}(h_d) \cap E \neq \phi$. Let B be the highest order element of $\mathrm{exs}(h_d) \cap E$ and assume that $B = \mathrm{lex}(f_k) + C$, that is

$$h_d = \tilde{h}_d^{(1)} + c_B z^B + h^{(2)},$$

where [exponent of any term of $\tilde{h}_d^{(1)}$] $\rhd B$ and $B \rhd \mathrm{lex}(h^{(2)})$. By definition, we have either $\tilde{h}_d^{(1)} = 0$ or $\mathrm{exs}(\tilde{h}_d^{(1)}) \cap E = \phi$. The rest part of h_d, or $c_B z^B + h^{(2)}$, can be reduced as

$$\begin{aligned} c_B z^B + h^{(2)} &\xrightarrow[F,M]{} c_B z^B + h^{(2)} - (c_B/a_{A_k}) z^C \cdot f_k \\ &= h^{(2)} - (c_B/a_{A_k}) z^C \cdot [f_k - \mathrm{ht}(f_k)]. \end{aligned}$$

Writing the right hand side expression as $h'^{(2)}$, we see $\mathrm{lex}(h'^{(2)}) \lhd B \trianglelefteq \mathrm{lex}(h_d)$. By induction assumption, there exist a finite sequence of reductions $h'^{(2)} \xrightarrow[F,M]{} \cdots \xrightarrow[F,M]{} \tilde{h}^{(2)}$, where the d-th degree part of $\tilde{h}^{(2)}$ is either 0 or M-irreducible w.r.t. F. Thus, Step 1 is proved.

<u>Step 2.</u> By the Step 1, we have $h \xrightarrow[F,M]{} \cdots \xrightarrow[F,M]{} \tilde{h}_d + h'_{d+1} + h'_{d+2} + \cdots$, where $\tilde{h}_d$ is either 0 or M-irreducible w.r.t. F. Next, we apply the reduction procedure of Step 1 to h'_{d+1}. This procedure does not alter the terms of degree less than d+1, hence $h \xrightarrow[F,M]{} \cdots \xrightarrow[F,M]{} \tilde{h}_d + \tilde{h}_{d+1} + h''_{d+2} + \cdots$, where $\tilde{h}_{d+1}$ is either 0 or M-irreducible w.r.t. F. Continuing this procedure, we can reduce h to $\tilde{h}$. □

Definition II-8 [M-Gröbner basis of power series ideal].

Let $I = (f_1,\dots,f_r)$ be an ideal in $C\{z\}$ and M a positive integer. A subset $G = \{g_1,\dots,g_s\}$ of $C\{z\}$ is called an M-Gröbner basis of I if the following conditions are satisfied:

(1) $(g_1,\dots,g_s) = I$,

(2) for any element f of I, $f \xrightarrow[G,M]{} 0$. □

Definition II-9 [S-power series].

Let f and g be power series in $C\{z\}$, and put $ht(f) = a_A x^A$ and $ht(g) = b_B x^B$. Let u and v be monomials satisfying $LCM(x^A,x^B) = u \cdot x^A = v \cdot x^B$, where LCM is the least common multiple. Then, S-power series of f and g, to be abbreviated to Sp(f,g), is defined by

$$Sp(f,g) = u \cdot f - (a_A/b_B) v \cdot g. \quad \square$$

Proposition II-2. (For the proof, refer to Corollary to Prop. I-4.)

Let G be an M-Gröbner basis of an ideal in $C\{z\}$, and h a power series in $C\{z\}$. Let $\tilde{h}_1$ and $\tilde{h}_2$ be M-normal forms of h w.r.t. G, then $\tilde{h}_1 = \tilde{h}_2$. □

Theorem 2. Let $I = (g_1,\dots,g_s)$ be an ideal in $C\{z\}$, G the set $\{g_1,\dots,g_s\}$, and M a positive integer. If

$$Sp(g_i,g_j) \xrightarrow[G,M]{} 0 \quad \text{for any pair } (g_i,g_j),\ i \neq j,\ 1 \leq i,j \leq s,$$

then G is an M-Gröbner basis of I. □

Proof. Let $E = \bigcup_{i=1}^{s} [lex(g_i) + Z_0^n]$ and f be any element of I with $lex(f) = A$. We have only to show $f \xrightarrow[G,M]{} 0$. If $A \in E$ then f can be reduced directly and we can replace f by f', $lex(f') \lhd A$, so we have only to consider the case of $A \notin E$.

Since $f \in I$, there exist $h_1, \dots, h_s$ in $C\{z\}$ satisfying

$$f = h_1 g_1 + \cdots + h_s g_s.$$

For i=1,...,s, put

$$ht(g_i) = a_{A_i} z^{A_i}, \quad ht(h_i) = b_{B_i} z^{B_i},$$

hence $lex(h_i g_i) = A_i + B_i$. Let D be the highest order element of $\{A_i+B_i \mid i=1,\dots,s,\ h_i \neq 0\}$. Without loss of generality, we assume $D = A_1+B_1 = \cdots = A_\sigma+B_\sigma$, $D \rhd A_j+B_j$ for all $j > \sigma$. Then, putting $h_i = b_{B_i} z^{B_i} + h_i'$, i=1,...,s, we decompose f as

$$f = f^{(1)} + f^{(2)},$$

$$f^{(1)} = \sum_{i=1}^{\sigma} b_{B_i} z^{B_i} g_i, \quad f^{(2)} = \sum_{i=1}^{\sigma} h_i' g_i + \sum_{i=\sigma+1}^{s} h_i g_i.$$

We see $\mathrm{lex}(f^{(2)}) \lhd D$. If $\sigma = 1$ then $D = A \in E$, contradicting to the assumption $A \notin E$. Hence, $\sigma > 1$ and we can rewrite $f^{(1)}$ as

$$\begin{aligned} f^{(1)} = & (a_{A_1} b_{B_1}) \cdot (z^{B_1} g_1 / a_{A_1} - z^{B_2} g_2 / a_{A_2}) \\ & + (a_{A_2} b_{B_2} + a_{A_1} b_{B_1}) \cdot (z^{B_2} g_2 / a_{A_2} - z^{B_3} g_3 / a_{A_3}) \\ & + \cdot\cdot\cdot\cdot\cdot \\ & + (a_{A_{\sigma-1}} b_{B_{\sigma-1}} + \cdots + a_{A_1} b_{B_1}) \cdot (z^{B_{\sigma-1}} g_{\sigma-1} / a_{A_{\sigma-1}} - z^{B_\sigma} g_\sigma / a_{A_\sigma}) \\ & + (a_{A_\sigma} b_{B_\sigma} + \cdots + a_{A_1} b_{B_1}) \cdot (z^{B_\sigma} g_\sigma / a_{A_\sigma}). \end{aligned}$$

We first note that the last term of the above expression is 0. To see this, we consider the sum of terms of exponent D in $f^{(1)}$, which is

$$\sum_{i=1}^{\sigma} a_{A_i} b_{B_i} z^{A_i+B_i} = (a_{A_1} b_{B_1} + \cdots + a_{A_\sigma} b_{B_\sigma}) \cdot z^D.$$

If this expression is not zero then $A = \mathrm{lex}(f) = A_1+B_1$, contradicting to the assumption $A \notin E$. We next consider the j-th term, $j \leq \sigma-1$, of the r.h.s. expression. Remembering the definition of S-power series, we see [the j-th term] = $u \cdot Sp(g_j, g_{j+1})$ with u a monomial. By the assumption of theorem, $Sp(g_j, g_{j+1}) \xrightarrow[G,M]{} 0$. Hence, we find $f^{(1)} \xrightarrow[G,M]{} 0$.

The $f^{(2)}$ is of the same form as f, so we can continue the above reduction making $f \xrightarrow[G,M]{} \cdots \xrightarrow[G,M]{} f'$, $\mathrm{lex}(f') \lhd \mathrm{lex}(g_i)$, $i=1,\dots,s$. That is $f \xrightarrow[G,M]{} 0$. □

Because the M-Gröbner basis is for truncated power series, it can be constructed by a finite number of steps. In fact, if we modify the line (*) in Proc. BUCHBERGER as

$$\tilde{g} := \text{an M-normal form of } Sp(g_i, g_j) \text{ w.r.t. } G \ ;$$

and if we calculate M-normal form and S-power series as truncated power series, then we obtain the required procedure. The M-Gröbner basis will be useful when we use truncated power series for approximate calculations.

§5. Gröbner basis of power series ideal

Now, we investigate the Gröbner basis of power series ideal for which we must consider the terms of arbitrarily high degree. This poses us an interesting problem when we stand on a viewpoint of constructive algebra. We discuss this point in the next section, and we first define a Gröbner basis of a power series ideal and investigate the properties generally.

Definition III-1 [tangential ideal].

Let $I = (f_1,\dots,f_r)$ be an ideal in $C\{z\}$. The tangential ideal of I, to be abbreviated to $\bar{I}$, is defined as

$$\bar{I} = in(I) = \langle in(f) \mid 0 \neq f \in I \rangle C[x]. \quad \square$$

Definition III-2 [Gröbner basis of power series ideal].

Let $I = (f_1,\dots,f_r)$ be an ideal in $C\{z\}$. A subset $G = \{g_1,\dots,g_s\}$ of I is called a Gröbner basis of I if the following conditions are satisfied:

(1) $(g_1,\dots,g_s) = I$,

(2) for any f in I and for any positive integer M, $f \xrightarrow[G,M]{} 0$. $\square$

Note that the Gröbner basis of power series is defined in terms of M-Gröbner basis which is constructive.

Now, we consider the following procedure.

Procedure PS-GRÖBNER

input: an ideal $I = (f_1,\dots,f_r)$ in $C\{z\}$.

output: a Gröbner basis $G = \{g_1,\dots,g_s\}$ of I.

```
        G_0 := {f_1,...,f_r};   M := 0;
LOOP:   M := M+1;   G_M := G_{M-1};
        P := {(g_i,g_j) | g_i,g_j ε G_M, g_i ≠ g_j};
        while P ≠ φ do begin
              p_ij := a pair (g_i,g_j) in P;
              P := P - {p_ij};
              g~ := an M-normal form of Sp(g_i,g_j) w.r.t. G_M;
              if g~ ≠_M 0 then begin
                    P := P ∪ {(g,g~) | g ε G_M};
                    G_M := G_M ∪ {g~};
                    end;
              end;
(**)    if G_M = G_{M-1} and [termination condition] is satisfied then return G_M;
```

goto LOOP.

Note that, in the line (**), the [termination condition] is not specified yet. In the following, we use the notation G_M defined above.

Proposition III-1. With the above notations, let f be any element of I. Then, for any positive integer L, $L \leq M$, we have $f \xrightarrow[G_M,L]{} 0$. □

Proof. The case $L = M$ is trivial, so we assume $L < M$. By the construction, $G_L \subseteq G_M$, so we write $G_L = \{g_1,\dots,g_\ell\}$ and $G_M = \{g_1,\dots,g_\ell,\dots,g_m\}$. Then, for all j in $\{\ell+1,\dots,m\}$, there exist $h'_{j1}, \dots, h'_{j\ell}$ in $C\{z\}$ satisfying

$$g_j = h'_{j1}g_1 + \cdots + h'_{j\ell}g_\ell.$$

Since $f \in I$, there exists $h_1, \dots, h_\ell, \dots h_m$ in $C\{z\}$ satisfying

$$f = h_1g_1 + \cdots + h_\ell g_\ell + \cdots + h_m g_m$$
$$= (h_1 + \textstyle\sum_j h'_{j1})g_1 + \cdots + (h_\ell + \sum_j h'_{j\ell})g_\ell.$$

Reducing f w.r.t. G_L as in the proof of Th. 2, we find $f \xrightarrow[G_M,L]{} 0$. □

Theorem 3. With the above notations, there exists a positive integer T such that G_T is a Gröbner basis of I. □

Proof. By virtue of Th. 2, we have only to show the existence of G_T such that $Sp(g_i,g_j) \xrightarrow[G_T,L]{} 0$ for any g_i and g_j in G_T and for any positive integer L. We put $E_M = \bigcup_{i=1}^{m} [lex(g_i) + Z_0^n]$. Since $E_1 \subseteq E_2 \subseteq \cdots$ is an increasing sequence of monoideals, Corollary to Prop. I-1 assures that there exists an integer T such that $E_T = E_{T+1} = \cdots$. Prop. III-1 assures that our claim is right for $L \leq T$. So, we consider the case $L > T$. Suppose $Sp(g_i,g_j) \xrightarrow[G_T,L]{} \tilde{g}$, $lex(\tilde{g}) \notin E_L$. Then, by the construction, there is an integer ℓ, $\ell > L$, such that $\tilde{g} \in G_\ell$. This means $lex(\tilde{g}) \in E_\ell$, but $E_\ell = E_T$ by definition of T, so we are lead to a contradiction. □

Theorem 4. Let $I = (f_1,\dots,f_r)$ be an ideal in $C\{z\}$ and $\bar{I} = in(I)$ a tangential ideal of I. Let $G = \{g_1,\dots,g_s\}$ be a Gröbner basis of I, and put $E = \bigcup_{i=1}^{s} [lex(g_i) + Z_0^n]$, then $lex(\bar{I}) = E$. (The lex in this equation is for polynomials.) □

Proof. Put $\bar{E} = lex(\bar{I})$, then $E \subseteq \bar{E}$ because $lex(g_i) \in \bar{E}$ for all i in $\{1,\dots,s\}$. Next, we show $A \in E$ for any element A of $\bar{E}$. Since $A \in \bar{E}$, there exists a homogeneous polynomial $\bar{f}$ in $\bar{I}$ such that $lex(\bar{f}) = A$. Since $\bar{I} = in(I)$, $\bar{f}$ can be expressed as $\bar{f} = in(h_1f_1 + \cdots + h_rf_r)$, with $h_1, \dots, h_r$ in $C\{z\}$. Putting f =

$\sum h_i f_i$, we see $f \in I$ and $lex(f) = A$. Since G is a Gröbner basis of I, we have $f \xrightarrow[G,M]{} 0$ for any integer $M \geq |A|$. This means $A = lex(f) \in E$, because if not so then $ht(f)$ cannot be reduced w.r.t. G. □

§6. On constructivity of a Gröbner basis for infinite power series

Let us consider Proc. PS-GRÖBNER from the constructive viewpoint. Actual construction of an ideal basis is possible only when the following conditions are satisfied:

(1) The number of basis elements is finite.

(2) We have a procedure of calculating the basis, where each step of the procedure can be executed constructively.

(3) We can decide constructively at which point we may stop the procedure.

We have proved that the above condition (1) is satisfied. However, Proc. PS-GRÖBNER does not satisfy conditions (2) and (3) in general. Even if condition (3) is satisfied, the procedure is not constructive because the arithmetic of general infinite power series is not constructive unless we are given an explicit construction procedure for any power series.

In the constructive algebra, it is quite common to assume that we are given an explicit construction procedure for every quantity to be input. Without such an assumption, it is impossible to discuss the infinite power series constructively. Therefore, we set an assumption.

> Assumption I. For each power series input to Proc. PS-GRÖBNER, we are given an explicit procedure which constructs the k-th degree term for any given integer $k \geq 0$.

With this assumption, we can construct $Sp(f,g)$ exactly as follows. We first construct $in(f)$ and $in(g)$ explicitly, which are homogeneous polynomials consisting of finite terms. Then, we calculate $ht(in(f))$ and $ht(in(g))$, let them be $a_A z^A$ and $b_B z^B$, respectively. Finally, we put $Sp(f,g) = u \cdot f - (a_A / b_B) v \cdot g$, where u and v are determined by $LCM(z^A, z^B) = u \cdot z^A = v \cdot z^B$. Note that we do not actually construct every term of $u \cdot f - (a_A / b_B) v \cdot g$ but define a new power series $Sp(f,g)$ by the above relation. Then, for any given $k \geqq 0$, we can construct the k-th degree term of $Sp(f,g)$ explicitly because we can for f and g. The construction of M-normal form can be done similarly.

With Assump. I, every step but the termination condition in Proc. PS-GRÖBNER becomes constructive. This is the reason why we defined the Gröbner basis for power series by using M-Gröbner basis of finite M. The problem is the termination condition. According to Def. III-2, in order to assert that G_M is a Gröbner basis of I, we must check $Sp(g_i,g_j) \xrightarrow[G_M,L]{} 0$ for every pair (g_i,g_j) in G_M and for every positive integer $L \geqq M$. If we perform this check by using the definition of M-normal form given in Def. II-7, then the check is not constructive in general because we must repeat the reduction infinitely many times. At present, it is unclear under what conditions there exists a constructive termination condition. So, the termination condition is an open problem now.

In some cases, however, we can constructively check the termination.

<u>Proposition III-2.</u> Let $G_M = \{g_1,\ldots,g_m\}$ be an M-Gröbner basis of an ideal I in $C\{z\}$, and put $E_M = \bigcup_{i=1}^{m} [\mathrm{lex}(g_i) + Z_0^n]$. If $E'_M = Z_0^n - E_M$ is a finite set and $M \geqq |A|$ for any element A of E'_M then G_M is a Gröbner basis of I. □

<u>Proof.</u> Let $Sp(g_i,g_j) \xrightarrow[G_M,\infty]{} \tilde{g}$, with g_i and g_j in G_M, and suppose $\tilde{g}$ is not zero. Then, $\mathrm{ord}(\tilde{g}) > M$ because G_M is an M-Gröbner basis. Hence $\mathrm{exs}(\tilde{g}) \in E_M$, contradicting to that $\tilde{g}$ is ∞-irreducible w.r.t. G_M. □

Figure 2 illustrates the case for which the above criteron applies.

|| Fig.2 ||

Another simple criterion, which also applies only in limited cases, is obtained by investigating the exponent set of power series to be handled.

<u>Definition III-3</u> [another monoideal $\mathcal{E}_M$].

For $\{g_1,\ldots,g_m\}$, which is a subset of $C\{z\}$, we define a monoideal $\mathcal{E}_M$ in Z_0^n by

$$\mathcal{E}_M = \bigcup_{i=1}^{m} [\mathrm{exs}(g_i) + Z_0^n]. \ \square$$

<u>Notes.</u> That $\mathcal{E}_M$ is a monoideal is obvious by the definition of monoideal. Since $\mathrm{lex}(g_i) \in \mathrm{exs}(g_i)$, $i=1,\ldots,m$, we see $E_M \subseteq \mathcal{E}_M$.

Proposition III-3. With the above notations, let h be in $C\{z\}$ and $h \xrightarrow[G_M,\infty]{} \tilde{h}$, then $\mathrm{exs}(\tilde{h}) \subseteq [\mathrm{exs}(h) \cup \mathcal{E}_M]$. □

Proof. If h is reduced by g_k as $h \xrightarrow[g_k]{} h' = h - cz^A g_k$, with $c \in C$, then $\mathrm{exs}(h') \subseteq [\mathrm{exs}(h) \cup \mathrm{exs}(z^A g_k)]$. Prop. III-3 is derived directly from this relation. □

Proposition III-4. With the above notations, let $\mathrm{ht}(g_i) = a_{A_i} z^{A_i}$, $i=1,\ldots,m$, and $E_M = \bigcup_{i=1}^{m} (A_i + Z_0^n)$. If $g_i - \mathrm{ht}(g_i)$, $1 \leq i \leq m$, is ∞-irreducible w.r.t. G_M, then g_i has the form

$$g_i = a_{A_i} z^{A_i} + \sum_{\ell} g_{i\ell},$$

$$g_{i\ell} = 0 \quad \text{or} \quad g_{i\ell} \in C\{z\} \text{ with } \mathrm{lex}(g_{i\ell}) \notin E_M. \ \square$$

Proof. Suppose $g_i - \mathrm{ht}(g_i)$ contains a monomial $t = cz^{A_k + A}$, $1 \leq k \leq m$, then we can reduce this term as $t \xrightarrow[G_M]{} t - (c/a_{A_k}) z^A g_k$. This contradicts to the assumption that $g_i - \mathrm{ht}(g_i)$ is ∞-irreducible w.r.t. G_M. Hence, $g_i - \mathrm{ht}(g_i)$ contains no term proportional to z^{A_k}. □

Corollary to Prop. III-4. With the above notations, if $\bigcup_{i=1}^{\mu} [\mathrm{lex}(g_i) + Z_0^n] = \bigcup_{i=1}^{\mu} [\mathrm{exs}(g_i) + Z_0^n]$, then we may set $g_i = z^{A_i}$ for $i=1,\ldots,\mu$. □

Proof. The above condition means that every monomial in $g_i - \mathrm{ht}(g_i)$, $i=1,\ldots,\mu$, is a multiple of z^{A_k}, $1 \leq k \leq \mu$, hence it can be reduced by g_k. Furthermore, Prop. III-3 assures that the above condition is not altered by the reduction. □

Let $G_M^{(1)} = \{g_1^{(1)},\ldots,g_\mu^{(1)}\}$ be the largest subset of G_M satisfying $E_M^{(1)} = \mathcal{E}_M^{(1)}$, where $E_M^{(1)} = \bigcup_{i=1}^{\mu} [\mathrm{lex}(g_i^{(1)}) + Z_0^n]$ and $\mathcal{E}_M^{(1)} = \bigcup_{i=1}^{\mu} [\mathrm{exs}(g_i^{(1)}) + Z_0^n]$. We put $G_M^{(2)} = G_M - G_M^{(1)} = \{g_1^{(2)},\ldots,g_\nu^{(2)}\}$, so $\mu + \nu = m$. We also put $E_M^{(2)} = \bigcup_{i=1}^{\nu} [\mathrm{lex}(g_i^{(2)}) + Z_0^n]$ and $\mathcal{E}_M^{(2)} = \bigcup_{i=1}^{\nu} [\mathrm{exs}(g_i^{(2)}) + Z_0^n]$, so $E_M^{(2)} \neq \mathcal{E}_M^{(2)}$.

Proposition III-5. With the above notations, if $\mathrm{exs}(\mathrm{Sp}(g_i,g_j)) \subseteq E_M^{(1)}$ for every i and j in $\{1,\ldots,m\}$ then G_M is a Gröbner basis of I. In particular, if $G_M^{(2)} = \phi$ then G_M is a Gröbner basis of I. If $\mathrm{lex}(\mathrm{Sp}(g_i,g_j)) \notin [E_M^{(1)} \cup E_M^{(2)}]$ for some i and j then G_M is not a Gröbner basis of I. □

Proof. Obvious from Prop. III-3 and Corollary to Prop. III-4. □

Note that $\mathrm{Sp}(g_i,g_j)$ may not be reduced to 0 if $\mathrm{exs}(\mathrm{Sp}(g_i,g_j)) \subseteq E_M^{(2)}$. For example, let $G_M = \{g_1,g_2\}$, $M >> 1$, with

$$g_1 = x^4y\cdot(1 + x + y + x^2 + y^2 + \cdots) + x^3y^2\cdot(1 + x^2 + y^2 + \cdots),$$

$$g_2 = x^3y^2\cdot(1 - x - y + x^2 + y^2 - \cdots) + x^2y^3\cdot(-1 + x^2 - y^2 + \cdots).$$

Then, $E_M = [(4,1)+Z_0^2] \cup [(3,2)+Z_0^2]$ and $\mathcal{E}_M = E_M \cup [(2,3)+Z_0^2]$. Hence, $G_M^{(1)} = \phi$, $E_M^{(1)} = \mathcal{E}_M^{(1)} = \phi$, $G_M^{(2)} = G_M$, $E_M^{(2)} = E_M$, and $\mathcal{E}_M^{(2)} = \mathcal{E}_M$. We see

$$\begin{aligned} Sp(g_1,g_2) &= y\cdot g_1 - x\cdot g_2 \\ &= x^4y^2\cdot(2x + 2y + \cdots) + x^3y^3\cdot(2 + 2y^2 + \cdots). \end{aligned}$$

Hence, $exs(Sp(g_1,g_2)) \subseteq E_M$. However, if we reduce the term $2x^3y^3$ of $Sp(g_1,g_2)$ by g_2, we obtain

$$\begin{aligned} Sp(g_1,g_2) \xrightarrow[g_2]{} & x^4y^2\cdot(2x + 2y + \cdots) + x^3y^3\cdot(2y^2 + \cdots) \\ &+ x^3y^3\cdot(2x + 2y - 2x^2 - 2y^2 + \cdots) \\ &+ x^2y^4\cdot(2 - 2x^2 + 2y^2 + \cdots). \end{aligned}$$

Thus, we obtain exponent (2,4) which is not in E_M.

In order to use Prop. III-5 as a termination condition (which is of course incomplete), we need some assumption on the constructivity of the monoideal $\mathcal{E}_M$. For example, we may set the following assumption:

Assumption II. For each power series f input to Proc. PS-GRÖBNER, we can construct the generators of the monoideal $[exs(f) + Z_0^n]$.

This assumption is not strange from the practical viewpoint. Suppose we represent a power series f in a form

$$f = \sum_{i=1}^{\lambda} z^{B_i}\cdot h_i, \quad B_i \notin \bigcup_{j\neq i}(B_j + Z_0^n).$$

(As explicit examples of this representation, see power series g_1 and g_2 presented above.) Then, Assump. II says that we can find $\{B_i \mid i=1,\dots,\lambda\}$ for every input power series f_k, $k=1,\dots,r$. The finite generation property of monoideal assures that $\{B_i \mid i=1,\dots,\lambda\}$ is a finite set, and we can usually find the set by scanning a finite number of leading terms. Thus, Assump. II is acceptable from the practical viewpoint.

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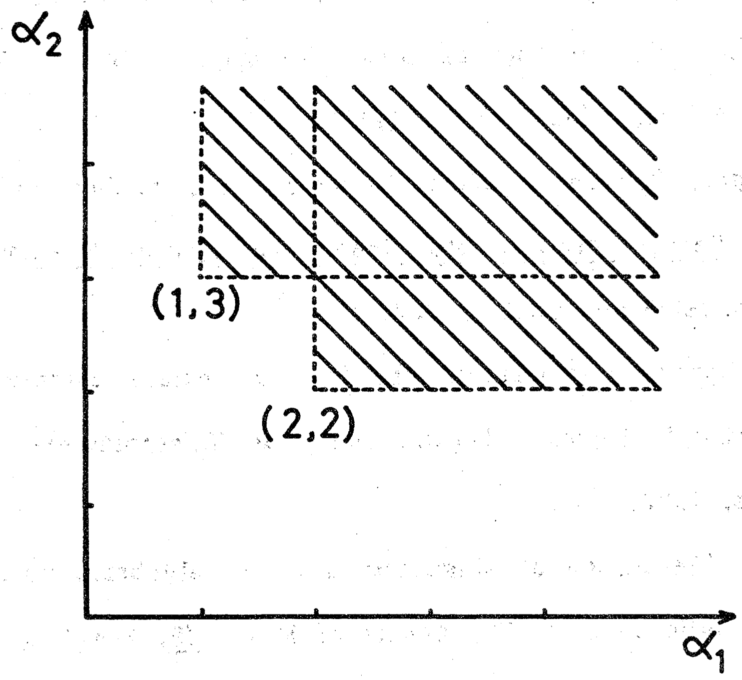

Fig. 1

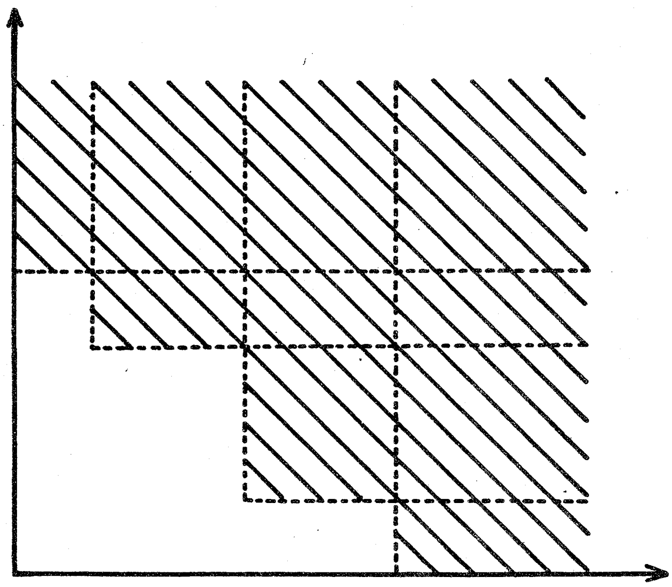

Fig. 2