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An Algorithm for the Normal Forms

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In a previous paper[8], we have given a recognition principle for a hypersurface isolated singularity of a certain type. In it, a normal form of homogeneous polynomial was constructed by the monomials of the lowest degree. However, the normal forms constructed by the principle were not unique. So, in this paper, we try to impose a condition to construct a unique normal form of homogeneous polynomial.

We consider it natural that normal form should be easy to write and remember; that is, the normal form should have the fewest monomials, and each monomial should be simple. The normal forms defined in this paper meet the above condition.

§ 1. Normal forms of singularities

In this section we review some theorems and definitions about normal forms of singularities which are given in [1,2,3].

Definition 1.1. A function $f:(\mathbf{C}^n,0) \longrightarrow (\mathbf{C},0)$ is said to be quasihomogeneous of degree d with exponents $a_1,\cdots,a_n$ if $f(\lambda^{a_1}x_1,\cdots,\lambda^{a_n}x_n) = \lambda^d f(x_1,\cdots,x_n)$ for all λ.

In terms of the function $f=\Sigma c_k x^k$, quasihomogeneity of degree 1 means that all exponents of non-zero terms lie on the hyperplane $\Gamma=\{k: a_1k_1+\cdots+a_nk_n=1\}$.

We call the hyperplane Γ the diagonal.

Definition 1.2. A quasihomogeneous function f is said to be Arnold non-degenerate if 0 is an isolated singular point.

Definition 1.3. We fix the set $\mathbf{a}$ of exponents. Then we say that a monomial x^K has generalized degree d if $\langle \mathbf{a}, \mathbf{K} \rangle = d$.

Definition 1.4. A polynomial has filtration d if all its monomials are of degree d or higher; when the generalized degree of all monomials is d , we call d the generalized degree of the polynomial; the degree of 0 is $+\infty$.

We denote the generalized degree of the polynomial by $\phi(f)$. The polynomials of filtration d form a linear space E_d . Let A be the polynomial ring. The E_d is an ideal in A .

Definition 1.5. A polynomial f is said to be semiquasihomogeneous of degree d with exponents $\mathbf{a}$ if $f=f_0+f'$, where f_0 is an Arnold non-degenerate quasihomogeneous polynomial of degree d with exponents $\mathbf{a}$, and $\phi(f')$ strictly greater than d .

Definition 1.6. Let $\mathbf{a}_1, \cdots\cdots, \mathbf{a}_p$ be a fixed collection of p quasihomogeneous types. We define the degree of x^k to be $\phi_i(K)=\langle \mathbf{a}, K\rangle$ in the i-th filtration. We define the piecewise degree of x^k to be $\phi(K)=\min[\phi_1(K), \cdots, \phi_p(K)]$.

Definition 1.7. A power series has piecewise filtration d if all its monomials have piecewise degree d or higher.

The equation $\phi(K)$ defines a polyhedron Γ in the space of exponents K that is convex towards 0. We denote $\{K \mid \phi(K)=d\}$ by $d\Gamma$. The sum of the terms of the lowest piecewise degree in a given power series is called the principal part of the series. A piecewise homogeneous function of degree d is a polynomial whose all monomials have piecewise degree d.

Definition 1.8. The multiplicity μ of the singular point 0 of a function $f:(\mathbf{C}^n,0) \longrightarrow (\mathbf{C},0)$ is defined as the dimension of the local ring

$$Q_f = \mathbf{C}[[x_1,\cdots,x_n]]/(\partial f/\partial x_1,\cdots,\partial f/\partial x_n).$$

Definition 1.9. A formal vector field $v = \Sigma v_i/\partial x_i$ has filtration d if the directional derivative of v raises the filtration which is not less than d: $L_v E_\lambda \subset E_{\lambda+d}$. We denote the set of all vector fields of filtration d by $\mathcal{G}_d$.

Proposition 1.10. Suppose that $d \geqq 0$. Then 1) the commutator of vector fields on $\mathcal{G}_d$ a Lie algebra structures ; 2) the commutator of elements $\mathcal{G}_{d1}$ and $\mathcal{G}_{d2}$ lies in $\mathcal{G}_{d1+d2}$ so that each $\mathcal{G}_d$ is an ideal in Lie algebra $\mathcal{G}_0$.

Definition 1.11. A piecewise homogeneous functions f of degree d satisfies condition A if for every function g of filtration $d+\delta > d$ in the ideal spanned by the derivatives of f there is a decomposition $g=(\partial f_a/\partial x_i)v_i + g'$,where the vector field v has filtration δ, and the function g has filtration greater than $d+\delta$.

We assume that a type of quasihomogeneity $\mathbf{a}=(a_1,\cdots,a_n)$ is given. Let $E_{>d}$ stand for the ideal of E_d consisting of polynomials of filtration strictly greater than d. We call the factor ring A/E_d the ring of d-jets, and its elements d-jets. A formal diffeomorphism $g:(\mathbf{C}^n,0) \longrightarrow (\mathbf{C},0)$ is given by a collection of n power series without constant terms and gives a ring isomorphism $g^*: A \longrightarrow A$ by the formula $g^*f=f \circ g$,where $\circ$ denotes the substitution of a series in a series.

Definition 1.12. A diffeomorphism g has filtration d if, for all λ, $(g^*-1)E_\lambda \subset E_{\lambda+d}$.

We denote the set of all diffeomorphisms of filtration d by $G_d=G_d(\mathbf{a})$.

Proposition 1.13. Let $d \geq 0$. Then G_d is a group under the operation $\circ$.

Proposition 1.14. For $q > p \geq 0$, G_q is a normal subgroup of G_p.

Let $G_{>d}$ be the subgroup consisting of the diffeomorphisms of filtration greater than d.

Definition 1.15. The group of d-jets of type $\mathbf{a}$ is the factor group of the group of diffeomorphisms by $G_{>d}$:

$$J_d = J_d(\mathbf{a}) = G_0 / G_{>d}.$$

There are natural factorizations $\Pi_{p,q}: J_p \longrightarrow J_q$ $(q > p \geq 0)$.

Proposition 1.16. The group J_p is obtained from J_0 by a chain of extensions with commutative factors.

Definition 1.17. A diffeomorphism $g \in G_0$ is said to be quasihomogeneous of type $\mathbf{a}$ if every space of quasihomogeneous functions of degree d and type $\mathbf{a}$ is mapped into itself by g.

Let V_d be the space of quasihomogeneous functions of degree d and type $\boldsymbol{a}$. Then $g \circ V_d \subset V_d$. The set of all quasihomogeneous diffeomorphisms of fixed type forms a group. We denote it by $H(=H(\boldsymbol{a}))$ and call it the group of quasihomogeneous diffeomorphisms.

Proposition 1.18. The group J_0 is naturally isomorphic to the group H.

Proposition 1.19. Suppose that $d \geq 0$. Then the group J_d of d-jets of diffeomorphisms acts as a group of linear transformations on the space $A / E_{>d}$ of d-jets of functions.

In this case of piecewise filtrations, the group of diffeomorphisms of filtration d, the group of d-jets of diffeomorphisms and the corresponding Lie algebras are defined just as in the case of quasihomogeneous filtrations. There is no analogue in the case of piecewise filtrations for the group of quasihomogeneous diffeomorphisms.

Definition 1.20. Let f_0 be a piecewise homogeneous function. Suppose that f_0 has finite multiplicity μ. Let $e_1, \cdots, e_\mu$ be a basis of Q_{f_0}. Then the basis $e_1, \cdots, e_\mu$ is said to be regular if, for each D, the elements of the basis of degree D are independent modulo the sum of the ideal $I = (\partial f_0 / \partial x)$ and the space $E_{>D}$ of functions of filtrations greater than D.

Proposition 1.21. There always exists a regular basis, in fact, one consisting of monomials.

The number of elements in a regular monomial basis having given piecewise homogeneous degree does not depend on the choice of a basis of the local ring. A monomial in a regular basis is said to be diagonal(superdiagonal) if its degree is equal to (greater than) the degree of the function f_0.

Theorem 1.22. If the principal part f_0 of a function f satisfies the condition A and has finite multiplicity μ, then f can be reduced by a diffeomorphism to the form $f_0 + c_1 e_1 + \cdots + c_s e_s$, where $e_1, \cdots, e_s$ are the superdiagonal monomials in a regular basis.

§ 2. Newton polyhedra and the recognition principle

Let f(x) be an analytic function in an open neighbourhood U of $\mathbf{C}^n$ (f(0)=0) and assume that f(x) has an isolated singular point at 0. We can take a positive number ε so that the sphere $S(r)=\{ x\in \mathbf{C}^n ;\ \| x \|^2=|x_1|^2+\cdots+|x_n|^2=r^2\}$ cuts the hypersurface $V_0=f^{-1}(0)$ transeversely for any $0< r\leq \varepsilon$. Fixing such an ε, we can take $\delta>0$ such that $V_\eta = f^{-1}(\eta)$ is non-singular in $D(\varepsilon)$ and is transverse to $S(\varepsilon)$ for $0<|\eta|\leq\delta$ where $D(\varepsilon) =\{ x\in \mathbf{C}^n ;\ x \leq \varepsilon\}$. Then we have a so-called Milnor fibration $f : X \rightarrow S$ where $S=\{\eta\in \mathbf{C};\ 0<|\eta|\leq\delta\}$ and $X=f^{-1}(S)\ \ D_\varepsilon$. This fibration does not depend on the choice of ε and δ up to a fibre preserving diffeomorphism. The fibre is (n-2)-connected and its (n-1)-th Betti number is the Milnor number μ of f(x) (Milnor[6]).

Definition 2.1. (see Lê and Ramanujam [5]) Let $\mathbf{N}\subset \mathbf{R}_+ \subset \mathbf{R}$ be the sets of all nonnegative integers, all nonnegative real numbers, and all real numbers respectively. Let $\mathbf{K}\subset \mathbf{N}^\kappa$ be a subset. Newton polyhedron of a set $\mathbf{K}$ is defined by the convex hull in R_+^K of the set $\bigcup_{n \in K} (n + R_+^K)$.

Newton boundary of a set K is defined by the union of all compact faces of Newton polyhedron of K. Newton polyhedron is denoted by $\Gamma_+(K)$ and Newton boundary by $\Gamma(K)$.

Let $f = \sum_{n \in N^K} a_n x^n$, $a_n \in C$. Let us write supp $f = \{n \in N^K \mid a_n \neq 0\}$.

Definition 2.2. (see Lê and Ramanujam [5]) Newton polyhedron of a series f (or Newton boundary) is defined by Newton polyhedron (Newton boundary) of the supp f. Newton polyhedron (Newton boundary) of the series f is denoted by $\Gamma_+(f)$ (and $\Gamma(f)$ respectively).

Definition 2.3. (see Oka [7]) The principal part of a series f is defined by the polynomial $f_0 = \sum_{n \in \Gamma(f)} a_n x^n$. For any closed face $\Delta \subset \Gamma(f)$ we shall denote by f_Δ the polynomial $\sum_{n \in \Delta} a_n x^n$. We say that f is non-degenerate on Δ if the equation $\partial f_\Delta / \partial x_1 = \partial f_\Delta / \partial x_2 = \cdots\cdots = \partial f_\Delta / \partial x_n = 0$ has no solution in $(C^*)^n$. When f is non-degenerate on every face Δ of $\Gamma(f)$, we say that f has a non-degenerate principal part.

Example 2.4. We consider the following equation :

$$f(x,y,z)=x^2+2xy+y^2+y^3+z^4.$$

Then f(x,y,z) is degenerate. We submit the following analytic transformation: $x=x'-y$. Then $f(x',y,z)$ is non-degenerate. (see Fig.1.)

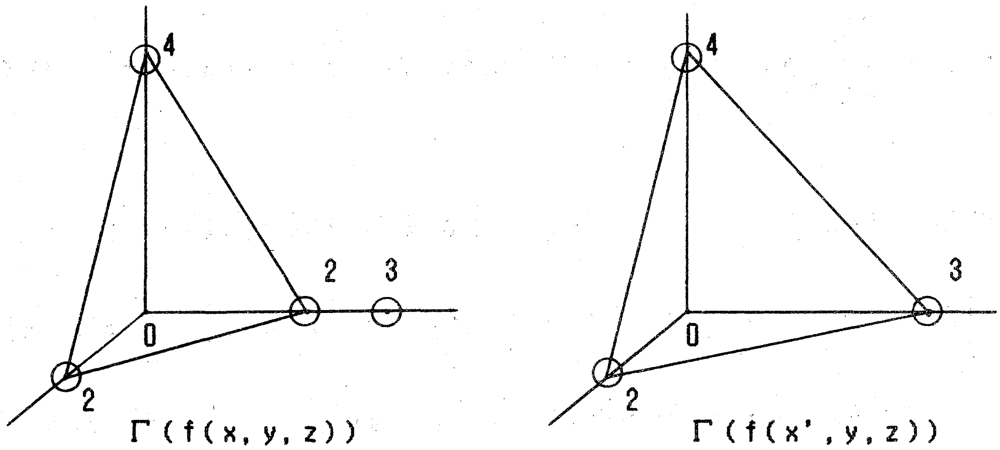

Fig.1.

Theorem 2.5. (Oka [7]) Suppose that f(x) has an isolated singularity at 0 and f has a non-degenerate principal part. Then the Milnor fibration at 0 is determined by the Newton boundary $\Gamma(f)$.

Corollary 2.6. (Kouchnirenco [4]) The topological type of singularity and the multiplicity μ are independent of the particular choice of f for a fixed $\Gamma(f)$.

Moreover, the author [8] proved that there exist a following canonical method as one of the methods to find the topological types of isolated singularities on hypersurfaces.

Lemma 2.7. Let f be a polynomial with an isolated singularity at 0. Let f_{θ} be a degenerate principal part of f. We assume that the piecewise degree (or the generalized degree) is equal to one. Then f may be non-degenerate after the following finite manipulations:

(1) We choose the monomials with the lowest degree (in usual sense) of the principal part f_{θ} of f. And we go to the next manipulation (2).

(2) We transform the monomials with the lowest degree to the normal form by a suitable linear transformation. If the new principal part is non-degenerate after this manipulation, then the manipulations are completed. If the new principal part is also degenerate , then we go to the next manipulation (3).

(3) We consider the monomials which have the degrees greater than the lowest degree (in usual sense). We choose the monomials which are the elements of the new principal part

with the lowest degree in the above monomials. And we try to delete the monomials by suitable analytic transformations. If the monomials are deleted after the suitable analytic transformations and the new principal part is non-degenerate, then the manipulations are completed. Otherwise we go to the nextmanipulation (4).

(4) We consider the monomials which are the elements of the new principal part and have the degrees greater than the lowest degree after the last manipulation. And we submit the similar manipulation. We repeat this manipulation.

Proof. Let $\mathcal{E}$ be a ring of germs of smooth functions at 0 $\in C^n$. We denote the maximal ideal of this ring by $\mathfrak{M}$. Let $f=\Sigma a_n x^n$, $x^n \in \mathfrak{M}^{d(n)}$ and $x^n \notin \mathfrak{M}^{d(n)+1}$. Then $(g^*-1)x^n \in \mathfrak{M}^{d(n)+\lambda}$, $(g^*-1)x^n \notin \mathfrak{M}^{d(n)+\lambda+1}$ for any $g \in G_\lambda$ ($\lambda>0$). From Definition 2.3., when we submit these manipulations we may obtain the non-degenerate Newton polyhedron after the finite manipulations. Then the proof of Lemma 2.7. is completed.

Let f be a non-degenerate polynomial with a singularity at 0.

Let f_0 be a (non-degenerate) pricipal part of f. We assume that the piecewise degree (or the generalized degree) of f_0 is equal to one. Let d be the lowest degree of the monomials of f which do not contain the superdiagonal monomials and have the piecewise degree (or the generalized degrees) greater than one. Then f is transformed into the following form by the suitable analytic transformations (or the suitable elements of the group of diffeomorphisms of filtration d):

$$f_0'+f'+c_1e_1+\cdots+c_re_r$$

where $e_1,\cdots,e_r$ are the superdiagonal monomials of f, $f'\in E_{>d}$ and $f_0'+c_1e_1+\cdots+c_re_r$ is the normal form.

In the same way as in Lemma 2.7.,we try to delete the monomials of $\mathfrak{M}^d$ (this d is the lowest degree of Lemma 2.7.(1)) in turn.

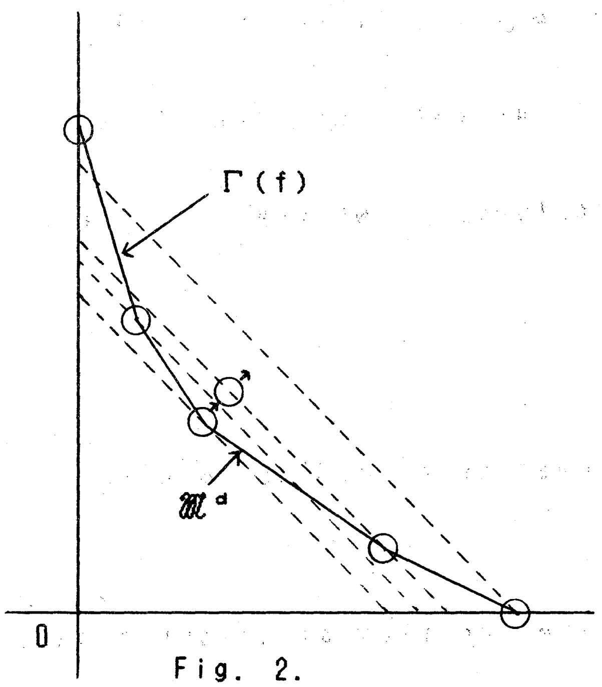

Fig. 2.

Then we can delete all the monomials which are not the superdiagonal monomials and are elements of $d\Gamma$ $(d\geq 1)$ by Theorem 1.22.. When f is a quasihomogeneous function, we can use J_d $(d\geq 0)$. (see Fig. 2.)

Example (Takahashi, Watanabe and Higuchi [9]) 2.8. Let $\mathbf{P}^3$ be a three-dimensional complex projective space with a coordinate $[x,y,z,w]$. Then a following equation has a P_8 $(T_{3,3,3})$ singularity at $[0,0,0,1]$ and A_{11} singularity at $[1,-1,0,0]$.

$$f=6(x^3+y^3+z^3+3xyz)w-3x^4-5x^3z+6x^2y^2-24x^2yz+6x^2z^2-24xy^2z+9xyz^2-8xz^3$$
$$-3y^4-5y^3z+6y^2z^2-8yz^3 .$$

§ 3. Textures

Definition 3.1. By an $n_1\times\cdots\cdots\times n_m$ texture in $\mathbf{C}$ one means a multi-indexed family of elements of $\mathbf{C}$. We abbreviate the notation for this texture by writing it $(a_{i_1\cdots i_m})$, $i_1=1,\cdots,n_1,\cdots,i_m=1,\cdots,n_m$. We call the element $a_{i_1\cdots i_m}$ the $i_1\cdots i_m$-component of the texture. We say that $(n_1,\cdots,n_m)$ is the size of the texture.

A texture $(a_{i_1i_2})$ may be viewed as an $n_1\times n_2$ matrix.

Definition 3.2. We define addition of textures only when they have the same size. If $A=(a_{i_1\cdots\cdots i_m})$ and $B=(b_{i_1\cdots\cdots i_m})$ are textures of the same size, we define A+B to be the texture whose $i_1\cdots\cdot i_m$-component is $a_{i_1\cdots\cdots i_m}+b_{i_1\cdots\cdots i_m}$. We define the multiplication of a texture A by an element $c \in \mathbf{C}$ to be the texture $(ca_{i_1\cdots\cdots i_m})$, whose $i_1\cdots\cdot i_m$-component is $ca_{i_1\cdots\cdots i_m}$. We have a zero texture in which $a_{i_1\cdots\cdots i_m}=0$ for all $i_1\cdots i_m$. We shall write it $\mathbf{O}$.

We see that the textures (of a given size $n_1\times\cdots\cdots\times n_m$) with components in a field $\mathbf{C}$ form a vector space over $\mathbf{C}$ which we may denote by $Tex_{n_1\times\cdots\times n_m}(\mathbf{C})$. We shall define the product of textures.

Definition 3.3. Let $A=(a_{i_1\cdots\cdots i_m})$, $i_1=1,\cdots\cdot,n_1$, $\cdots$, $i_m=1,\cdots,n_m$,be an $n_1\times\cdots\cdots\times n_m$ texture. Let $B=(b_{i_m j_1\cdots j_k})$, $i_m=1,\cdots\cdot,n_m$, $j_1=1,\cdots\cdot,n'_1$, $\cdots\cdot$, $j_k=1,\cdots\cdot,n'_k$, be an $n_m\times n'_1\cdots\cdot\times n'_k$ texture. We define the product AB to be the $n_1\times\cdots\cdots\times n_{m-1}\times n'_1\cdots\cdot\times n'_k$ texture whose $i_1\cdots i_{m-1}j_1\cdots j_k$-component is

$$\sum_{i_m=1}^{n_m} a_{i_1\cdots\cdots i_m}\cdot b_{i_m j_1\cdots j_k} .$$

Multiplication of textures is therefore a generalization of the product of matrices. If A, B, C are textures such that AB is defined and BC is defined, then so is (AB)C and A(BC) and we can easily see (AB)C=A(BC).

Definition 3.4. An $n_1 \times \cdots\cdots \times n_m$ texture is said to be an n^m texture if $n_1=\cdots\cdots=n_m=n$.

We defined one more notion related to textures.

Definition 3.5. Let $A=(a_{i_1 \cdots\cdots i_m})$ be an $n_1 \times \cdots\cdots \times n_j \times \cdots\cdots n_k \times \cdots\cdots \times n_m$ texture, and let $B=(b_{i_1 \cdots i_k \cdots i_j \cdots i_m})$ an $n_1 \times \cdots \times n_k \cdots \times n_j \times \cdots \times n_m$ texture such that

$$b_{i_1 \cdots i_k \cdots i_j \cdots i_m} = a_{i_1 \cdots i_j \cdots i_k \cdots i_m} \quad (j \neq k\ ,\ 1 \leqq j, k\ \leqq m)$$

is called the (j, k)-transpose of A, and is also denoted by ${}^{t(j,k)}A$.

A texture A is said to be symmetric if ${}^{t(j,k)}A=A$ for all j, k such that $j \neq k\ ,\ 1 \leqq j, k\ \leqq m$.

A symmetric texture is necessarily an n^m texture.

§ 4. Multilinear Maps and Textures

Let $V_1, \cdots, V_p$ $(p \geq 1)$, W be vector spaces over $\mathbf{K}$, and let $\phi : V_1 \times \cdots \times V_p \rightarrow W$ be a map. We say that ϕ is multilinear over $V_1 \times \cdots \times V_p$ if for every $(u_1, \cdots, u_p) \in V_1 \times \cdots \times V_p$ the map $\phi(u_1, \cdots, u_{i-1}, \lambda u_i + \mu u_i, u_{i+1}, \cdots, u_p) = \lambda \phi(u_1, \cdots, u_p) + \mu \phi(u_1, \cdots, u_p)$ $(\lambda, \mu \in \mathbf{K}, i=1, \cdots, n)$

Let A be an $n_1 \times \cdots \times n_m$ texture. We can define a map $\phi_A : K^{n_1} \times \cdots \times K^{n_m} \rightarrow K$ where K^{n_j} is n_j-dimensional vector $(1 \leq j \leq m)$ by letting

$\phi_A(X_1, \cdots, X_m) = \cdots X_{m-1} X_m A X_1 X_2 \cdots$ where $X_j = (a_{i_{m+1-j}})$, $a_{i_{m+1-j}} =: z_{m+1-j\ i_{m+1-j}}$.

Thus ϕ_A maps sets of vectors into $\mathbf{K}$. Note that

$$\cdots X_{m-1} X_m A X_1 X_2 \cdots = \sum_{i_m=1}^{n_m} \cdots \sum_{i_1=1}^{n_1} a_{i_1 \cdots i_m} z_{1\ i_1} \cdots z_{m\ i_m}.$$

Theorem 4.1. Given a multilinear map $\phi : \mathbf{K}^{n_1} \times \cdots \times \mathbf{K}^{n_m} \rightarrow K$, there exists a unique texture A such that $\phi = \phi_A$, i.e. such that $\phi(X_1, \cdots, X_m) = \cdots X_{m-1} X_m A X_1 X_2 \cdots$. The set of multilinear maps of $K^{n_1} \cdots K^{n_m}$ into $\mathbf{K}$ is a vector space, denote by $\mathcal{L}(K^{n_1} \times \cdots \times K^{n_m}, K)$, and association $A \mapsto \phi_A$ gives an isomorphism between $Tex_{n_1 \times \cdots \times n_m}(\mathbf{K})$ and $\mathcal{L}(K^{n_1} \times \cdots \times K^{n_m}, K)$.

Proof. We first prove the first statement, concerning the existence of a unique texture A such that $\phi = \phi_A$. Let $E_i^1, \cdots, E_i^{n_i}$ be the standard unit vectors for K^{n_i} $(i=1,\cdots,m)$. We can then write any $X_i \in K^{n_i}$ as $X_i = \sum_{j=1}^{n_i} z_{i\ j} E_i^j$ $(i=1,\cdots,m)$. Then $\phi(X_1,\cdots,X_m) = \phi(z_{1\ 1} E_1^1 + \cdots + z_{1\ n_1} E_1^{n_1}, \cdots\cdots, z_{m\ 1} E_m^1 + \cdots + z_{m\ n_m} E_m^{n_m})$. By its linearity, we find

$$\phi(X_1,\cdots,X_m) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} z_{1\ i_1} \cdots z_{m\ i_m} \phi(E_1^{i_1}, \cdots, E_m^{i_m}).$$

Let $a_{i_1,\cdots,i_m} = \phi(E_1^{i_1}, \cdots, E_m^{i_m})$. Then we see that

$$\phi(X_1,\cdots,X_m) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} a_{i_1 \cdots i_m} z_{1\ i_1} \cdots z_{m\ i_m},$$

which is precisely the expression we obtained for the product $\cdots X_{m-1} X_m A X_1 X_2 \cdots$, where A is the texture $(a_{i_1,\cdots,i_m})$. Suppose that B is a texture such that $\phi = \phi_B$. Then for all vectors $X_1,\cdots,X_m$ we must have $\cdots X_{m-1} X_m A X_1 X_2 \cdots = \cdots X_{m-1} X_m B X_1 X_2 \cdots$. Subtracting, we find $\cdots X_{m-1} X_m (A - B) X_1 X_2 \cdots = 0$ for all $X_1,\cdots,X_m$. Let $C = A - B = (c_{i_1,\cdots,i_m})$. Then $c_{i_1,\cdots,i_m} = 0$ for all $i_1,\cdots,i_m$. The second statement, concerning the isomorphism between the spaces of textures and multilinear maps is clear.

Let M be a linear space over a field K . A mapping $H: M \to K$ is called a homogeneous form with a degree p if the following two conditions hold: (1) $H(\alpha x) = \alpha^p H(x)$ $(\alpha \in K,\ x \in M)$; and (2) the mapping $\phi_p : \underbrace{M \times \cdots\cdots \times M}_{(p\text{-times})} \to \mathbf{K}$ defined by $\phi_p(x_1, \cdots, x_p) = H(x_1 + \cdots + x_p) - H(x_1) \cdots - H(x_p)$ $(x_1, \cdots, x_p \in M)$ is a multilinear form on $\underbrace{M \times \cdots\cdots \times M}_{(p\text{-times})}$. In this case, ϕ_p is called the multilinear form associated with the homogeneous form with the degree p H, and it can be shown to be symmetric. We have $\phi_p(x, \cdots, x) = p\ H(x)$ $(x \in M)$ and $H(x) = (1/p)\ \phi_p(x, \cdots, x)$ if the characteristic of $\mathbf{K} \neq p$. In general, for any multilinear form $f : \underbrace{M \times \cdots\cdots \times M}_{(p\text{-times})} \to \mathbf{K}$, the mapping $H: M \to K$ defined by $H(x) = f\underbrace{(x, \cdots\cdot, x)}_{(p\text{-times})}$ is a homogeneous form with a degree p . And moreover, assigning $x_1 \otimes \cdots\cdot \otimes x_n$ to $(x_1, \cdots, x_n)$, we obtain the canonical multilinear mapping $M_1 \times \cdots\cdot \times M_n \to M_1 \otimes \cdots\cdot \otimes M_n$. Thus, given any linear space L, we have the natural isomorphism $\mathrm{Hom}(M_1 \otimes \cdots\cdot \otimes M_n,\ L) \cong \mathcal{L}(M_1, \cdots\cdot, M_n ;\ L)$.

§ 5. A condition for the normal form to be unique

Let's recall the algorithm of Lemmma 2.7., in it first we fixed our eyes upon the monomials with the lowest degree, and we have taken the normal form of the homogeneous polynomial which was generated from the monomials. However, the normal form is not unique. If the way to take the normal form of Lemma 2.7.(2) is different, naturally the normal form produced by the last manipulation is different. So, we try to impose a condition on the way to take the unique normal form of homogeneous polynomial.

Definition 5.1. Let A be an $n_1 \times \cdots \times n_m$ texture in **C**. By the rank of i-codimension one $n_1 \times \cdot\cdot \times n_{i-1} \times n_{i+1} \times \cdot\cdot \times n_m$ texture ($1 \leq i \leq m$) of A we shall mean the maximum number of linearly independent i-codimension one $n_1 \times \cdot\cdot \times n_{i-1} \times n_{i+1} \times \cdot\cdot \times n_m$ textures ($1 \leq i \leq m$) of A. Thus these ranks are the dimensions of the vector spaces generated respectively by i-codimension one $n_1 \times \cdot\cdot \times n_{i-1} \times n_{i+1} \times \cdot\cdot \times n_m$ textures ($1 \leq i \leq m$) of A.

Definition 5.2. Let P^{n-1} be a (n-1)-dimensional complex projective space with a coordinate $[x_1, \cdots, x_n]$ and let $f = \Sigma c_I x^{K_I}$ be a homogeneous polynomial with a degree m (in usual sense) in P^{n-1}. We define $a_{i1,\cdots,im} = \partial^m f / \partial x_{i1} \cdots \partial x_{im}$ where $1 \leq i_m \leq n$, $i_m \in N_+$. We denote the rank of i-codimension one texture by $T_i(f)$. We define the vector rank of a each monomial $x^{K1} \in f$ to be $(T_1(x^{K1}), \cdots, T_n(x^{K1}))$.

And $T(f) := \sum_{l=1}^{m} \sum_{i=1}^{n} T_i(x^{Kl})$.

We give the following order to the monomials of f.

Definition 5.3. For the exponents $K_i = k_{i1}, \cdots, k_{in}$ and $K_j = k_{j1}, \cdots, k_{jn}$, K_i is greater than K_j if $k_{i1} > k_{j1}$ or $k_{ip} = k_{jp}$ $(1 \leq p < m)$, $k_{ip+1} > k_{jp+1}$.

Condition 5.4. We try to delete a monomials x^{Ki} by suitable linear transformations. Then if we can delete the monomial x^{Ki} (K_i is the minimal number of the exponents) without generating a monomial x^{KJ} ($K_i < K_j$), we delete the monomial x^{Ki}. Otherwise, we don't submit the linear transformations.

From these Definitions and the Condition, we can define a following unique invariant on the normal form of f.

Definition 5.5. We define the vector rank of the homogeneous polynomial f to be

$V_f = (T_1(x^{K_1}), \cdots, T_n(x^{K_1}), T_1(x^{K_2}), \cdots, T_n(x^{K_2}), \cdots, T_n(x^{K_n}))$

where $K_i > K_{i+1}$ $(i=1, \cdots, m-1)$.

Here, we define the normal forms of homogeneous polynomials.

Definition 5.6. Let f be a homogeneous polynomial in $\mathbf{C}$. Then f is said to be the nomal form if, for every g which is linearly equivalent to f, $T(f) \leqq T(g)$ and satisfies the condition 5.3..

Let $\tau_j(f) = \sum_{l=1}^{m} T_l(x^{K_l})$. And assume that $\tau_j(f) \leqq \tau_{j+1}(f)$ $(j=1, \cdots, n-1)$. Then f is linearly equivalent to g if and only if $T(f)=T(g)$, $\tau_i(f) = \tau_i(g)$ $(i=1, \cdots, n)$ and $V_f = V_g$.

Example 5.7. Let f be a non-singular elliptic curve and let g be a nodal curve in $\mathbf{P}^2$. Then we obtain the following table

Nomal form	T(f)	(τ_1, τ_2, τ_3)
$x^2z+y^3+ay^2z+z^3=0$ $(a \in \mathbf{C},\ 4a^3+27 \neq 0)$	8 $(a \neq 0)$ 5 $(a=0)$	(2, 3, 3) (1, 2, 2)
$xyz+y^3+z^3=0$	8	(2, 3, 3)

V_f
(2, 0, 1, 0, 1, 0, 0, 2, 1, 0, 0, 1) (2, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1)
(2, 2, 2, 0, 1, 0, 0, 0, 1)

Table 1.

We consider it natural that normal form should be easy to write and remember; that is, the normal form should have the fewest monomials, and each monomial should be simple. The normal forms defined in this section meet the above condition.

Condition 5.8. When we try to delete the monomials by suitable analytic transformations in Lemma 2.7., we consider the part of the monomials with the lowest degree under Definition 5.6.. If the part has normal form, then we consider the next part of the monomials with the lowest degree except for the above part under Definition 5.6.. We repeat this manipulations.

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