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On Balanced Complementation for Regular t-wise Balanced Designs

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Abstract

Vanstone has shown a procedure, called r-complementation, to construct a regular pairwise balanced design from an existing regular pairwise balanced design. In this paper, we give a generalization of r-complementation, called balanced complementation. Necessary and sufficient conditions for balanced complementation which gives a regular t-wise balanced design from an existing regular t-wise balanced design are shown. Properties on designs for applying balanced complementation are discussed in detail. Results obtained here will be applied to construct regular t-wise balanced designs which are useful in Statistics.

1. Introduction.

A <u>t-wise balanced design</u> (denoted by t-BD) is a pair $(V,\mathcal{B})$, where V is a v-set (called points) and $\mathcal{B}$ is a collection of subsets of V (called blocks), satisfying the following condition:

For any t-subset T of V, the number of blocks containing T is λ_t which is independent of the t-subset T chosen.

If, for any s-subset S ($s \leqq t$), the number of blocks containing S is λ_s which is independent of the s-subset S chosen, then the design is called a regular t-wise balanced design. When t=2, the design is called a regular pairwise balanced design (regular PBD) or an (r,λ)-design ($r=\lambda_1, \lambda=\lambda_2$).

Vanstone[3] has shown a procedure, called r-complementation, to construct a regular PBD from an existing regular PBD. The r-complementation is the procedure defined as follows:

Let $(V,\mathcal{B})$ be a regular PBD. For any point $x \in V$, let $\mathcal{B}_x$ be a collection of blocks containing x. Consider

$$V^*=V-\{x\}$$

and

$$\mathcal{B}^*=\{V-B: B \in \mathcal{B}_x\} \cup (\mathcal{B}-\mathcal{B}_x).$$

Then the pair $(V^*,\mathcal{B}^*)$ is also a regular PBD with new parameters $v^*=v-1$, $r^*=2(r-\lambda)$ and $\lambda^*=r-\lambda$.

The r-complementation is useful to construct new (r,λ)-designs (see, for example, Stinson and van Rees[2]).

In this paper, we give a generalization of r-complementation in Sections 2 and 3, called balanced complementation. Its definition is seen in Section 2 for regular PBD's and in Section 3 for regular t-BD's ($t \geqq 3$), respectively. Necessary and sufficient conditions for balanced complementation which gives a regular t-BD from an existing regular t-BD are shown in Section 2 for t=2 and in Section 3 for $t\geq 3$, respectively. Properties on designs for applying balanced complementation are discussed in detail in Section 3. Results obtained here will be applied to construct regular t-BD's which are useful in Statistics (see, for example, Raktoe, Hedayat and Federer[1]).

2. Balanced complementation for a regular PBD.

We generalize r-complementation by the following theorem:

Theorem 2.1.

Let $(V,\mathcal{B})$ be a regular PBD. Consider

$$V^*=V$$

and

$$\mathcal{B}^*=\{V-B:B\in\mathcal{B}'\}\cup(\mathcal{B}-\mathcal{B}'),$$

where $\mathcal{B}'\subset\mathcal{B}$. Then the pair $(V^*,\mathcal{B}^*)$ is also a regular PBD if and only if each point of V is contained in exactly the same number of blocks in $\mathcal{B}'$.

Proof.

Now assume that each point of V is contained in exactly r' blocks in $\mathcal{B}'$. Let $|\mathcal{B}'|=b'$. It is easy to see that each point of V^* is contained in exactly r+b'-2r' blocks in $\mathcal{B}^*$. For any pair {x,y} of V, let b_1 be the number of blocks in $\mathcal{B}'$ containing x and y and let b_2 be the number of blocks in $\mathcal{B}'$ containing neither x nor y, and let b_3 be the number of blocks in $\mathcal{B}-\mathcal{B}'$ containing x and y. Then we have

$$b_1+b_3=\lambda$$

and

$$b_2-b_1=b'-2r'.$$

From these equations, we can show that each pair of V^* is contained in exactly $\lambda+b'-2r'$ blocks in $\mathcal{B}^*$. Therefore, the above pair $(V^*,\mathcal{B}^*)$ is a regular PBD.

Let $(V^*,\mathcal{B}^*)$ be a regular PBD. For some $x \in V$, let c_x be the number of blocks in $\mathcal{B}'$ containing x and let d_x be the number of blocks in $\mathcal{B}-\mathcal{B}'$ containing x. Since $(V,\mathcal{B})$ is a regular PBD, c_x+d_x is independent of the chosen x. The number of blocks in $\mathcal{B}^*$ containing x is $b'-c_x+d_x$, which is also independent of the chosen x, since $(V^*,\mathcal{B}^*)$ is a regular PBD. Hence, each point of V is contained in exactly the same number of blocks in $\mathcal{B}'$. □

We, in this paper, call this procedure <u>balanced complementation</u>. A <u>spread</u> (or resolution class) of a PBD is a set of blocks, in which each point appears in exactly one block of the set. If the blocks of the design are partitioned into spreads, then the partition is called a <u>resolution</u> and the design is said to be <u>resolvable</u>. There are many examples of resolvable designs. We can apply Theorem 2.1 to the designs with spreads.

<u>Corollary 2.2.</u>

Let $(V,\mathcal{B})$ be a regular PBD with m disjoint spreads. Then there exists a regular PBD $(V^*,\mathcal{B}^*)$ with parameters $v^*=v$, $r^*=r+b'-2m$ and $\lambda^*=\lambda+b'-2m$, where b' is the total number of blocks in the m spreads. (If block size of the design is constant k, then $b'=mv/k$.)

In a regular PBD $(V,\mathcal{B})$, $r-\lambda$ is called <u>order</u> and denoted by n. From the proof of Theorem 2.1, we have the following corollary:

<u>Corollary 2.3.</u>

The order $n=r-\lambda$ is invariant under any balanced complementation.

3. Balanced complementation for a regular t-BD.

Let $(V,\mathcal{B})$ be a pair, where V is a finite set (called points) and $\mathcal{B}$ is a collection of subsets of V (called blocks). For subsets T and S of V such that $S\subset T$, let $\lambda(T,S)$ be the number of blocks in $\mathcal{B}$ which contain S but not contain any point of T-S. The following lemma is used through this section.

<u>Lemma 3.1 (Basic Lemma).</u>

Let T and S be subsets of V such that $S\subset T$. Then, for a point e of V-T,

$$\lambda(T,S)=\lambda(T\cup\{e\},S\cup\{e\})+\lambda(T\cup\{e\},S)$$

holds.

<u>Proof.</u>

Let $\mathcal{B}'$ be a collection of blocks which contain S but not contain any point of T-S. $\mathcal{B}'$ will be partitioned into $\mathcal{B}_1$ and $\mathcal{B}_2$, where each block of $\mathcal{B}_1$ contains e and each of $\mathcal{B}_2$ does not contain e. The number of blocks of $\mathcal{B}'$ is $\lambda(T,S)$, the number of blocks of $\mathcal{B}_1$ is $\lambda(T\cup\{e\},S\cup\{e\})$ and the number of blocks of $\mathcal{B}_2$ is $\lambda(T\cup\{e\},S)$. □

We consider two propositions on designs for applying balanced complementation.

Proposition L(t,s).

Let T and S be a t-subset and an s-subset of V, respectively, such that $S\subset T$. $\lambda(T,S)$ is $\lambda_{t,s}$ which is independent of the t-subset T and the s-subset S chosen.

If a pair $(V,\mathcal{B})$ satisfies the propositions L(i,i) for $i\leqq t$, then it is a regular t-BD.

The following lemma is an immediate consequence of Basic Lemma.

Lemma 3.2.

If two of the propositions L(t,s), L(t+1,s+1) and L(t+1,s) are true, then the rest of the propositions is also true.

Note that, from Lemma 3.2, if the propositions L(i,i) are true for every $i\leqq t$, then the propositions L(i,j) are also true for every $j\leqq i\leqq t$.

Proposition M(t,s).

Let T and S be a t-subset and an s-subset of V, respectively, such that $S\subset T$. $\lambda(T,S)-\lambda(T,T-S)$ is $\delta_{t,s}$ which is independent of the t-subset T and the s-subset S chosen.

If a pair $(V,\mathcal{B})$ is a regular t-BD, then it satisfies the propositions $M(i,j)$ for $j \leq i \leq t$.

On the proposition $M(t,s)$, we will show some results.

<u>Lemma 3.3.</u>

If two of the propositions $M(t,s)$, $M(t+1,s+1)$ and $M(t+1,s)$ are true, then the rest of the propositions is also true.

<u>Proof.</u>

This is clear from Basic Lemma. □

Note that $\delta_{t,s} = \delta_{t+1,s+1} + \delta_{t+1,s}$, when two of the propositions $M(t,s)$, $M(t+1,s+1)$ and $M(t+1,s)$ are true.

<u>Lemma 3.4.</u>

If the proposition $M(t,s)$ is true, then the proposition $M(t,t-s)$ is also true.

<u>Proof.</u>

This is also clear from the definition of the proposition $M(t,s)$. □

Note that $\delta_{t,s} + \delta_{t,t-s} = 0$, when the proposition $M(t,s)$ is true.

<u>Lemma 3.5.</u>

If the propositions $M(i,i)$ are true for every $i \leq t$, then

$$\delta_{2d,d}=0,$$

for $d=0,1,\ldots,[t/2]$, where $[a]$ denotes the largest integer $\leq a$.

Proof.

Since the propositions $M(i,i)$ are true for every $i \leqq t$, the propositions $M(i,j)$ are also true for every $j \leqq i \leqq t$, from Lemma 3.3. Then, from the note of Lemma 3.4, we have $\delta_{2d,d}=0$ for $d \leqq [t/2]$. □

Theorem 3.6.

If the propositions $M(t-1,j)$ are true for every $j \leqq t-1$ and t is even, then the propositions $M(t,s)$ are also true for every $s \leqq t$.

Proof.

Let $S_0, S_1, \ldots, S_t$ be subsets of V such that $S_0(=\phi) \subset S_1 \subset \ldots \subset S_t$ with $|S_j|=j$, $j=0,1,\ldots,t$, respectively. Define variables d_j as

$$d_j = \lambda(S_t, S_j) - \lambda(S_t, S_t - S_j).$$

Since the propositions $M(t-1,j)$ are true for every $j \leqq t-1$, we have, from Basic Lemma,

$$d_j + d_{j+1} = \delta_{t-1,j},$$

for $j=0,1,\ldots,t-1$. Since t is even, from these equations, we have

$$\sum_{j=0}^{t-1} (-1)^j \delta_{t-1,j} = d_0 - d_t$$

$$=2\{\lambda(S_t,\phi) - \lambda(S_t,S_t)\}.$$

This implies that the proposition $M(t,0)$ is true and $\delta_{t,0} = \{\sum_{j=0}^{t-1} (-1)^j \delta_{t-1,j}\}/2$. Thus, from Lemma 3.3, the propositions $M(t,s)$ are true for every $s \leqq t$. □

When block size is constant, it is well known that, if the proposition L(t,t) is true, then the propositions L(i,j) are also true for every $j \leqq i \leqq t$. But it is not generally true for the proposition M(i,j). We can say that in the following special case.

Lemma 3.7.

If the proposition M(t,s) is true and block size is k=v/2 ($\geqq s$), then the proposition M(t-1,s-1) is also true.

Proof.

Let T and S be a (t-1)-subset and an (s-1)-subset of V, respectively, such that $S \subset T$. Since M(t,s) is true, we have

$$\lambda(T\cup\{e\},S\cup\{e\})-\lambda(T\cup\{e\},T-S)=\delta_{t,s},$$

for any point e of V-T. Let $\mathbf{B}_e$ and $\mathbf{C}_e$ be a collection of blocks counted in the first term and in the second term of the above equation, respectively. Since block size is constant k, we have $|B-T|=k-(s-1)$ for a block B which contains S but not contain any point of T-S. Such a block appears in exactly k-(s-1) collections of $\mathbf{B}_{e_1},\mathbf{B}_{e_2},\ldots,\mathbf{B}_{e_{v-(t-1)}}$, where $V-T=\{e_1,e_2,\ldots,e_{v-(t-1)}\}$. Similarly, if a block B appears in one of the collections $\mathbf{C}_{e_1},\mathbf{C}_{e_2},\ldots,\mathbf{C}_{e_{v-(t-1)}}$, then B is contained in exactly v-k-(s-1) collections of $\mathbf{C}_{e_1},\mathbf{C}_{e_2},\ldots,\mathbf{C}_{e_{v-(t-1)}}$. Then we have

$$\{k-(s-1)\}\lambda(T,S)-\{v-k-(s-1)\}\lambda(T,T-S)=\{v-(t-1)\}\delta_{t,s}.$$

Substituting the equation into $\lambda(T,S)-\lambda(T,T-S)$, we have

$$\lambda(T,S)-\lambda(T,T-S)=\{(v-t+1)\delta_{t,s}+(v-2k)\lambda(T,T-S)\}/(k-s+1).$$

So, if $v=2k$, then $\lambda(T,S)-\lambda(T,T-S)$ is independent of the $(t-1)$-subset T and the $(s-1)$-subset S chosen. This implies that the proposition $M(t-1,s-1)$ is true. □

From Lemmas 3.3, 3.4 and 3.7, we have the following theorem:

Theorem 3.8.

If the proposition $M(t,s)$ is true and block size is $k=v/2$ $(\geq s)$, then the propositions $M(i,j)$ are also true for every $j\leq i\leq t$.

Now we consider balanced complementation for a regular t-BD.

Theorem 3.9.

Let $(V,\mathcal{B})$ be a regular t-BD. Consider

$$V^{*}=V$$

and

$$\mathcal{B}^{*}=\{V-B:B\in\mathcal{B}'\}\cup(\mathcal{B}-\mathcal{B}'),$$

where $\mathcal{B}'\subset\mathcal{B}$. Then the pair $(V^{*},\mathcal{B}^{*})$ is also a regular t-BD if and only if the pair $(V,\mathcal{B}')$ satisfies the propositions $M(t,s)$ for $s\leq t$.

Proof.

Let $\mathcal{B}_1=\{V-B:B\in\mathcal{B}'\}$ and $\mathcal{B}_2=\mathcal{B}-\mathcal{B}'$. For subsets T and S of V such that $S\subset T$, let $\lambda^{(i)}(T,S)$ be the number of blocks in $\mathcal{B}_i$ which contain S but not contain any point of $T-S$. Since $(V,\mathcal{B})$ is a regular t-BD, it satisfies the propositions $L(t,s)$, that is,

$$\lambda^{(1)}(T,T-S)+\lambda^{(2)}(T,S)=\lambda_{t,s},$$

for $s\leq t$, where $t=|T|$ and $s=|S|$.

If $(V^*,\mathcal{B}^*)$ is a regular t-BD, then it satisfies the propositions L(t,s), that is,

$$\lambda^{(1)}(T,S)+\lambda^{(2)}(T,S)=\lambda^*_{t,s}, \text{ say,}$$

for $s\leqq t$. Therefore, we have

$$\lambda^{(1)}(T,T-S)-\lambda^{(1)}(T,S)=\lambda_{t,s}-\lambda^*_{t,s},$$

for $s\leqq t$. This implies that the pair $(V,\mathcal{B}')$ satisfies the propositions M(t,s) for $s\leqq t$.

If $(V,\mathcal{B}')$ satisfies the propositions M(t,s) for $s\leqq t$, then we have

$$\lambda^{(1)}(T,T-S)-\lambda^{(1)}(T,S)=\delta^{(1)}_{t,s}, \text{ say,}$$

for $s\leqq t$. Therefore, we have

$$\lambda^{(1)}(T,S)+\lambda^{(2)}(T,S)=\lambda_{t,s}-\delta^{(1)}_{t,s},$$

for $s\leqq t$. This implies that the pair $(V^*,\mathcal{B}^*)$ satisfies the propositions L(t,s) for $s\leqq t$ and it is a regular t-BD. □

It is easily seen, from the above proof, that $\lambda^*_{i,j} = \lambda_{i,j} - \delta^{(1)}_{i,j}$ for $j\leqq i\leqq t$, when $(V^*,\mathcal{B}^*)$ is a regular t-BD. Especially, from Lemma 3.5, we have $\lambda^*_{2d,d}=\lambda_{2d,d}$ for $d\leqq[t/2]$.

From Theorems 3.6 and 3.9, we have the following theorem:

<u>Theorem 3.10.</u>

If $(V,\mathcal{B})$ is a regular t-BD with a sub regular (t-1)-BD $(V,\mathcal{B}')$, $\mathcal{B}'\subset\mathcal{B}$, and t is even, then $(V^*,\mathcal{B}^*)$ is also a regular t-BD, where $(V^*,\mathcal{B}^*)$ is defined in Theorem 3.9.

References

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