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On Bouquets of Matroids and Orientations

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Abstract. The notion of squashed geometries introduced by Deza and Frankl is a common generalization of matroids and permutation geometries. We study different axiomatizations for squashed geometries. Some new classes of squashed geometries, including bouquets of graphic matroids, are given. We introduce a notion of orientability of squashed geometries, which arises naturally in our examples. Finally, some future research problems are discussed.

1. Introduction

Let X be a finite set. A family $\mathcal{D}$ of subsets of X is called a clutter if it is not nested, i.e.,

(M1) $D, D' \in \mathcal{D}$ and $D \neq D'$ ==> $D \not\subseteq D'$.

Each member of a clutter is called a circuit.

A matroid is a clutter $\mathcal{D}$ satisfying the elimination property:

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(M2) $D, D' \in \mathcal{D}$, $D \neq D'$ and $x \in D \cap D'$

$\Longrightarrow \exists\ D'' \in \mathcal{D}$ such that $D'' \subseteq D \cup D' \setminus x$.

A typical example of matroid comes from a graph where the clutter $\mathcal{D}$ corresponds to the set of all simple cycles. A more general example arises from a vector subspace of a vector space F^X over some field F. It is widely known that matroids capture much of fundamental properties of graphs and vector spaces, and it is recognized that matroid theory provides a fundamental framework for many branches of combinatorics and optimization. Furthermore, more recently, the study of "oriented" matroids (BL, C, FL, F, Ma, Mu), which abstract the sets of directed cycles in a digraph and vector subspaces over an ordered field, has enhanced the importance of matroid theory even further.

While matroids offer an ideal setting for many combinatorial problems, sometimes the second condition (M2) is too strong. For example, one might want to study certain collection of matroids or even any clutters, which one often comes across in combinatorial mathematics. The notion of <u>squashed geometries</u> or <u>bouquets of matroids</u>, which has been introduced by Deza and Frankl (DF1) as a common generalization of matroids and permutation geometries (CD), appears to be appropriate for such purposes. (Actually, (DF1, DF2) consider more specified notion of $\mathcal{F}$-squashed geometry.) A successful application of squashed geometry has been reported by Conforti and Laurent (CL) on the evaluation of the greedy algorithm applied to weighted independent systems.

The present paper is a first attempt to study orientations of squashed geometries. We start with the known circuit axioms of squashed geometry given in [L]. Then we give different axiomatizations of squashed geometries. Some new examples of squashed geometries, including bouquets of graphic matroids, will be introduced. The orientability of a squashed geometry comes rather automatically from the new axiomatizations, and we explain how this orientation arises in our examples. Finally, some open problems for future research will be discussed.

The reader is assumed to be familiar with the matroid theory. An appropriate reference is [W]. For the orientation of matroids, see [BL, FL, F, Ma].

2. Circuit Axioms

Let X be a finite set. Suppose that $\mathcal{D}$ is a clutter of subsets of X. In general $\mathcal{D}$ may not satisfy the elimination property (M2) and thus not a matroid. But the recent work on squashed geometries by Deza and Frankl [DF1], Laurent [L] has shown that there is a very natural way to partition a clutter into a "matroidal" part and a "nonmatroidal" part.

A <u>squashed geometry</u> or <u>bouquet of matroids</u> is a clutter $\mathcal{D}$ together with its partition $\mathcal{D} = \mathcal{S} \cup \mathcal{C}$

satisfying the following axioms:

(S1) $S \in \mathcal{S}$, $C \in \mathcal{C}$, and $x \in S \cap C$

$\Longrightarrow \exists\, C' \in \mathcal{C}$ such that $C' \subseteq S \cup C \setminus x$;

(S2) $S, S' \in \mathcal{S}$, $S \neq S'$ and $x \in S \cap S'$

$\Longrightarrow \exists\, D \in \mathcal{D}$ such that $D \subseteq S \cup S' \setminus x$.

We often denote a squashed geometry simply by the pair $(\mathcal{S}, \mathcal{C})$. A member of $\mathcal{S}$ is called a <u>stigme</u>, and a member of $\mathcal{C}$ is called a <u>critical set</u>. A matroid can be considered as a squashed geometry $(\mathcal{S}=\mathcal{D}, \emptyset)$. Let us associate with $\mathcal{C}$ the clutter

$$f(\mathcal{C}) = \{ R \subseteq X : R \text{ is maximal with respect to } C \not\subseteq R \text{ for all } C \in \mathcal{C} \}.$$

Then we immediately obtain

<u>Proposition 2.1</u>. Let $(\mathcal{S}, \mathcal{C})$ be a squashed geometry. Then for any $R \in f(\mathcal{C})$, the restriction of $\mathcal{S}$ to the set R

$$\mathcal{S} \mid R \equiv \{ S \in \mathcal{S} : S \subseteq R \}$$

is the set of circuits of a matroid.

In fact, we can easily prove (using the axiom (S1)) that the axiom (S2) can be replaced by slightly weaker conditions

(R2) for any $R \in f(\mathcal{C})$, $\mathcal{S} \mid R$ is a matroid, or more explicitly,

$$S, S' \in \mathcal{S}, \quad S \neq S', \; x \in S \cap S' \text{ and}$$
$$S \cup S \subseteq R \in f(\mathcal{C})$$
$$\Longrightarrow \exists\, S'' \in \mathcal{S} \text{ such that } S'' \subseteq S \cup S' \setminus x$$

(This fact was observed in [L]).

Let $f(\mathcal{C}) = \{R_1, \ldots, R_t\}$. Each member R_i of $f(\mathcal{C})$ is called a <u>roof</u> and each matroid $\mathcal{S}_i \equiv \mathcal{S} \mid R_i$ is called a <u>flower</u> of the squashed geometry. Since each S is contained in at least one R_i, the union of all flowers is $\mathcal{S}$.

<u>Proposition 2.2</u>. Let $(\mathcal{S}, \mathcal{C})$ be a squashed geometry with the roofs $R_1, \ldots, R_t$. Then for any i, j, the set $R_{ij} \equiv R_i \cap R_j$ is a closed set in the matroid $\mathcal{S}_i$ (i.e. a flat of $\mathcal{S}_i$).

<u>Proof</u>. We may assume $i \neq j$. Suppose R_{ij} is not closed in $\mathcal{S}_i$. This implies that there exists $x \in R_j \setminus R_i$ and $S \in \mathcal{S}_i$ such that $S \setminus x \subseteq R_{ij}$. Since R_i is maximal with respect to the property $C \not\subseteq R_i$ for all $C \in \mathcal{C}$, there exists some critical set $C \in \mathcal{C}$ with $x \in C \subseteq R_i \cup x$. Using the elimination property (S1) for S, C and x, we obtain $C' \in \mathcal{C}$ with $C' \subseteq S \cup C \setminus x \subseteq R_i$. This contradicts $R_i \in f(\mathcal{C})$. □

There is a way to define squashed geometry without using critical sets. For this purpose we need some definition. For any family $\mathcal{R}$ of subsets of X, we associate the clutter

$$g(\mathcal{R}) = \{ C \subseteq X : C \text{ is minimal with respect to } C \not\subseteq R \text{ for all } R \in \mathcal{R} \}.$$

The next proposition says that this function g is the inverse of the function f defined before.

<u>Proposition 2.3</u>. For any clutter $\mathcal{C}$ of subsets of X, $\mathcal{C} = g(f(\mathcal{C})) = f(g(\mathcal{C}))$.

<u>Proof</u>. The proof is left for the reader. □

The above proposition shows that the set of critical sets can be replaced by the set of roofs for defining a squashed geometry. Here is one definition of such kind.

<u>Theorem 2.4</u>. Let $\mathcal{S}$ and $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ be clutters of subsets of a finite set X, and let $\mathcal{C} = g(\mathcal{R})$. Then the pair $(\mathcal{S}, \mathcal{C})$ is a squashed geometry iff the following conditions (R1), (R2) and (R3) hold:

(R1) $S \in \mathcal{S} \implies \exists i$ such that $S \subseteq R_i$;

(R2) $\mathcal{S} \mid R_i$ is a matroid for each i;

(R3) $R_i \cap R_j$ is closed in $\mathcal{S} \mid R_i$ for any i, j.

<u>Proof</u>. The necessity is clear from Proposition 2.1 and 2.2. We will prove the sufficiency. Suppose the conditions (R1), (R2) and (R3) are satisfied.

Claim 1. $\mathcal{D} \equiv \mathcal{S} \cup \mathcal{C}$ is a clutter.

Suppose $\mathcal{D}$ is not a clutter. Since both $\mathcal{S}$ and $\mathcal{C}$ are clutters and $\mathcal{S} \cap \mathcal{C} = \emptyset$ by (R1), we have either $S \subsetneq C$ or $C \subsetneq S$ for some $S \in \mathcal{S}$ and some $C \in \mathcal{C}$. However, $C \subsetneq S$ cannot hold because of (R1) and the definition of $\mathcal{C}$. So we must have $S \subsetneq C$. Without loss of generality, we can assume $S \subseteq R_1$. Take any $x \in S$. Since $x \in C \in \mathcal{C}$, there is some set in $\mathcal{R}$, say R_2, such that $x \notin R_2$ and $S \setminus x \subseteq C \setminus x \subseteq R_2$. This implies that $S \setminus x \subseteq R_1 \cap R_2$, contradicting $R_1 \cap R_2$ being closed in the matroid $\mathcal{S} \mid R_1$.

Remark that the axiom (S2) is clearly satisfied by $\mathcal{D}$. To complete the proof we must show

Claim 2. The axiom (S1) holds for $\mathcal{D}$.

Suppose the contrary: there exist some $S \in \mathcal{S}$, $C \in \mathcal{C}$ and $x \in S \cap C$ and some R_i such that $S \cup C \setminus x \subseteq R_i$. By (R1), there is $R_j \neq R_i$ with $S \subseteq R_j$. Since $S \cup C \setminus x \subseteq R_i \cap R_j$ and $S \in \mathcal{S} \mid R_j$, $R_i \cap R_j$ is not a closed set in $\mathcal{S} \mid R_j$. This contradicts the assumption (R3). □

Using Theorem 2.4, we can define a squashed geometry as a special collection of matroids.

Corollary 2.5. Let $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ be a clutter of subsets of a finite set X, and let $\mathcal{S}_i$ be a matroid on R_i for each i. Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \ldots \cup \mathcal{S}_t$ and $\mathcal{C} = g(\mathcal{R})$. Then the pair $(\mathcal{S}, \mathcal{C})$ is a squashed geometry with the flowers $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_t$ iff

(B1) $S_i | R_j = S_j | R_i$ for any i, j;

(B2) $R_i \cap R_j$ is closed in S_i for any i, j.

<u>Proof</u>. Suppose the pair $(\mathcal{S}, \mathcal{C})$ is a squashed geometry with the flowers $S_1, S_2, \ldots, S_t$. Then we can easily verify that $S_i = S | R_i$ for each i. This implies that $S_i | R_j = S | (R_i \cap R_j)$ for any i, j, and hence (B1) follows. This together with (R3) immediately yields (B2). Conversely, suppose the conditions (B1) and (B2) are satisfied. Note that (R1) is clearly satisfied. Using (B1), we have $S | R_i = \cup \{ S_j | R_i : j=1,\ldots,t\} = \cup \{ S_i | R_j : j=1,\ldots,t\} = S_i$. Since S_i is a matroid, and by (B2), we know that (R2) and (R3) also hold. Hence, by Theorem 2.4, $(\mathcal{S}, \mathcal{C})$ is a squashed geometry with the flowers $S_1, S_2, \ldots, S_t$. □

At the end of this section, we mention some interesting result given in [CL]. Let $\mathcal{D}$ be a clutter of subsets of X. A subset I of X is called <u>independent</u> if it does not contain any circuit of $\mathcal{D}$. Now consider the set of all possible pairs $(\mathcal{S}, \mathcal{C})$ of partitions $\mathcal{D} = \mathcal{S} \cup \mathcal{C}$ yielding a squashed geometry. It was shown that this set forms a semi-lattice in which $(\mathcal{S}, \mathcal{C}) < (\mathcal{S}', \mathcal{C}')$ if $\mathcal{S} \subseteq \mathcal{S}'$. Trivially, the least element in this semi-lattice is $(\emptyset, \mathcal{D})$. If it is a lattice then the greatest element $(\hat{\mathcal{S}}, \hat{\mathcal{C}})$ is given by

$$\hat{C} = \{ C \in \mathcal{D} : \exists C' \in \mathcal{D} \text{ such that } C \cap C' \setminus x \text{ is independent in } \mathcal{D} \},$$

$$\hat{S} = \mathcal{D} \setminus \hat{C}.$$

For instance, it is a lattice if $\mathcal{D}$ is the set of edges of a graph.

3. Examples of Squashed Geometries

Example 3.1. Squashed Geometry induced from a Matroid

Let $\mathcal{M}$ be a matroid on a finite set X, and let $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ be a clutter of subsets of X such that for any i and j, $R_i \cap R_j$ is closed in $\mathcal{M}$. Then, setting

$$\mathcal{S}_i = \mathcal{M} \mid R_i$$

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \ldots \cup \mathcal{S}_t$$

$$\mathcal{C} = f(\mathcal{R}),$$

the pair $(\mathcal{S}, \mathcal{C})$ is a squashed geometry. We say that the squashed geometry $(\mathcal{S}, \mathcal{C})$ is induced from the matroid $\mathcal{S}$ by the clutter $\mathcal{R}$. A squashed matroid obtained this way is called a matroidal squashed geometry. We can consider the natural special case of the above in which each R_i is chosen to be a flat of $\mathcal{S}$. In this case, $R_i \cap R_j$ is automatically closed in $\mathcal{S}$. Such a squashed geometry is called strongly matroidal.

There are various important classes of matroids, such as graphic, binary, representable (over certain field) matroids. Accordingly matroidal squashed geometries can be classified as follows.

<u>Example 3.1.1</u>. Graphic Squashed Geometry

Consider $\mathcal{M}$ to be the polygon (cycle) matroid of a graph G with the edge set X. Let $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ be a clutter of subsets of edges such that for any i, j there is no cycle S and an edge in S such that $S \setminus x \subseteq R_{ij}$ and $S \not\subseteq R_{ij}$. A <u>graphic</u> squashed geometry is a matroidal squashed geometry obtained this way.

Fig. 3.1 shows an example of graphic squashed geometry on the edge set $X = \{a_i, b_i, c_i, d_i, e_i : i=1,2,3\}$. Here the clutter is given by

R_1 = {all egdes of subscript 1 and $a_2, b_2, c_2\}$

R_2 = {all egdes of subscript 2 and $a_3, b_3, c_3\}$

R_3 = {all egdes of subscript 3 and $a_1, b_1, c_1\}$.

Thus, for instance, $R_{12} = \{a_2, b_2, c_2\}$,

$S_1 = \{ \{a_1,b_1,c_1\}, \{a_2,b_2,c_2\}, \{a_1,d_1,c_2,e_1,c_1\},$
$\{a_1,d_1,a_2,b_2,e_1,c_1\}, \{b_1,d_1,c_2,e_1\}, \{b_1,d_1,a_2,b_2,e_1\}$

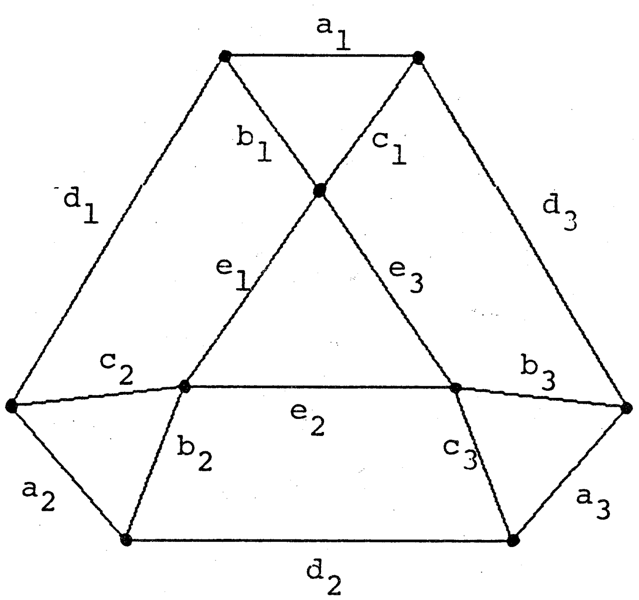

Fig. 3.1

<u>Example 3.1.2</u>. Representable Squashed Geometry

Let F be a field, let m be a positive integer, and let X be a finite subset of F^m. Let $\mathcal{M}$ be the matroid on X determined by linear (affine) dependency over F, i.e., $S \in \mathcal{S}$ iff S is a minimal linearly (affinely) dependent subset of X. A squashed geometry arising this way is called <u>representable over F</u>.

Fig. 3.2 describes an example of a representable squashed geometry on $\{p_1, p_2, \ldots, p_7\}$, a set of 7 points in R^3 determined by affine dependency. Let us denote p_i by i for simplicity. In this example, there are two roofs

$$R_1 = \{1,2,3,4,5\} \quad \text{and} \quad R_2 = \{1,2,3,6,7\},$$

and

$$\mathcal{S}_1 = \{\{1,2,3\},\ \{3,4,5\},\ \{1,2,4,5\}\}$$

$$\mathcal{S}_2 = \{\{1,2,3\},\ \{2,6,7\},\ \{1,3,6,7\}\}.$$

Note that $R_1 \cap R_2 = \{1,2,3\}$ is a common flat of $\mathcal{S}_1$ and $\mathcal{S}_2$.

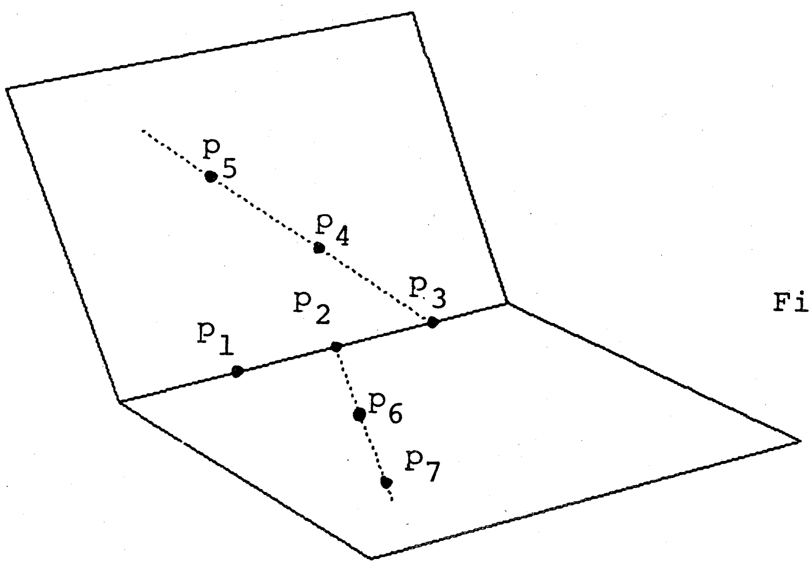

Fig. 3.2

<u>Example 3.2.</u> Bouquets of Particular Matroids

Let $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ be a clutter of subsets of a finite set X, and let $\mathcal{S}_i$ be a matroid on R_i for each i such that $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \ldots \cup \mathcal{S}_t$ is a squashed union of matroids (recall Corollary 2.5). One fundamental question is the following:

<u>Question 3.2.1</u>. Can a bouquet of matroids in one class (e.g., graphic) be induced from a matroid in the same class?

In some cases one can easily see that the answer is no.

<u>Proposition 3.2.2</u>. There is a bouquet of graphic matroids that cannot be induced from any graphic matroid.

<u>Proof</u>. Consider the bouquet of the four graphic matroids $\mathcal{S}_1 = \{e_1, e_2, e_3\}$, $\mathcal{S}_2 = \{e_3, e_4, e_5\}$, $\mathcal{S}_3 = \{e_1, e_5, e_6\}$ and $\mathcal{S}_4 = \{e_3, e_6, e_7\}$, each corresponding to the triangle graph. Then it is routine to check that there is no graph with 7 egdes $\{e_1, e_2, \ldots, e_7\}$ inducing this squashed geometry.

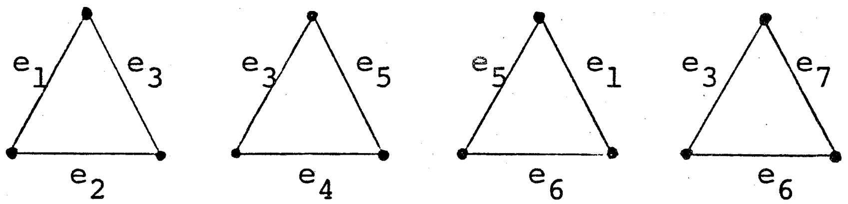

Fig. 3.3

<u>Proposition 3.2.3</u>. There is a bouquet of matroids representable over some field that cannot be induced by any

matroid representable over some field.

Proof. An example can be constructed using exactly the same way as Lazarson's construction of infinite class of non-representable matroids [Laz]. Namely, take a bouquet of two matroids, one being representable only over fields of characteristic p and the other being representable only over fields of characteristic $p' \neq p$. Such a squashed geometry cannot be induced by any representable matroid.

4. Orientations of Squashed Geometries

Let X be a finite set. A signed set (on X) is a pair $S=(S^+, S^-)$ of disjoint subsets S^+ and S^- of X. The negative $-S$ of S is the signed set (S^-, S^+). The support $\underline{S}$ of S is $S^+ \cup S^-$. For a set $\mathcal{O}$ of signed sets, $\underline{\mathcal{O}}$ denotes the underlying set $\{\underline{S} : S \in \mathcal{O}\}$.

For a family $\mathcal{S}$ of subsets of X, a family $\mathcal{O}$ of signed sets is called an orientation of $\mathcal{S}$ if the following conditions are satisfied:

(O1) $\underline{\mathcal{O}} = \mathcal{S}$;

(O2) $S \in \mathcal{O} \implies -S \in \mathcal{O}$;

(O3) $S, S' \in \mathcal{O}$ and $\underline{S} \subseteq \underline{S'} \implies S = \pm S'$;

where $S = \pm S'$ means either $S = S'$ or $S = -S'$.

An oriented matroid $\mathcal{O}$ is a family of signed sets on

X satisfying the following axioms:

(OM1) $S \in \mathcal{O} \implies -S \in \mathcal{O}$;

(OM2) $S, S' \in \mathcal{O}$ and $\underline{S} \subseteq \underline{S'} \implies S = \pm S'$;

(OM3) $S, S' \in \mathcal{O}$ and $x \in (S^+ \cap S'^-) \cup (S^- \cap S'^+)$

$\implies \exists\, T \in \mathcal{O}$ such that

$$T^+ \subseteq S^+ \cup S'^+ \setminus x \quad \text{and} \quad T^- \subseteq S^- \cup S'^- \setminus x.$$

We call each member of $\mathcal{O}$ a <u>circuit</u>. The set $\underline{\mathcal{O}}$ is a matroid called the <u>underlying</u> matroid of $\mathcal{O}$.

A digraph G yields an oriented matroid in the following way. For any (simple) cycle of G, fix any edge e in the cycle, and let $S^+ (S^-)$ be the set of edges in the cycle having the same (opposite) direction as e. Let $\mathcal{O}$ be the set of all such signed sets (S^+, S^-) and their negatives. The resulting set $\mathcal{O}$ is an oriented matroid.

A squashed geometry $(\mathcal{S}, \mathcal{C})$ is called <u>orientable</u> if there is an orientation $\mathcal{O}$ of the set $\mathcal{S}$ of stigmes such that for any $R \in f(\mathcal{C})$, $\mathcal{O} \mid R \equiv \{ S \in \mathcal{O} : \underline{S} \subseteq R \}$ is an oriented matroid. The squashed geometry $(\mathcal{S}, \mathcal{C})$ together with the orientation $\mathcal{O}$ is called an <u>oriented</u> squashed geometry. We will be simply denoted it by $(\mathcal{O}, \mathcal{C})$. Each oriented matroid $\mathcal{O} \mid R$, $R \in f(C)$ is called a <u>flower</u>.

Using the equivalent definitions of squashed geometries, Theorem 2.4 and Corollary 2.5, we obtain the different axiomatizations of oriented squashed geometries as follows.

Axioms 4.1. Let $\mathcal{O}$ be a family of signed vectors on X and let $\mathcal{R}$ be a clutter of subsets of X. $\mathcal{O}$ and $\mathcal{R}$ is respectively the orientation and the set of roofs of an oriented squashed geometry if

(OR1) $S \in \mathcal{O} \implies \exists i$ such that $\underline{S} \subseteq R_i$;

(OR2) $\mathcal{O} \mid R_i$ is an oriented matroid for each i;

(OR3) $S \in \mathcal{O}$, $x \in \underline{S}$ and $\underline{S}\setminus x \subseteq R_i \cap R_j$

$\implies \underline{S} \subseteq R_i \cap R_j$

(i.e., $R_i \cap R_j$ is closed in $\mathcal{O} \mid R_i$ for any i, j).

Axioms 4.2. Let $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ be a clutter of subsets of X, and let $\mathcal{O}_i$ be an oriented matroid on R_i for each i. Let $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \ldots \cup \mathcal{O}_t$. $\mathcal{O}$ and $\mathcal{R}$ is respectively the orientation and the set of roofs of an oriented squashed geometry if

(OB1) $\mathcal{O}_i \mid R_j = \mathcal{O}_j \mid R_i$ for any i, j;

(OB2) $S \in \mathcal{O}_i$, $x \in \underline{S}$ and $\underline{S}\setminus x \subseteq R_i \cap R_j$

$\implies \underline{S} \subseteq R_i \cap R_j$

(i.e., $R_i \cap R_j$ is closed in $\mathcal{O}_i$ for any i, j).

The equivalence of the axiomatizations 4.1 and 4.2 can be easily proved. From 4.1 to 4.2, we simply set $\mathcal{O}_i = \mathcal{O} \mid R_i$. The reverse direction can be shown by checking $\mathcal{O}_i = \mathcal{O} \mid R_i$.

5. Examples of Orientable Squashed Geometries

We can easily verify the following:

Proposition 5.1. Every graphic squashed geometry is orientable.

Proof. Let G be a graph inducing the squashed geometry. It is clear that any orientation of edges induces an orientation we need. □

Proposition 5.2. Every squashed geometry representable over an ordered field is orientable.

One natural question arises.

Question 5.3. Is a squashed geometry $(S, \mathcal{C})$ orientable if all the flowers $S \mid R_i$, $R \in f(\mathcal{C})$ are graphic (or representable over an ordered field) ?

We conjecture that there is a non-orientable squashed geometry whose flowers are graphic. In such an example, if it exists, there must be enough flowers being "tight" together in order to restrict possible orientations of each other. It seems rather difficult to find an example of non-orientable squashed geometry whose flowers are orientable but non-graphic, because non-graphic oriented matroids tend to have many different orientations (while graphic oriented matroids have unique orientations up to certain trivial transformations, see [BL]). If the answer to Question 5.3 is yes, then one may be able to go even further.

<u>Question 5.4</u>. Is a squashed geometry $(\mathcal{S}, \mathcal{C})$ orientable if all the flowers $\mathcal{S} \mid R_i$, $R_i \in f(\mathcal{C})$ are orientable?

6. Some Operations on Squashed Geometries

Let $(\mathcal{S}, \mathcal{C})$ be a squashed geometry with roofs $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$. Instead of considering the set of circuits of each flower $\mathcal{S}_i = \mathcal{S} \mid R_i$, let us take the set $\mathcal{F}_i$ of all flats of the matroid $\mathcal{S}_i$ for $i=1,\ldots,t$, and let $\mathcal{F}$ be the union of all $\mathcal{F}_i$'s. Clearly, $\mathcal{F}$ is closed under taking intersections of any elements. Therefore $\mathcal{F}$ forms a meet semi-lattice ordered by inclusion, whose maximal intervals $\mathcal{F} \mid R_i$ are (the sets of flats of) matroids. In fact these two properties define a squashed geometry, see [DF1], [L], [S].

Many operations on the set of all squashed geometries were considered in [DF1,DF2, DL]. We present here large class of operations coming from following notion of <u>elementary cut</u> – removing of exactly one roof. This operation of removing some R_i creates a new squashed geometry whose roofs are $\mathcal{R} \setminus R_i$ together with hyperplanes of $\mathcal{S}_i$ not contained in any of other roofs. Let us call <u>cut</u> any sequence of elementary cuts and call <u>$\mathcal{P}$-cut</u> any cut satisfying to given property $\mathcal{P}$. We obtain poset, denoted by $(\mathcal{P})$, of squashed geometries, where $G < G'$ means that G can be obtained from G' by $\mathcal{P}$-cut. Call <u>$\mathcal{P}$-enlargement</u> inverse operation (if possible) to $\mathcal{P}$-cut.

The problem will be to find maximal and minimal elements of the poset $(\mathcal{P})$ as, in local form, the maximal and minimal elements compatible (by $<$) with a given squashed geometry G.

Examples of interesting $\mathcal{P}$-cut are:

a) Uniform cut - removing of all roofs.
It coincides with elementary cut iff the squashed geometry is a matroid. The partial order $<$ become linear, $G = \{\emptyset\}$ is the smallest element so the problem is to find largest uniform enlargements of given G.

b) j-unfolding - removing of roof R_o such that

$R_o \subseteq (R_{i_1} \cup R_{i_2} \cup \ldots \cup R_{i_j})$ for some fixed j.

For any given G, the maximal element G' (compatible by $<$ with G) will have property: no roof belongs to a union of j other roofs.

7. Final Remarks

At the end of this note, we list up some open problems of interest.

a) How to define a dual of a squashed geometry in meaningful way?

b) For roof S_i in a squashed geometry, let T_i be its dual matroid. Since the set T_i of co-stigmes has exactly the same information as S_i, we must be able to define squashed geometries by using co-stigmes and something else (e.g. critical sets, roofs,...). Find simple axioms.

c) Single-element extensions (Lv, Ma), perturbations and surgeries (F, Ma) have been studied for oriented matroids. These operations should be generalized to squashed geometries. What conditions do we need to perform these operations?

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