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Metric Diophantine Approximation on some Fuchsian Groups

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Let Γ be a finitely generated Fuchsian group acting on the upper half complex plane H^2, L the set of limit points of Γ and P the set of parabolic cusps. We assume that $\infty \in P$.

An element $g \in \Gamma$ can be viewed as a 2×2 real matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of determinant 1. We write $a = a(g)$, $b = b(g)$, $c = c(g)$ and $d = d(g)$ for convenience.

In Lehner [4], he proved that there exists a positive number k depending on Γ such that

$$\#\{\ g(\infty)\ :\ |\alpha - g(\infty)| < \frac{k}{c^2(g)}\ ,\ g \in \Gamma\ \} = \infty$$

for any $\alpha \in L \setminus P$. He also proved that if Γ is of the first kind ($L = \mathrm{R}$), then for any sequence $\{\varepsilon_n\}$ of positive numbers and almost all $\alpha \in L \setminus P$, there exists a sequence $\{g_n\} \subset \Gamma$ such that

$$|\alpha - g_n(\infty)| < \frac{\varepsilon_n}{c^2(g_n)}\ .$$

Moreover, Patterson [7] proved a kind of Khintchine theorem when Γ is of the first kind: for example, his result implies that

$$\#\{\ g(\infty)\ :\ |\alpha - g(\infty)| < \frac{1}{c^2(g)\cdot \log |c(g)|},\ g \in \Gamma\ \} = \infty$$

for almost all $\alpha \in \mathrm{R} \setminus P$.

In this note, we shall calculate the asymptotic number of

$$\tilde{g}(\infty) : |\alpha - g(\infty)| < \frac{k}{c^2(g)} , \ g \in \Gamma$$

for some positive real number k and almost all $\alpha \in R \setminus P$. To do this, we consider a relation among the Diophantine inequality, geodesics of H^2 and geodesics of H^2/Γ. We show that the ergodicity of the geodesic flow on H^2/Γ with the hyperbolic measure is closely related to the quantitative theory of the Diophantine approximation on Γ. The relation between the Diophantine inequality and geodesics of H^2 also have been considered by Haas [2] and Haas and Series [3] to determine Lagrange spectrum of the approximations on Γ. They have pointed out that the spectrum is related to the "height" of a Γ-congruent family of geodesics.

1. Main Theorem We assume that Γ is of the first kind and $\infty \in P$. Since $\infty \in P$, there exists

$$U_\lambda = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \in \Gamma, \qquad \lambda \in R_+,$$

such that

$$\{ U^k : k \in Z \} = \Gamma_\infty$$

where Γ_∞ denotes the subgroup of Γ that fixes ∞. We define the fundamental region F of Γ by

$$F = \{ z = x + iy : -\frac{\lambda}{2} < x < \frac{\lambda}{2} , \ y > 0 \}$$

$$\bigcap \Big[\bigcap_{g \in \Gamma \setminus \Gamma_\infty} \{ z : |c(g) \cdot z + d(g)| > 1 \} \Big].$$

It is well-known that the hyperbolic metric $ds = \sqrt{dx^2 + dy^2}/y$ and the hyperbolic measure $d\mu = dx\,dy / y^2$ on H^2 are invariant under Γ-action over H^2.

Theorem. *Let*

$$k_0 = \frac{1}{2} \min_{g \in \Gamma \setminus \Gamma_\infty} |c(g)|,$$

then we have for almost all $\alpha \in R \setminus P$,

$$\lim_{N\to\infty} \frac{\#\{\, g(\infty) \;:\; |\alpha - g(\infty)| < \frac{k}{C^2(g)} \,,\; |c(g)| \le N,\; g \in \Gamma \,\}}{\log N}$$

$$= \frac{2\lambda \cdot k}{\pi \cdot \mu(F)}$$

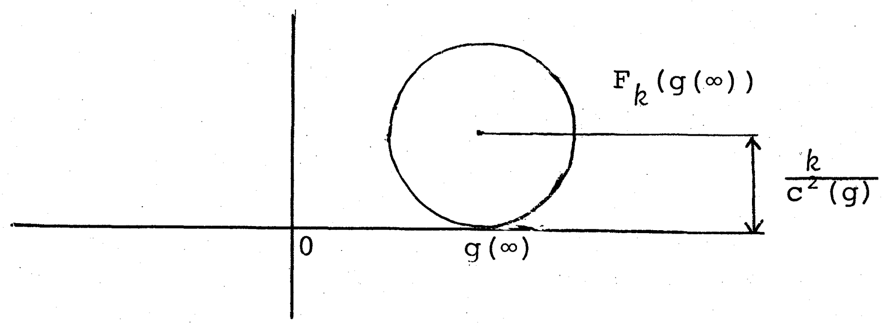
for any k, $0 < k < k_0$.

In the sequel, we outline the proof of this theorem.

2. Some lemmas We denote by $\gamma(\alpha, \beta)$ the geodesic curve which starts from α and ends at β for $(\alpha, \beta) \in (R \cup \{\infty\})^2 \setminus \{\text{diagonal}\}$. We also denote by

$$F_k(g(\infty)),\ k > 0,$$

the circle tangent to the real line at $\frac{a(g)}{c(g)}$ with the radius $\frac{k}{c^2(g)}$ for $g \notin \Gamma_\infty$ and $\{\, x + iy : y = 1/2k\}$ for $g \in \Gamma_\infty$.



It is possible to show the following:

Lemma 1. *If we fix* $k > 0$, *then*

$$g'(F_k(g(\infty))) = F_k(g'g(\infty))$$

for any g' *and* $g \in \Gamma$.

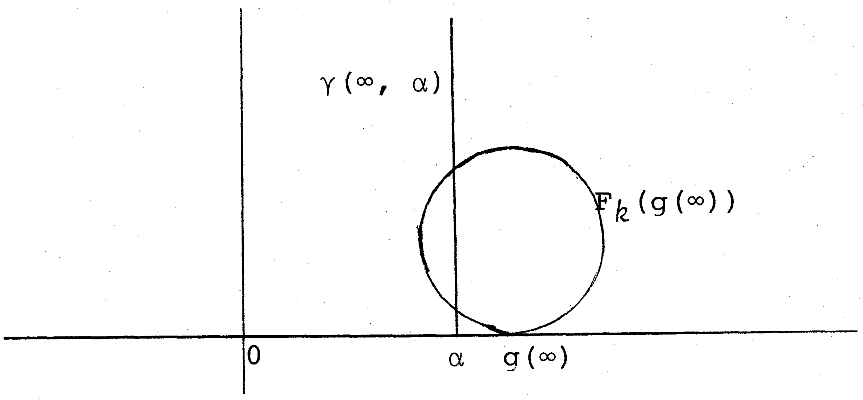
This lemma implies that $\{ F_k(g(\infty)) : g \in \Gamma \}$ is an invariant family of circles under Γ-action. The next lemma is essential.

<u>Lemma 2.</u> *For any* $k > 0$,

$$|\alpha - g(\infty)| < \frac{k}{c^2(g)}$$

holds if and only if

$$\gamma(\infty, \alpha) \cap F_k(g(\infty)) \neq \emptyset.$$



If $k < k_0$, then we see that $\{ F_k(g(\infty)) \}$ is a disjoint family of circles, that is,

$$F_k(g(\infty)) \cap F_k(g'(\infty)) = \emptyset$$

if $g(\infty) \neq g'(\infty)$. Thus we have the following:

<u>Lemma 3.</u> *If* $0 < k < k_0$, *then every point of* $F_k(g(\infty)) \setminus (R \cup \{\infty\})$ *is congruent to some point of* $F_k(\infty) \cap (F \setminus \{\infty\})$, *that is, if* $p \in F_k(g(\infty)) \setminus (R \cup \{\infty\})$, *then there exists* $g' \in \Gamma$ *such that* $g'(p) = x + iy$, $-\frac{\lambda}{2} < x < \frac{\lambda}{2}$ and $y = 1/2k$.

3. <u>Sketch of the proof</u> Let $T(H^2)$ and $T(F)$ be the unit tangent bundles of H^2 and F, respectively. We consider the geodesic flows f_s and $\hat{f}_s$ on $T(H^2)$ and $T(F)$, respectively. For $\omega^* \in T(H^2)$, there is a unique geodesic (α, β) passing tangentially through ω^*. If $\alpha \neq \infty$ and $\beta \neq \infty$, then we denote by s the (directed)

hyperbolic length from the top of the geodesic arc (α, β) to ω, which is the base point of ω^*. If $\alpha=\infty$ (or $\beta=\infty$), then we denote by s the hyperbolic length from the point $\beta + i$ (or $\alpha + i$) to ω, (respectively). Thus we can parametrize $\omega^* \in T(H^2)$ by (α, β, s) $(R \cup \{\infty\})^2 \setminus \{\text{diagonal}\}) \times R$. So if $0 < k < k_0$, we see from lemmas 2 and 3 that

$$\#\{ g(\infty) : |\alpha - g(\infty)| < \frac{k}{c^2(g)},\ |c(g)| \leq N,\ g \in \Gamma \}$$

$$= \#\left\{ s : \begin{array}{l} f_s(\infty, \alpha, -\log(k_0+1)) \text{ crosses a circle } F_k(g(\infty)) \\ \text{from outside at time } s,\ 0 < s \leq \log(k_0+1) - \log k \\ \qquad + 2 \log N \end{array} \right\}$$

$$= \#\left\{ s : \begin{array}{l} \hat{f}_s(\omega^*) \text{ crosses } F_k(\infty) \text{ from below at a time } s, \\ 0 < s \leq \log(k_0+1) - \log k + 2 \log N \end{array} \right\}$$

where $\omega^* \in T(F)$ is the congruent point to $(\infty, \alpha, -\log(k_0+1)) \in T(H^2)$.

Now we apply the individual ergodic theorem for $(T(F), \hat{f}, \hat{\mu})$ to our problem. Here, the hyperbolic measure $\hat{\mu}$ on $T(F)$ induced from μ is defined by

$$\hat{\mu} = \frac{d\alpha\, d\beta\, ds}{(\alpha - \beta)^2}$$

if we parametrize a point in $T(F)$ by (α, β, s).

Proposition 4. *If we fix k, $0 < k < k_0$, then*

$$\lim_{u\to\infty} \frac{\#\{ s : f_s(\omega^*) \text{ crosses } F_k(\infty) \text{ from below, } 0 < s < u\}}{u}$$

$$= \frac{\mu\{ x + iy \in F : y > 1/2k \}}{2\pi \cdot \mu(F)}$$

for almost all $\omega^* \in T(F)$.

Moreover, by using an approximation method on k, we have

Proposition 4'. *For almost all* $\omega^* \in T(F)$,

$$\lim_{u\to\infty} \frac{\#\{ s : \hat{f}_s(\omega^*) \text{ crosses } F_k(\infty) \text{ from below}, 0 < s < u\}}{u}$$

$$= \frac{\mu\{ x + iy \in F : y > 1/2k \}}{2\pi \cdot \mu(F)}$$

for any k, $0 < k < k_0$.

Furthermore, it is possible to show that if $\omega^* = (\alpha, \beta, s) \in T(F)$ has the above property, then for any $\alpha' \in R \cup \{\infty\}$ and $s' \in R$, $\omega^{**} = (\alpha', \beta, s')$ also has the same property. Since the hyperbolic length between $\alpha + (k_0+1)i$ and $\alpha + \frac{1}{N}i$ is equal to $\log N + \log (k_0+1)$, we have

$$\lim_{N\to\infty} \frac{\#\{ g(\infty) : |\alpha - g(\infty)| < \frac{k}{c^2(g)}, \; |c(g)| \leq N, g \in \Gamma \}}{\log N}$$

$$= 2 \cdot \frac{\mu\{ x + iy \in F : y > 1/2k \}}{2\pi \cdot \mu(F)}$$

$$= \frac{2\lambda \cdot k}{\pi \cdot \mu(F)}$$

for any k, $0 < k < k_0$ and almost all $\alpha \in R \setminus P$.

4. Some remarks It is possible to apply the theorem to Hecke group G_n, $n \geq 3$, and its principal congruence subgroups $G_n(m)$.

Let G_n be the group generated by

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & \lambda_n \\ 0 & 1 \end{bmatrix}$$

where $\lambda_n = 2\cdot\cos\frac{\pi}{n}$ for $n \geqq 3$, and $G_n(m)$ be the subgroup of G_n defined by

$$G_n(m) = \{g \in G_n : g \equiv \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \text{ mod.} (m\cdot\lambda_n)\}$$

where $(m\cdot\lambda_n)$ denotes the ideal generated by $m\cdot\lambda_n$ with positive integer m.

A fundamental region F_n of G_n is given by

$$F_n = \{ x + iy : -\cos\frac{\pi}{n} < x \leq \cos\frac{\pi}{n},\ x^2 + y^2 > 1,\ y > 0\}.$$

Thus we see that G_n is of the first kind and

$$P = P_n = G_n(\infty) = \{ g(\infty) : g \in G_n \}$$

if $n \neq \infty$. In this case, we have

$$\lim_{N\to\infty} \frac{\#\{ g(\infty) : g \in G_n,\ |\alpha - g(\infty)| < \frac{k}{c^2(g)},\ |c(g)| \leqq N \}}{\log N}$$

$$= \frac{2\cdot n\cdot\lambda_n\cdot k}{(n-2)\cdot\pi^2}$$

for any k, $0 < k < 1/2$ and almost all $\alpha \in R \setminus P_n$. By using the normality of $G_n(m)$, we can prove that the above inequality holds for all $k > 0$.

It is also possible to prove a theorem of the same type for some Kleinian groups of the first kind acting on

$$H^3 = \{ (x, y, z) : x, y \in R,\ z > 0 \}.$$

For example, we can prove the following: Let d be a square free negative integer and $\mathcal{O}(d)$ the set of integers in $Q(\sqrt{d})$. Then we have for almost all complex number α,

$$\lim_{N\to\infty} \frac{\#\{ \frac{p}{q} : |\alpha - \frac{p}{q}| < \frac{k}{|q|^2},\ p,q \in \mathcal{O}(d),\ |q| \leqq N,\ (p, q) = I\}}{\log N}$$

$$= C_d \cdot k^2$$

for all $k > 0$, where C_d is a constant depending only on d.

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