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ASYMPTOTIC K - TH MEANS AND OSCILLATIONS OF ARITHMETICAL CONVOLUTIONS

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First of all I wish to thank the persons who took part in the organisation of this symposium for the opportunity to speak they gave me.

I want to discuss a certain class of real functions that can be written as convolutions, and among which we find error terms related to arithmetical functions as the Euler ϕ - function or the sum-of-divisors function.

1. Notation and examples.

We denote by α a real bounded sequence that satisfies, for some real constant K,

$$\sum_{n\leq x} \alpha(n) \sim Kx \qquad (x\to\infty) , \tag{1}$$

and by f a real periodic function, of bounded variation, and such that if T is the period,

$$\int_0^T f(t)\,dt = 0 . \tag{2}$$

If now the real function h, defined on $[1,\infty)$, satisfies

$$h(x) = \sum_{n\le x} \frac{\alpha(n)}{n} f(x/n) + o(1) \quad (x\to\infty) , \tag{3}$$

we shall say that $h \in C(\alpha,f)$.

We now give a few examples of functions belonging to some $C(\alpha,f)$.

1) It is well known that if

$$H(x) := \sum_{n\le x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}x , \tag{4}$$

where ϕ denotes the Euler function, then

$$H(x) = -\sum_{n\le x} \frac{\mu(n)}{n} \psi_1(x/n) + o(1) \quad (x\to\infty) , \tag{5}$$

where μ is the Moebius function and ψ_1 the first Bernoulli polynomial modulo 1 : $\psi_1(y) = \{y\} - 1/2$, $\{y\}$ denoting the fractional part of y.

[Note : it seems more natural to study first the error term

$$R(x) := \sum_{n\le x} \phi(n) - \frac{3}{\pi^2}x^2 ; \tag{6}$$

since however we know [PiC 1] that

$$R(x) \sim xH(x) \qquad (x\to\infty), \tag{7}$$

information concerning H can often be translated into an equivalent information concerning R.]

2) We also know that if

$$F(x) := \sum_{n\le x} \frac{\sigma(n)}{n} - \frac{\pi^2}{6}x + \frac{1}{2}\log x + \frac{\gamma}{2} + 1 \quad , \tag{8}$$

where $\sigma(n)$ denotes the sum of the positive divisors of n and γ Euler's constant, then

$$F(x) = -\sum_{n\le x} \frac{1}{n}\,\psi_1(x/n) + o(1) \qquad (x\to\infty). \tag{9}$$

[As in the first example, if we define

$$E(x) := \sum_{n\le x} \sigma(n) - \frac{\pi^2}{12}x^2 \quad , \tag{10}$$

it is known [La] that

$$E(x) \sim x(F(x) + C) \qquad (x\to\infty) \quad , \tag{11}$$

where C is a constant.]

3) The functions

$$G_{a,\ell}(x) := \sum_{n\le\sqrt{x}} n^a \psi_\ell(x/n) \tag{12}$$

and

$$G^*_{a,\ell}(x) := \sum_{n\le x} n^a \psi_\ell(x/n) \quad , \tag{13}$$

where ψ_ℓ denotes the ℓ - th Bernoulli polynomial modulo 1 (for an introduction to the ψ_ℓ see [R]), have close relations with various divisor problems (see e.g. [IK,I] ; the most famous example of such a relation is

$$\sum_{n\le x} d(n) = x\log x + (2\gamma-1)x - 2G_{0,1}(x) + O(1), \tag{14}$$

where $d(n)$ is the number of the positive divisors of n).
$G_{a,\ell}$ and $G^*_{a,\ell}$ belong to some $C(\alpha,\psi_\ell)$ if $a \le -1$.

4) We finally mention the functions

$$P(x) := \sum_{n\le x} \frac{1}{n} \cos(x/n) \tag{15}$$

and

$$Q(x) := \sum_{n\le x} \frac{1}{n} \sin(x/n) \quad , \tag{16}$$

first studied by Hardy and Littlewood [HL].

As we shall see below, much information can be obtained for functions in $C(\alpha,f)$ whose corresponding sums (3) can be truncated. If $g\in C(\alpha,f)$ satisfies

$$g(x) = \sum_{n\le z} \frac{\alpha(n)}{n} f(\frac{x}{n}) + K\int_1^\infty \frac{f(u)}{u}\, du + o(1) \qquad (x\to\infty) \quad , \tag{17}$$

where K is defined by (1), and where $z = z(x)$ is an increasing and unbounded function such that $z = o(x)$ as $x\to\infty$, we shall say that $g \in C_z(\alpha,f)$ and denote the sum on the right side of (17) by $g_z(x)$.

2. Means, distribution functions, changes of sign.

Paolo Codecà proved in 1984 [Co 2]

THEOREM 1. If $g \in C_z(\alpha,f)$ where z is a slowly varying function (i.e. $z = o(x^\varepsilon)$ for all positive ε), then the asymptotic k - th mean of g,

$$M(g^k) := \lim_{x\to\infty} \frac{1}{x}\int_1^x (g(t))^k dt \quad , \qquad (1)$$

exists for each positive integer k.

This is applicable [P3] to H , F , P and Q by refining estimates of Walfisz' [W 2], Flett's [F] and Codecà's [Co 2]. It is also applicable to the $G_{a,\ell}$ and $G^*_{a,\ell}$: Walfisz' argument for $G^*_{-1,1}$ [W 2, Chapter III] can easily be generalised if one uses the Fourier expansion of ψ_ℓ instead of that of ψ_1.

But Theorem 1 is an existence theorem, and it is of interest to evaluate $M(g^k)$. For k larger than 2 however, this problem appears to be difficult : it seems that, for any k - tuple of positive integers $(n_1,\ldots,n_k)$, one requires the value of

$$\lim_{x\to\infty} \frac{1}{x}\int_1^x f(t/n_1)f(t/n_2)\ldots f(t/n_k)dt \; . \qquad (2)$$

As for the examples given in the preceding section we know that [PiC 1, C]

$$M(H) = 0 \qquad \text{and} \qquad M(H^2) = \frac{1}{2\pi^2} . \tag{3}$$

The values of $M(F)$ [PiC 2], of $M(F^2)$ [W 1] and of $M(Q)$ [S] are also known, and in general it shouldn't be very difficult to evaluate $M(g)$ and $M(g^2)$ if g is any of the examples discussed in Section 1.

Higher odd asymptotic means are known in certain cases :

THEOREM 2 [P 3]. If $g \in C_z(\alpha, f)$ for a slowly varying z, and if

$$f(t) = -f(-t) \tag{4}$$

except possibly on a set of measure zero, then

$$M\left((g - K\int_1^{\infty} \frac{f(u)}{u}\, du)^{2k+1} \right) = 0 \tag{5}$$

for all non-negative integers k (K is defined by (1.1)).

As a corollary we have, for any non-negative integer k,

$$M(H^{2k+1}) = 0 . \tag{6}$$

Theorem 2 is also applicable to F, Q, and $G_{a,2\ell+1}$.

But it would be of the greatest interest to be able to evaluate, or even estimate with a good accuracy, the higher even asymptotic means $M(g^{2k})$: it would provide information on

1) the distribution function

$$D_g(u) := \lim_{x\to\infty} \frac{1}{x} \left|\{n\le x \ , \ g(n)\ge u\}\right| \tag{7}$$

if it exists - it does for instance [ES 2] if $g = H$, and the method Erdös and Shapiro use to prove it is probably applicable to $g = F$ or equivalently to $g = G_{-1,1}$; the case $g = G_{-1,2}$, however, already looks difficult ;

2) the function

$$X_g(x) := \begin{cases} \text{the number of changes of sign of} \\ g(t) \ (t \in \mathbb{R}) \text{ in the interval } [1,x). \end{cases} \tag{8}$$

The following result provides an illustration.

THEOREM 3. Let $g \in C_z(\alpha,f)$ for a slowly varying z, and suppose that

$$g(x) = g([x]) - C\{x\} + o(1) \quad (x\to\infty) \tag{9}$$

for some real constant $C \neq 0$. Then

$$X_g(x) \ge \frac{2^{k+1}}{2^{2k}-1}\left(1 - \frac{(2k+1)M(g^{2k})}{C^{2k}}\right)x + o(x) . \tag{10}$$

If, in addition, the distribution function D_g exists and is continuous, then

$$X_g(x) = 2\left|D_g(0) - D_g(C)\right|x + o(x) . \tag{11}$$

COROLLARY [P 1].

$$X_H(x) \sim 2(D_H(0)-D_H(6/\pi^2))x \geq \frac{8}{3}(1-\frac{\pi^2}{24})x + o(x). \qquad (12)$$

Proof : (3), (9) with $k=1$, and (10).

For other applications of Theorem 3 with $k=1$, see [P 1].

REMARK 1. The case $k=1$ of Theorem 3 (and (10)) is proved in [P 1]; the general case can be obtained by following the same argument with the principal term of

$$\int_n^{n+1} (g(t))^{2k} dt = \frac{1}{(2k+1)C}\left((g(n))^{2k+1}-(g(n)-C)^{2k+1}\right) \qquad (13)$$

that can be written as a polynomial of degree k in the argument $(g(n) - C/2)^2$, and whose constant term is then $(2k+1)^{-1}(C/2)^{2k}$.

REMARK 2. If g satisfies (9) we have the trivial upper bound

$$X_g(x) \leq 2x + O(1), \qquad (14)$$

valid for all $x \geq 1$. Is it true that

$$\frac{(2k+1)M(H^{2k})}{(6/\pi^2)^{2k}} \to 0 \qquad (k\to\infty) \ ? \qquad (15)$$

With (10) and (14), it would imply that

$$X_H(x) \sim 2x \qquad (x\to\infty). \qquad (16)$$

3. Oscillations.

It follows from the discussion in Section 2 that the existence of $M(g^k)$ for all k, which looks first like a strong condition of regularity, might in fact imply very wild irregularities, as the existence of many changes in sign. In this section we discuss the amplitude of oscillations for functions belonging to a $C(\alpha,f)$. We first recall

DEFINITIONS. Let h be a real positive function, and g a real function. Then, as $x\to\infty$,

1) $g(x) = \Omega(h(x))$ means that $\limsup_{x\to\infty} |g(x)|/h(x) > 0$;

2) $g(x) = \Omega_+(h(x))$ means that $\limsup_{x\to\infty} g(x)/h(x) > 0$;

3) $g(x) = \Omega_-(h(x))$ means that $\liminf_{x\to\infty} g(x)/h(x) < 0$;

and

4) $g(x) = \Omega_\pm(h(x))$ means that $g(x) = \Omega_+(h(x))$ and $g(x) = \Omega_-(h(x))$.

We know for instance that [PiC 1]

$$H(x) = \Omega(\log\log\log x) , \tag{1}$$

and that [ES 1]

$$H(x) = \Omega_\pm(\log\log\log\log x) . \tag{2}$$

The idea of Erdös and Shapiro to prove (2) consists in evaluating the mean of H over certain arithmetical progressions. They obtain

$$\frac{1}{x} \sum_{n \le x} H(An-B) = -H(B) + O(1) \tag{3}$$

for the special choice $A = \prod_{p \le B} p = x^{1/2}$ (the symbol p is as usual the prime numbers' monopoly). With (3), (2) is an easy consequence of (1).

In [Co 1] Codecà proves an interesting formula for

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} g(An+B) =: M(A,B) , \tag{4}$$

valid for any fixed pair of positive integers A and B, and applicable to $g \in C(\alpha, f)$ if the error term in (1.1) is not too large. By then making arbitrarily large special choices of A he uses this formula to show that the functions H , F and Q are not bounded above nor below. But since, unlike the parameters A and B of (3), those of (4) are not allowed to increase with x, no explicit Ω - result is obtained.

If however we give up the existence of the limit on the left side of (4), A and B can be let loose under certain conditions. Here is one version, usable to seek Ω - results.

THEOREM 4 [P 4]. Let $A = A(x) > 0$ and $B = B(x) \geq 0$ be integers, and $z(x)$ be a real, increasing, unbounded function such that

$$\limsup_{x \to \infty} \frac{z(2x)}{z(x)} < \infty \tag{5}$$

and that

$$u(x) := z(Ax+B) = o(x) \quad (x\to\infty) . \tag{6}$$

Then, if $g \in C_z(\alpha,f)$, we have

$$\frac{1}{x}\sum_{n\leq x} g(An+B) = \sum_{k\leq u}\frac{\alpha(k)}{k}\left(\frac{1}{k^*}\sum_{n\leq k^*} f\left(\frac{n}{k^*}+\frac{B}{k}\right)\right) + O(1) , \tag{7}$$

where k^* denotes $k/(A,k)$, (A,k) being as usual the greatest common divisor of A and k.

In order to use the right hand side of (7) to obtain Ω-results, we must first estimate the inside sum. This is possible if for instance f satisfies a distribution property as

LEMMA.

$$\sum_{n\leq k^*} \begin{matrix}\sin\\ \cos\end{matrix}\left(2\pi\left(\frac{n}{k^*}+\frac{B}{k}\right)\right) = \begin{cases}\begin{matrix}\sin\\ \cos\end{matrix}(2\pi B/k) & \text{if } k^*=1\\ 0 & \text{otherwise} ;\end{cases} \tag{8}$$

$$\frac{1}{k^*}\sum_{n\leq k^*}\psi_\ell\left(\frac{n}{k^*}+\frac{B}{k}\right) = \frac{1}{k^{*\ell}}\,\psi_\ell\left(\frac{B}{(A,k)}\right) \quad \text{(see e.g. [R]).} \tag{9}$$

From (7), (8) and (9) we can obtain

COROLLARY [P 4].

$$H(x) = \Omega_\pm(\log\log\log\log x) ; \tag{2}$$

$$F(x) = \Omega_\pm(\log\log x) ; \tag{10}$$

$$P(x) = \Omega_+(\log\log x) ; \tag{11}$$

$$Q(x) = \Omega_\pm((\log\log x)^{1/2}) ; \tag{12}$$

$$G_{-1,2\ell}(x) = \Omega_*(\log\log x) , \tag{13}$$

where in (13) $_*$ denotes the sign of $B_{2\ell}$, the 2ℓ-th Bernoulli number.

NOTE 1. (11) and (12) are essentially due to Hardy and Littlewood [HL]. (10) and (13) we believe are new.

NOTE 2. A more precise form of Theorem 4 [P 4] leads to

$$G_{a,2\ell}(x) = \Omega_*(1) \quad (a<-1) , \tag{14}$$

where $_*$ is defined as in (13).

4. Changes of sign on integers.

In this section we use the letter c freely to denote in general any positive constant.

In Section 2 we have obtained, as a consequence of Theorem 3, that

$$X_H(x) \sim cx \quad (x\to\infty) . \tag{1}$$

The function $H(t)$ decreases linearly by $6/\pi^2$ on each interval $[n,n+1)$ and jumps (upwards) by $\phi(n+1)/(n+1)$ at $t = n+1$ (n being any positive integer). The problem of estimating the function $N_H(x)$, that counts only the changes in sign of the restricted $H(n)$ $(n \in \mathbb{N})$ in the interval $[1,x)$, appears to be more difficult than that of estimating $X_H(x)$. Erdös conjectures [E] that

$$N_H(x) \sim cx \quad (x\to\infty) . \tag{2}$$

By exploiting the special choice of A and B for which (3.3) is valid and [PiC 1]'s method of proving (3.2) I have shown [P 2] that

$$N_H(x) \geq c \log\log x + O(1) . \qquad (3)$$

As it is pointed out in [P 4], this can be slightly improved to

$$N_H(x) \geq c(\log\log x)^{3/2}(\log\log\log x)^{-2} + O(1) \quad (4)$$

by exploiting the special choices of A and B that one is allowed to make when one applies Theorem 4 to the function H , which we know [W 2, P 3] belongs to $C_z(\mu,\psi_1)$ for $z(x) = \exp(c(\log x)^{2/3}(\log\log x)^{4/3})$.

Unfortunately, if we believe in Conjecture (2), this method is hopeless : we cannot expect to obtain better than

$$N_H(x) \geq c(\log\log x)e(x) + O(1) , \qquad (5)$$

where $e(x)$ is such that

$$\sum_{n\leq x} H(n) = \frac{3}{\pi^2}x + O(x/e(x)) ; \qquad (6)$$

and even under the assumption of the Riemann Hypothesis we only know to date that [Su]

$$(e(x))^{-1} = O(x^{-1/2+\varepsilon}) \qquad (7)$$

for all positive ε .

Similar results [P 2, P 4] and a similar remark apply to the function F .

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