

HOMEOMORPHISMS WITH MARKOV PARTITIONS

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ABSTRACT

Necessary and sufficient conditions for expansive homeomorphisms to have Markov partitions are given.

It is known that the existence of Markov partitions supplies us important informations in dynamical systems (for examples, studies of equilibrium states [5] and zeta functions [24]).

The construction of Markov partitions first was done for Anosov diffeomorphisms by Ja. G. Sinai [35]. After that R. Bowen [5] constructed Markov partitions for nonwandering sets of Axiom A diffeomorphisms. Following [5], [35], we see that the notion so called canonical coordinates plays an important role to construct Markov partitions. In topological setting, K. Hiraide [20] proved that every expansive homeomorphism with POTP has canonical coordinates and further such a homeomorphism has Markov partitions. It is known (N. Aoki [2] and [3]) that every expansive automorphism of a solenoidal group has POTP and such an automorphism has canonical coordinates. Thus the automorphism has Markov partitions ([20]).

However homeomorphisms with Markov partitions do not necessarily have canonical coordinates. In fact, every pseudo-Anosov map has Markov partitions and does not have canonical coordinates (see paragraph 9 and 10 of [17]).

It is natural to ask what kind of expansive homeomorphisms have Markov partitions. The purpose of this paper is to give necessary and sufficient conditions for expansive homeomorphisms to have Markov partitions. More precisely we can describe our result as follows;

THEOREM. Let X be a compact metric space and f be an expansive self-homeomorphism of X . Then the following conditions are equivalent;

(I) there exists $c > 0$ with $2c$ is an expansive constant such that for every $x \in X$ there exists an $\eta = \eta(x) > 0$ such that $\{Y_c(y) \cap B_\eta(x) \mid y \in B_\eta(x)\}$ is finite,

(II) there exists $c > 0$ with $2c$ is an expansive constant such that for every $x \in X$ there exists a $\delta = \delta(x) > 0$ such that $\{Z_c(y) \cap B_\delta(x) \mid y \in B_\delta(x)\}$ is finite,

(III) (X, f) has TPOTP,

(IV) (X, f) has a Markov partition.

In the remainder, we shall give some definitions which are used in our theorem. The proof will appear elsewhere.

Let X be a compact metric space with metric d , and f be a self-homeomorphism of X . For $x \in X$, $B_\varepsilon(x)$ will denote the closed ε -ball in X centered at x . For $x \in X$ and $\varepsilon > 0$ define subsets $W_\varepsilon^s(x)$ and $W_\varepsilon^u(x)$ of $B_\varepsilon(x)$ by $W_\varepsilon^s(x) = \bigcap_{n \geq 0} f^{-n} B_\varepsilon(f^n x)$ and $W_\varepsilon^u(x) = \bigcap_{n \leq 0} f^{-n} B_\varepsilon(f^n x)$. Then we have

- (1) $f W_\varepsilon^s(x) \subset W_\varepsilon^s(fx)$, $f^{-1} W_\varepsilon^u(x) \subset W_\varepsilon^u(f^{-1}x)$,
- (2) $y \in W_\varepsilon^\sigma(x)$ if and only if $x \in W_\varepsilon^\sigma(y)$ ($\sigma = s, u$),
and
- (3) $z \in W_{\varepsilon_1 + \varepsilon_2}^\sigma(x)$ when $y \in W_{\varepsilon_1}^\sigma(x)$ and $z \in W_{\varepsilon_2}^\sigma(y)$
($\sigma = s, u$).

Definition 1. (X, f) is said to be expansive if there exists a constant $c^* > 0$ such that

$$\{x\} = \bigcap_{n \in \mathbb{Z}} f^{-n}(B_{c^*}(f^n x)) \quad (= W_{c^*}^s(x) \cap W_{c^*}^u(x))$$

for all $x \in X$, and such a c^* is said to be an expansive constant for f .

For every $\varepsilon > 0$ define subsets Y_ε and Z_ε of $X \times X$ by

$$Y_\varepsilon = \{(x, y) \in X \times X \mid W_\varepsilon^s(x) \cap W_\varepsilon^u(y) \neq \emptyset\}$$

and

$$Z_\varepsilon = \{(x, y) \in X \times X \mid (x, y) \in Y_\varepsilon \text{ and } (y, x) \in Y_\varepsilon\} .$$

For $x \in X$ and $\varepsilon > 0$, subsets $Y_\varepsilon(x)$ and $Z_\varepsilon(x)$ of X are defined by

$$Y_\varepsilon(x) = \{y \in X \mid (x, y) \in Y_\varepsilon\}$$

and

$$Z_\varepsilon(x) = \{y \in X \mid (x, y) \in Z_\varepsilon\} .$$

Then we have

$$(4) \quad y \in Z_\varepsilon(x) \text{ if and only if } x \in Z_\varepsilon(y)$$

and

$$(5) \quad W_\varepsilon^S(x) \cup W_\varepsilon^U(x) \subset Z_\varepsilon(x) \subset Y_\varepsilon(x) .$$

Definition 2. Let \mathcal{D} be a finite partition of X , i.e., a finite family of subsets in X whose elements are mutually disjoint and $\bigcup_{D \in \mathcal{D}} D = X$. A sequence $\{x_i\}_{i \in \mathbb{Z}}$ of points in X is said to be an α -pseudo orbit with respect to \mathcal{D} if $d(fx_i, x_{i+1}) \leq \alpha$ and $fx_i \overset{\mathcal{D}}{\sim} x_{i+1}$ for all $i \in \mathbb{Z}$ where $x \overset{\mathcal{D}}{\sim} y$ denotes that x and y are in the same element of \mathcal{D} . A sequence $\{x_i\}_{i \in \mathbb{Z}}$ of points in X is said to be β -traced if $\bigcap_{i \in \mathbb{Z}} f^{-i}(B_\beta(x_i)) \neq \emptyset$. (X, f) is said to have Takahashi pseudo orbit tracing property (abbrev. TPOTP) if there exists a finite partition \mathcal{D} such that for every $\beta > 0$, there is $\alpha > 0$ such that every α -pseudo orbit with respect to \mathcal{D} is β -traced.

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Especially (X, f) is said to have the pseudo orbit tracing property (abbrev. POTP) if \mathcal{D} can be chosen so that $\mathcal{D} = \{X\}$. The notion of TPOTP is firstly used in M. Yuri 39] with a suggestion of Y. Takahashi. It seems likely that for every homeomorphism of a torus, TPOTP implies POTP. However the author does not have the proof.

Let (X, f) be expansive and $c > 0$ be a number such that $2c$ is an expansive constant for f . Then for $(x, y) \in Y_c$, $W_c^s(x) \cap W_c^u(y) \neq \emptyset$ and the set $W_c^s(x) \cap W_c^u(y)$ consists only of one point by expansiveness. Therefore we can define the map $[\ , \] : Y_c \longrightarrow X$ by $(x, y) \longrightarrow [x, y] \in W_c^s(x) \cap W_c^u(y)$ ($(x, y) \in Y_c$), and we have the following;

$$(6) \quad [x, x] = x ,$$

$$(7) \quad y \in W_c^s(x) \text{ and } z \in W_c^u(x) \text{ imply that} \\ (y, z) \in Y_c \text{ and } [y, z] = x ,$$

$$(8) \quad [x, [y, z]] = [x, z] \text{ if } (y, z), (x, z) \in Y_c \\ \text{and } (x, [y, z]) \in Y_c ,$$

$$(9) \quad [[x, y], z] = [x, z] \text{ if } (x, y), (x, z) \in Y_c \\ \text{and } ([x, y], z) \in Y_c ,$$

and

$$(10) \quad W_\varepsilon^s(x) \cap W_\varepsilon^u(y) = \{[x, y]\} \text{ if } (x, y) \in Y_\varepsilon \text{ for } \varepsilon \leq c .$$

Definition 3. Under the above notations, a subset E of X is said to be a rectangle if $(x,y) \in Y_C$ and $[x,y] \in E$ for all $x,y \in E$.

Definition 4. A family \mathcal{P} of closed rectangles of X is said to be a Markov partition for (X,f) if \mathcal{P} satisfies the following conditions;

$$1) \quad P = \overline{\text{int } P} \text{ for all } P \in \mathcal{P},$$

$$2) \quad \bigcup_{P \in \mathcal{P}} P = X,$$

$$3) \quad \text{int } P \cap \text{int } Q = \emptyset \text{ for all } P, Q \in \mathcal{P}$$

with $P \neq Q$,

$$4) \quad \text{for every sequence } \{P_n\}_{n \in \mathbb{Z}} \text{ of elements of } \mathcal{P},$$

$\bigcup_{n \in \mathbb{Z}} f^{-n} P_n$ consists at most of one point,

$$5) \quad f(W_C^S(x) \cap \text{int } P) \subset W_C^S(fx) \cap \text{int } Q$$

and

$$f^{-1}(W_C^U(fx) \cap \text{int } Q) \subset W_C^U(x) \cap \text{int } P$$

whenever $x \in \text{int } P \cap f^{-1}(\text{int } Q)$ ($P, Q \in \mathcal{P}$), and

$$6) \quad \text{there exist subsets } B^S \text{ and } B^U \text{ of } X \text{ such that}$$

$$f B^S \subset B^S, \quad f^{-1} B^U \subset B^U, \text{ and } B^S \cup B^U = \bigcup_{P \in \mathcal{P}} \partial P.$$

Markov partitions are not partitions in strict sense. However we use the word conventionally.

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