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On the weak simultaneous resolution of a negligible truncation of the Newton bondary

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§1. Introduction

Let $f(z_1,\ldots,z_n,t)$ be an analytic function defined in a open set $U \times D$ where U is a neighbourhood of $\vec{0} \in \mathbf{C}^n$ and D is a open disk in $\mathbf{C}$ containing the unit interval I and let $V = \{ (z,\mathbf{t}) \in U \times D : f(z,t) = 0 \}$ and let $\pi : V \to D$ be the projection. We also use the notation $f_t(z) = f(z,t)$. We assume that for each $t \in D$, $V_t = \pi^{-1}(t) = f_t^{-1}(0)$ has an isolated singularities at the origin and that the Milnor number $\mu(f_t)$ is constant $(= \mu(f_0))$. The question which we are interested in this paper is the following.

<u>Are</u> V_t $(t \in D)$ <u>toplogically equivalent to</u> V_0?

The assertion is true for $n \neq 3$ by Lê and Ramanujam [8]. Thus the question is open only for $n = 3$. If $\mu^*(f_t)$ $(t \in D)$ are constant, the assertion is true by Teissier [6]. In particular, if the Newton boundary $\Gamma(f_t)$ is non-degenerate in the sense of Kouchnirenko [1] and if $\Gamma(f_t) = \Gamma(f_0)$, the assertion is true by [6]. See also [3].

$\Psi : \tilde{V} \to V$ is called a <u>weak simultaneous resolution</u> of $\pi : V \to D$ if the following conditions are satisfied.

(i) Ψ is a proper modification.

(ii) $\pi \circ \Psi : \tilde{V} \to D$ is a flat map.

(iii) $\Psi : \tilde{V}_t \to V_t$ is a resolution of V_t.

(iv) Let $E = \Psi^{-1}(\vec{0} \times D)$. Then $\pi \circ \Psi : E \to D$ is simple.

See Teissier [7] and Laufer [2] for further detail.

In our case, the existence of a weak resolution is equivalent to the topological stability of $\{V_t\}$ by Theorem 6.4 of [2].

Briançon and Speder gave the following example of μ-constant family which is not a μ^*-constant family.

(1.1) $$z_1^5 + tz_1z_3^6 + z_2z_3^7 + z_2^{15} = 0.$$

This has a weak simultaneous resolution by [5, 10]. The purpose of this paper is to generalize this in the following case.

Let $f_1(z) = \sum_\nu b_\nu z^\nu$ be an analytic function defined in a neighbourhood of the origin. We assume that f_1 has a non-degenerate Newton boundary which is convenient and let A be a vertex of $\Gamma(f_1)$. Let $f_t(z) = f_1(z) - (1-t)\, b_A z^A$. We say that $f_t(z)$ a <u>negligible truncation</u>. if the following conditions are satisfied.

(i) There exists an open disk D in **C** containing the unit interval $[0,1]$ such that $f_t(z)$ has a non-degenerate Newton boundary $\Gamma(f_t)$ for each $t \in D$.

(ii) f_0 is convenient and $\Gamma_-(f_1)$ is a proper subset of $\Gamma_-(f_0)$.

(iii) $\nu(\Gamma_-(f_1)) = \nu(\Gamma_-(f_0))$.

Here $\nu(W)$ is the Newton number of the polyhedron W. See §2.

The example of Biançon-Speder does not satisfies the convenience condition in (ii). But this can be modified by adding y^N for a sufficiently large N for which the isomorphism class of f_t does not change.

Let $f_t(z)$ be a negligible truncation and let $\pi : V \to D$ be as above. Then the following is the result.

Theorem (1.2) $\pi : V \to D$ <u>has a weak simultaneous resolution</u>.

In general, the family of a negligible truncation is not μ^*-constant.

§2. Positivity of the Newton numbers

Let W be a polyhedron in $\mathbf{R}^n_+ = \{ (x_i) \in \mathbf{R}^n ; x_i \geq 0 \}$. Recall that <u>the Newton number</u> $\nu(W)$ is defined by

$$\sum_I (-1)^{n-|I|} \; |I|! \; |I|\text{-dim.volume } (W^I)$$

where the sum is taken for every subset I of $\{1,\ldots, n\}$ and $W_I = \{ (x_i) ; \; x_i = 0 \text{ for } i \notin I \}$. The corresponding term for $I = \phi$, is $(-1)^n$ or 0 according to $\vec{0} \in W$ or not. Let

$W = W_1 \cup W_2$ be a polyhedral decomposition of W. Then we have

$$(2.1) \qquad \nu(W) = \nu(W_1) + \nu(W_2) - \nu(W_1 \cap W_2).$$

Now we consider the case that $n = 3$ and W is a three dimensional simplex with integral vertices A, B, C and D. Let $A = (a_1, a_2, a_3), \ldots, D = (d_1, d_2, d_3)$. We assume that $\vec{0} \notin W$. Let h be the number of I's such that $|I| = 2$ and $\dim W^I = 2$. Let ℓ the number of i's such that $\dim W^{\{i\}} = 1$.

Lemma 2.2. <u>Assume that $\vec{0}$ is not in W. Then</u> $\nu(W) \geq 0$. <u>The equality holds if and only if one of the following conditions are satisfied (up to a permutation of the coordinates).</u>

(i) $h = 2$, $\ell = 1$. <u>We assume that A, C be in</u> $\mathbf{R}^{\{3\}}$, <u>B be in</u> $\mathbf{R}^{\{1,3\}}$ <u>and D be in</u> $\mathbf{R}^{\{2,3\}}$. <u>Then either</u> $b_1 = 1$ or $d_2 = 1$.

(ii) $h = 1$ <u>and</u> $\ell = 0$. <u>Assume that A, B and C be in</u> $\mathbf{R}^{\{1,3\}}$. <u>Then</u> $d_2 = 1$.

<u>Proof</u>. If $h = 0$, it is obvious that $\nu(W) > 0$. Assume that $h = 1$ and A, B and C be in $\mathbf{R}^{\{1,3\}}$. Then we have

$$\nu(W) \geq s\, d_2 - s = s\,(d_2 - 1) \geq 0$$

where $s = 2$ volume $W^{\{1,3\}}$. The first equality holds if and only if $\ell = 0$. Thus $\nu(W) = 0$ if and only if $d_2 = 1$ and $\ell = 0$. Assume that $h = 2$ and $b_2 = d_1 = 0$ and $a_i = c_i = 0$ for $i = 1,2$ and that $c_3 > a_3$. Then we have

$\nu(W) = (c_3-a_3)(d_2b_1-b_1-d_2+1) = (c_3-a_3)(d_2-1)(b_1-1).$

Thus $\nu(W) = 0$ if and only if b_1 or d_2 is 1. The case $h = 3$ is eliminated by the assumption on W. This completes the proof.

Remark 2.3. The analogous assertion in Lemma 2.2 is not true for the higher dimensions. For example, let $n = 4$ and let W be the simplex spun by $A = (1+t,0,0,0)$, $B = (1,0,0,0)$, $C = (1,2,3,0)$, $D = (1,3,2,0)$ and $E = (1,1,1,1)$. Then $\nu(W)$ is $-t$.

§3. **Negligible truncations**

Let $f_1(z_1,z_2,z_3) = \sum_\nu b_\nu z^\nu$ be an analytic function defined in a neighbourhood of the origin and assume that $f_1(z)$ has a non-degenerate Newton boundary $\Gamma(f_1)$. We also assume that f_1 is convenient in the sense of Kouchnirenko [1] where $\mu(f_1)$ is the Milnor number of f_1 at $\vec{0}$. Namely $\Gamma(f_1)^{\{i\}}$ is non-empty for each i. Let $\Gamma_-(f_1)$ be the cone of $\Gamma(f_1)$ with the origin. Recall that $\nu(\Gamma_-(f_1)) = \mu(f_1)$ by Kouchnirenko [1]. Let $A = (a_1,a_2,a_3)$ be a vertex of $\Gamma(f)$ and let $f_t(z) = f(z) - (1-t)\, b_A\, z^A$ be a negligible truncation. Let W be the closure of $\Gamma(f_0)-\Gamma(f_1)$ and let $\hat{W}$ be the cone of W with A. Then it is easy to see that $\Gamma_-(f_0) = \Gamma_-(f_1) \cup \hat{W}$. Thus the assumption that $\nu(\Gamma_-(f_1)) = \nu(\Gamma_-(f_0))$ and (2.1) implies $\nu(\hat{W}) = 0$. We can triangulate W in a finite two-simplices so that any vertex on a coordinate ax is contained in a unique two-simplex.

Then by the equality $\nu(\hat{W}) = 0$ and (2.1) and Lemma 2.2, we conclude that W is a simplex spun by three integral vertices B, C, D and $\hat{W}$ satisfies one of the conditions in Lemma 2.2. We say that $f_t(z)$ a negligible truncation of type (i) or of (ii) when $\hat{W}$ is of type (i) or of (ii) of Lemma 2.2 respectively. Before we proceed to prove Theorem 1.2, we give some examples.

Example 3.1 (E_7,type (i)) Let $f_t(x,y,z) = x^2 + y^3 + y z^3 + tz^5 + z^k$ for $k > 5$. Then f_t is a μ^*-constant family.
More generally, assume that $f_t(z)$ be a negligible truncation of type (i). Then one can show that $f_t(z)$ is a μ^*-constant family.

Example 3.2 (Type (ii), Briançon-Speder) Let

$$f_t(x,y,z) = z^5 + t\, y^6 z + y^7 x + y^k + x^{15} \quad (k \geq 8)$$

This is not a μ^*-constant family.

Example 3.3.(Type (ii)) Let

$$f_t(x,y,z) = x^8 + y^\ell + z^\ell + t\, x^5 z^2 + x^3 y z^3$$

where $\ell \geq 16$. Then μ^* is not constant. In fact, $\mu(f_t) = 2\ell^2 + 18\ell + 7$ and the Milnor numbers of the generic hyperplane sections of f_1 and f_0 are $2\ell + 23$ and $2\ell + 24$ respectively. These examples show that negligible truncations of type (ii) are not generally μ^*-constant.

§4. Resolution by Toroidal embedding

In this section, we recall the resolution of $V_1 = f_1^{-1}(0)$ briefly. The dual vector space of $\mathbf{R}^3$ can be identified canonically with itself. To distinguish vectors in $\mathbf{R}^3$ and in the dual space, we write dual vectors by column vectors. Let N^+ be the subset of the dual vectors which are non-negative. Let P be a vector in N^+. We denote by $\Delta(P)$ the face of $\Gamma_+(f_1)$ where P takes its minimal value d(P) as a function on $\Gamma_+(f)$. Here $\Gamma_+(f_1)$ is the upper half space in $(\mathbf{R}^+)^3$ with boundary $\Gamma(f_1)$. We introduce an equivalence relation $\sim$ in N^+ by $P \sim Q$ if and only if $\Delta(P) = \Delta(Q)$. This gives a conical subdivision of N^+ which we call <u>the dual Newton diagram</u> and denote it by $\Gamma^*(f_1)$. We can subdivide $\Gamma^*(f_1)$ into a cone over a simplicial complex Σ^* such that each three simplex $\sigma = (P_1, P_2, P_3)$ of Σ^* is a unimodular matrix. We call Σ^* a <u>unimodular simplicial subdivision</u> of $\Gamma^*(f_1)$. See Varchenko [9] and Oka [4] for detail. Let Σ^* be a unimodular simplicial subdivision. For each three simplex $\sigma = (P_1, P_2, P_3) = (p_{ij})$ of Σ^*, we associate a tree space $\mathbf{C}^3_\sigma$ with coordinates $\mathbf{y}_\sigma = (y_{\sigma 1}, y_{\sigma 2}, y_{\sigma 3})$ and the birational morphism $\pi_\sigma : \mathbf{C}^3_\sigma \to \mathbf{C}^3$ which is defined by $z_i = y_{\sigma 1}^{p_{i1}} y_{\sigma 2}^{p_{i2}} y_{\sigma 3}^{p_{i3}}$ $(i = 1, 2, 3)$. Then we glue $\mathbf{C}^3_\sigma$'s in a canonical way to get a smooth complex manifold X and proper birational morphism $\hat{\pi} : X \to \mathbf{C}^3$. Let $\tilde{V}_1$ be the proper transform of V_1 and let $\pi : \tilde{V}_1 \to V_1$. Then $\pi : \tilde{V}_1 \to V_1$ is a resolution of the singularity $\vec{0}$ of V_1. In $\mathbf{C}^3_\sigma$, $\tilde{V}_1$ is defined by $f_{1,\sigma}(\mathbf{y}_\sigma) = 0$ where

$$f_{1,\sigma}(\mathbf{y}_\sigma) = f_1(\pi_\sigma(\mathbf{y}_\sigma)) \,/\, \prod_{i=1}^{3} y_{\sigma i}^{d(P_i)} .$$

For each vertex P such that $\dim \Delta(P) \geq 1$, there is a corresponding exceptional divisor E(P). Suppose that $P = P_1$. Then E(P) is defined in $\mathbf{C}_\sigma^3$ by

$$h_\sigma(\mathbf{y}_\sigma) = f_{1,\Delta(P)}(\mathbf{y}_\sigma) \,/\, \prod_{i=1}^{3} y_{\sigma i}^{d(P_i)} .$$

For a further detail, we refer Oka [4].

§5. **Proof of Theorem 1.2**

Let $f_1(\mathbf{z}) = \sum_\nu b_\nu \mathbf{z}^\nu$ and let $f(\mathbf{z},t) = f_1(\mathbf{z}) - (1-t)\, b_A \mathbf{z}^A$ be a negligible truncation as in Theorem 1.2 of §1. Let $V = \{(\mathbf{z},t) \in \mathbf{C}^3 \times D : f_t(\mathbf{z}) = 0\}$. Let Σ^* be a unimodular simplicial subdivision. We apply the mapping $\pi : X \to \mathbf{C}^3$ simultaneously to f_t to resolve the singularities of V. Namely let $\hat{\Pi} : X \times D \to \mathbf{C}^3 \times D$ be the projection defined by $\hat{\Pi}(\mathbf{y},t) = (\hat{\pi}(\mathbf{y}),t)$ and let $\tilde{V}$ be the proper transform of V. Let $\omega : V \to D$ be the projection into D. We denote the restriction of $\hat{\Pi}$ to $\tilde{V}$ by Π. Let $\tilde{V}_t$ be $\Pi^{-1}(\omega^{-1}(t))$. We are going to show that $\Pi : \tilde{V} \to V$ is a simultaneous resolution of V if we choose a suitable unimodular simplicial subdivision Σ^*.

Let $\hat{W}$ and A, B, C and D be as in §3. We first prove Theorem 1.2 assuming $\hat{W}$ is of type (ii) of Lemma 2.2. The case of (i) can be proved in a similar way. Let $A = (a_1,0,a_3)$, $B = (b_1,0,b_3)$, $C = (c_1,0,c_3)$ and

$D = (d_1, 1, d_3)$. Let Δ_1 and Δ_2 be the faces of $\Gamma(f_1)$ which are spun by A, B, D and A, C, D respectively. Let $P = {}^t(p_1, p_2, p_3)$ and $Q = {}^t(q_1, q_2, q_3)$ be the respective weight vector of Δ_1 and Δ_2. By the definition, $\Delta(P) = \Delta_1$ and $\Delta(Q) = \Delta_2$. (Strictly speaking, Δ_i may be a subset of $\Delta(P)$ and Δ_2 may be a subset of $\Delta(Q)$.) P and Q must satisfy the following equalities and inequalities.

(5.1) $$p_1a_1 + p_3a_3 = p_1b_1 + p_3b_3 = p_1d_1 + p_2 + p_3d_3 < p_1c_1 + p_3c_3$$

(5.2) $$q_1a_1 + q_3a_3 = q_1c_1 + q_3c_3 = q_1d_1 + q_2 + q_3d_3 < q_1b_1 + q_3b_3.$$

From (5.1) and (5.2), we have

(5.3) $$p_2 = p_1(a_1 - d_1) + p_3(a_3 - d_3) \quad \text{and}$$

(5.4) $$q_2 = q_1(a_1 - d_1) + q_3(a_3 - d_3).$$

As P and Q are primitive vectors, (5.3) and (5.4) implies

(5.6) $$\text{G.C.D.}(p_1, p_3) = \text{G.C.D.}(q_1, q_3) = 1$$

We may assume that $b_1 > a_1 > c_1$, or equivalently $b_3 < a_3 < c_3$. From (5.1), (5.2) and (5.6), we have

(5.7) $$(b_1 - a_1) = rp_3, \quad (a_3 - b_3) = rp_1$$

(5.8) $$(a_1 - c_1) = sq_3, \quad (c_3 - a_3) = sq_1$$

for some positive integers r and s. Thus by the inequality of (5.1), we have

(5.9) $q_1p_3 - q_3p_1 > 0.$

Let $R = {}^t(0,1,0)$. Then the interior T of the triangle T(P, Q, R) is an equivalence class in $\Gamma^*(f_1)$. Namely for a dual vector S, $\Delta(S) = \{A\}$ if and only if $S \in T$. By (5.6), we have det (P, R) = det (Q, R) = 1. (For the definition det(P,Q), see Oka [4].) Thus we do not need any other vertex on the line segment $\overline{PR}$ and $\overline{QR}$. On the other hand, using (5.3) and (5.4), we easily see that

(5.10) $\det(P,Q) = \det(P,Q,R) = q_1\, p_3 - q_3\, p_1.$

This implies the following. Let $T_1, \dots, T_k$ be the canonical primitive sequence of $\overline{PQ}$ in the sense of [4]. Then two-simplices (T_i, T_{i+1}, R) ($i = 0, \dots, k$) are already unimodular. Thus we do not need any new vertices in T to subdivide $\Gamma^*(f_1)$. This is the key to the proof. We take a unimodular simplicial subdivision Σ^* which is, restricted on two-simplex T(P,Q,R), the one described above. Now we consider $\Pi : \tilde{V} \to V$ which is associated to Σ^*. It is easy to see that $\tilde{V} - \tilde{V}_0$ and $\tilde{V}_t$ ($t \neq 0$) are non-singular. Let $\sigma = (P_1, P_2, P_3)$. Then in the coordinate chart $\mathbf{C}^3_\sigma \times D$, $\tilde{V}$ is defined by

$$f_\sigma(\mathbf{y}_\sigma, t) = f(\pi_\sigma(\mathbf{y}_\sigma), t) \,/\, \prod_{i=1}^{3} y_{\sigma i}^{\,d(P_i)} = 0.$$

and $E(P_1)$ is defined by

$$h_\sigma(\mathbf{y}_\sigma, t) = f_{\Delta(P_1)}(\pi_\sigma(\mathbf{y}_\sigma), t) \,/\, \prod_{i=1}^{3} \mathbf{y}_{\sigma i}^{\,d(P_i)} = 0.$$

This is a polynomial of $y_{\sigma 2}$, $y_{\sigma 3}$ and t and

$f_\sigma(\mathbf{y}_\sigma, t) \equiv h_\sigma(\mathbf{y}_\sigma, t)$ modulo $(y_{\sigma 1})$.

Let $\xi(\sigma) = \bigcap_{i=1}^{3} \Delta(P_i)$. Then the constant term of $h_\sigma(\mathbf{y}_\sigma, t)$ with respect to $\mathbf{y}_\sigma$ is $b_{\xi(\sigma)}$ if $\xi(\sigma) \neq A$. Thus in this case, $\tilde{V}$ and $\tilde{V}_0$ is non-singular and $\Pi : E \to D$ is simple in this chart where $E = \Pi^{-1}(\vec{0} \times D)$. Assume that $\sigma = (T_i, T_{i+1}, R)$ ($T_0 = P$ and $T_{k+1} = Q$). (This is the most essential chart to be studied carefully.) Then $E(T_i)$ is defined by

$$h_\sigma(y_{\sigma 2}, y_{\sigma 3}, t) = b_A\, t + b_D\, y_{\sigma 3} = 0. \quad (0 < i < k-1)$$

$$= b_A t + b_D y_{\sigma 3} + b_B y_{\sigma 2}^r = 0 \quad (i = 0)$$

This is easy to see by a direct calculation using (5.3),..., (5.9) and by the fact that

$$T_1 = (Q + \alpha P) \,/\, (q_1 p_3 - q_3 p_1)$$

for a (unique) integer α such that $0 < \alpha < \det(P,Q)$. See Oka [4]. Thus $E(T_i)$ is non-singular in any case by the existence of the linear term $b_D y_{\sigma 3}$. By the same reason, $\tilde{V}_0$ and V are non-singular and $\Pi : E \to D$ is simple over D. The smoothness of $E(Q)$ is proved in a similar way.

Now we consider the case of a negligible truncation of type (i). Then by the same notation as above, we have to replace $A = (0,0,a_3)$, $B = (b_1,0,b_3)$, $C = (0,0,c_3)$ and $D = (0,1,d_3)$. Also the weight vector Q is simply ${}^t(1,0,0)$. The rest of the argument is completely parallel to the above argument. This completes the proof of Theorem 1.2.

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