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UNIQUENESS OF PRODUCTS IN HIGHER ALGEBRAIC K-THEORY

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Let **E** be a higher algebraic K-theory defined on rings, that is to say, **E** assigns to each ring R a spectrum **E**R of algebraic K-theory of R. Fiedorowicz uniqueness theorem [2] says that if **E** has an external tensor product, then there is a natural map of spectra

$$f : \mathbf{E}R \to \mathbf{GW}R$$

which induces an equivalence between (-1)-connected covers of **E**R and the Gersten-Wagoner spectrum **GW**R ([3] and [13]). May [6] has given a similar uniqueness theorem for higher algebraic K-theories (or, infinite loop space machines) defined on permutative categories: given an infinite loop space machine **E** defined on permutative categories, there exists a natural equivalence of spectra between **E**U and the spectrum SB$\bar{U}$ constructed by Segal [9].

In this article we study the multiplicativity of such natural transformations between various higher algebraic K-theories defined on permutative categories, or exact categories, or rings. Here the term 'multiplicativity' is used in the following sense. Let **E** and **E'** be functors $\mathcal{C} \to \mathcal{S}$ from permutative categories (or exact categories, or rings) to CW-spectra, and suppose that **E** (resp. **E'**) functorially associates to each pairing $U \times V \to W$ in $\mathcal{C}$ a pairing $\mathbf{E}U \wedge \mathbf{E}V \to \mathbf{E}W$ (resp. $\mathbf{E}'U \wedge \mathbf{E}'V \to \mathbf{E}'W$) of CW-spectra. Then a natural transformation $f : \mathbf{E} \to \mathbf{E}'$ is called multiplicative if the following square commutes in the homotopy category $H\mathcal{S}$;

$$\begin{array}{ccc} \mathbf{E}U \wedge \mathbf{E}V & \longrightarrow & \mathbf{E}W \\ {\scriptstyle f \wedge f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{E}'U \wedge \mathbf{E}'V & \longrightarrow & \mathbf{E}'W. \end{array}$$

Notice that most of the constructions of products in higher algebraic K-theory, except for May's [7], provide only weak pairings, i.e., pairings in the sense of G. W. Whitehead. This notion of a weak pairing is inadequate for sophisticated spectrum level analysis. Hence we want to find a condition, as generous as possible, which ensures that a given machine functorially associates 'true' pairings. Thus we introduce a notion of a pairing of S_*-spectra which generalizes May's notion of a pairing of $\mathcal{I}_*$-prespectra [7].

We now state the results.

A CW-spectrum $E = \{E_n | n \geq 0\}$ is called an S_*-spectrum if each E_n has an action by the symmetric group S_n which is compatible with the structure maps and restricts to a homotopically trivial A_n-action. There is a relevant notion of a pairing of S_*-spectra and we can show that pairings $(E,F) \to G$ of S_*-spectra functorially determine pairings $E \wedge F \to G$ in $H\mathcal{S}$.

We use the term higher algebraic K-theory defined on permutative categories to denote a functor E which assigns to every permutative category U a connective CW-spectrum $EU = \{E_nU \mid n \geq 0\}$ together with a natural map $\lambda : BU \to E_0U$ such that the composite $BU \to \Omega^\infty E_\infty U = \bigcup_n \Omega^n E_n U$ is a group completion.

Definition. $\mathbf{E}$ is called a multiplicative higher algebraic K-theory if (i) $\mathbf{E}U$ has a natural structure of an S_*-spectrum, and (ii) there associated, to every bipermutative functor $f : U \times V \to W$, a natural pairing $Ef = \{E_{m,n}f\} : (EU, EV) \to EW$ of S_*-spectra such that the following square commutes;

$$\begin{array}{ccc} BU \wedge BV & \xrightarrow{Bf} & BW \\ {\scriptstyle \lambda\wedge\lambda}\big\downarrow & & \big\downarrow{\scriptstyle \lambda} \\ \mathbf{E}_0U \wedge \mathbf{E}_0V & \xrightarrow{E_{0,0}f} & \mathbf{E}_0W. \end{array}$$

Thus a multiplicative higher algebraic K-theory $\mathbf{E}$ functorially associates a true pairing $Ef : \mathbf{E}U \wedge \mathbf{E}V \to \mathbf{E}W$.

It will be shown that both May machine $\mathbf{M}$ [7] and Shimada-Shimakawa machine $\mathbf{C}$ [10] are multiplicative higher algebraic K-theories defined on permutative categories. (But Segal's machine [9] is not.)

Now our first theorem is

THEOREM A. Let $\mathbf{E}$ be a higher algebraic K-theory defined on permutative categories. Then there is a natural equivalence $\gamma : \mathbf{E}U \to \mathbf{C}U$ which is multiplicative when $\mathbf{E}$ is a multiplicative higher algebraic K-theory.

Next let $\mathbf{K}$ denote the Waldhausen machine [14] which assigns to each exact category U a CW-spectrum $\mathbf{K}U = \{BQ^n U^{[n]} \mid n \geq 0\}$ (cf. [11]). Then $\mathbf{K}$ associates to any biexact functor $f : U \times V \to W$ a pairing $\mathbf{K}f : (\mathbf{K}U, \mathbf{K}V) \to \mathbf{K}W$ of S_*-spectra. (This is essentially the result of [11].) Let us denote by IsU the subcategory of all isomorphisms in a category U.

THEOREM B. There is a multiplicative natural transformation $\kappa : \mathbf{C}IsU \to \mathbf{K}U$ defined as the composite of a natural equivalence $\eta : \Omega\mathbf{C}QU \cong \mathbf{K}U$ with a natural map $\nu : \mathbf{C}IsU \to \Omega\mathbf{C}QU$ which deloops the familiar map $BIsU \to \Omega BQU$.

Note that by the "+ = Q" theorem [4], κ becomes an equivalence if every short exact sequence in U splits.

Finally we consider higher algebraic K-theories defined on rings. We do not know whether Loday's pairing $(\mathbf{GW}R, \mathbf{GW}R') \to \mathbf{GW}(R \otimes R')$ induces a 'true' pairing $\mathbf{GW}R \wedge \mathbf{GW}R' \to \mathbf{GW}(R \otimes R')$ or not. However we have

THEOREM C. There exists a functor $\mathbf{A}$ from rings to S_*-spectra which satisfies the followings:

(1) $\mathbf{A}$ assignes to every pair of rings R and R' a natural pairing $\mu : (\mathbf{A}R, \mathbf{A}R') \to \mathbf{A}(R \otimes R')$ of S_*-spectra.

(2) For each $n \geq 1$, there exists a natural group completion $f_n : BIsP(S^n R) \to \mathbf{A}_n R \simeq K_0 S^n R \times BGLS^n R^+ = \mathbf{GW}_n R$ such that

$$\begin{array}{ccc} BIsP(S^mR) \wedge BIsP(S^nR') & \longrightarrow & BIsP(S^{m+n}(R \otimes R')) \\ f_m \wedge f_n \downarrow & & \downarrow f_{m+n} \\ \mathbf{A}_m R \wedge \mathbf{A}_n R' & \xrightarrow{\mu_{m,n}} & \mathbf{A}_{m+n}(R \otimes R') \end{array}$$

commutes. (Here $P(R)$ denotes the category of finitely generated projective modules over R.)

(3) The structure map $\mathbf{A}_n R \wedge S^1 \to \mathbf{A}_{n+1} R$ is given by the composite

$$A_n R \wedge S^1 \xrightarrow{1 \wedge \iota} A_n R \wedge A_1 Z \xrightarrow{\mu_{n,1}} A_{n+1}(R \otimes Z) = \mathbf{A}_{n+1} R$$

where $\iota : S^1 \to \mathbf{A}_1 Z$ represents the standard generator of $K_1 S\mathbf{Z} = \mathbf{Z}$ (cf. [5, Chapitre II]).

(4) There is a multiplicative natural transformation $\alpha : \mathbf{C}IsP(R) \to \mathbf{A}R$ such that the induced map $\Omega^\infty \mathbf{C}_\infty IsP(R) \to \Omega^\infty \mathbf{A}_\infty R$ is an equivalence.

Note that the condition (3) is similar to the description of the structure map of $\mathbf{GW}R$ given by Loday [5]. From (2) we see that $\mu_{m,n}$ is weakly homotopic to Loday's map $\mathbf{GW}_m R \wedge \mathbf{GW}_n R' \to \mathbf{GW}_{m,n}(R \otimes R')$.

As a consequence we have

COROLLARY. The product structures in higher algebraic K-theory of rings constructed by Waldhausen [14], May [7], Shimada-Shimakawa [10], and Loday [5] (modified as in Theorem C) all agree with each other.

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