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CONFIGURATION OF HERMAN RINGS AND DYNAMICAL SYSTEMS ON TREES

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ABSTRACT. The configurations of Herman rings of rational functions are represented in terms of trees and "piecewise linear" maps on them. Their properties are investigated. A sufficient condition for trees to be such configurations is obtained by means of surgery.

0. INTRODUCTION.- JULIA SETS AND HERMAN RINGS.

Let $f(z)$ be a rational function with complex coefficients of degree greater than one. Consider the dynamical system $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$, where $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is the Riemann sphere. We write $f^n = \overbrace{f \circ \ldots \circ f}^{n}$.

The *Julia set* of f is

$J_f = \{z \in \overline{\mathbf{C}} \mid \{f^n | n \geq 0\}$ is not equicontinuous in any neighborhood of $z\}$.

The complement $\overline{\mathbf{C}} - J_f$ is called the *stable set* and its connected component a *stable region*. Every stable region is preperiodic under f and every periodic stable region is one of five types- attractive domain, superattractive domain, parabolic domain, Siegel disk and Herman ring. (For details, see [B].)

We are particularly interested in the Herman ring. A periodic stable region D of period p is a *Herman ring*, if f^p is conformally conjugate to an irrational rotation on a concentric annulus, i.e. if there exist a conformal mapping $\phi : D \to A_r = \{z \in \mathbf{C} \mid r < |z| < 1\}$ with $0 < r < 1$ and an irrational $\theta \in \mathbf{R} - \mathbf{Q}$ such that

$$\begin{array}{ccc} D & \xrightarrow{f^p} & D \\ \phi\downarrow & \overset{\curvearrowright}{R_\theta} & \downarrow\phi \\ A_r & \longrightarrow & A_r \end{array} \qquad \text{where } R_\theta(z) = e^{2\pi i\theta}\cdot z.$$

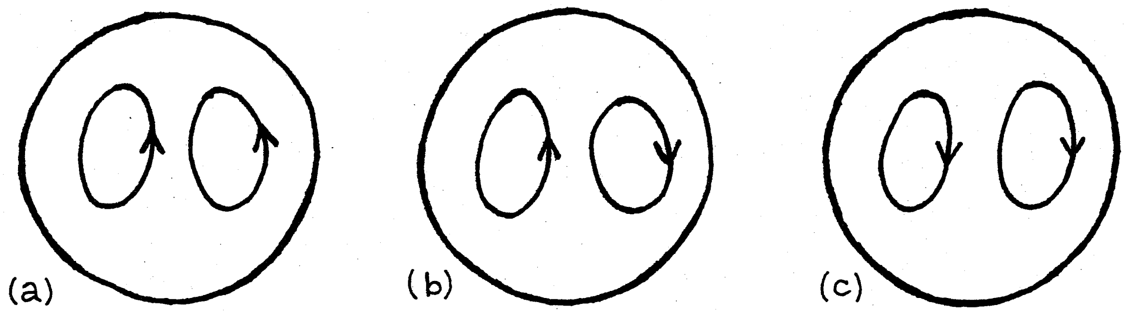
The irrational θ is called the rotation number. For the definition of the *Siegel disk*, replace A_r by $\{z\in\mathbf{C} \mid |z|<1\}$.

REMARK ON THE HERMAN RINGS.

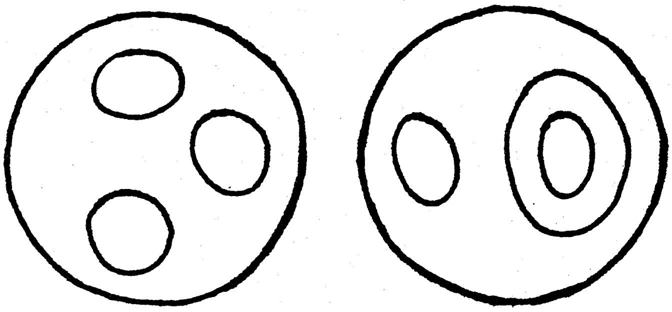
1° The existence of Herman rings was proved by Herman[H], by means of Arnold's theorem or the "Newton's method". After that, it was shown in [S] that a Herman ring can be constructed from a couple of Siegel disks by "surgery". The converse procedure is also possible. This method was used to prove that a rational function of degree d has at most $d-2$ cycles of Herman rings.

2° Note that (super)attractive domains, parabolic domains and Siegel disks are related to periodic points in them or on their boundaries. That is to say, one can deduce their existence from the condition on the eigenvalues of a periodic point (not completely in the case of Siegel disk). On the contrary, the Herman ring has nothing to do with periodic points. So it is rather difficult to know whether a rational function has a Herman ring.

3° Configuration. Suppose f has more than one Herman rings. Choose an invariant curve from each of them so that f preserves those oriented curves. There naturally arises a problem of the *configuration*, that is, how those curves are located. For example, if there is a cycle of Herman rings of period 2, there are three possibilities (up to homeomorphism). See the figures below, in which the arrows denote the orientations of the invariant curves.



Changing the orientations of both curves, (c) can be identified with (a). However (a) and (b) cannot be identified by any means. If there are three Herman rings, there still are two possibilities apart from the orientations. Furthermore, it was observed in [S] that not only Herman rings themselves but also their successive pre-images under f play an important role in the surgery decomposing Herman rings into Siegel disks.



Now we come to the subject of this paper:

Problem 1. Describe the configuration of Herman rings and their pre-images by something easier to handle.

Problem 2. Characterize the possible configurations for rational functions.

It turns out that certain kind of trees are nice objects for this purpose. (Compare with Douady-Hubbard's work on "Hubbard's tree".) We may say, in some sense, we have succeeded in extracting only a property concerning the configuration from some "fractals".

All the proof of theorems and lemmas will be given in another paper.

1. ANNULUS.

To define the trees, we need some terminologies about annuli.

A connected open set A of $\overline{\mathbf{C}}$ is an *annulus*, if its complement $\overline{\mathbf{C}}-A$ has exactly two connected components, neither of which is a point. Let X be a subset of $\overline{\mathbf{C}}$ and A an annulus (resp. γ a simple closed curve). We say that A (resp. γ) *separates* X if both components of $\overline{\mathbf{C}}-A$ (resp. $\overline{\mathbf{C}}-\gamma$) intersect with X.

For an annulus A, there exist $0<r<1$ and a conformal mapping $\phi_A : A \to \{z\in\mathbf{C} \mid r<|z|<1\}$. (See, for example, [A].) Here, r is unique and $m(A) = -\log r$ is called the *modulus* of A.

Define for $x,y \in \overline{\mathbf{C}}$,

$$A(x,y) = \bigcup_{\substack{S_r(A) \text{ separates } \{x,y\} \\ e^{-m(A)}<r<1}} S_r(A),$$

where $S_r(A) = \phi_A^{-1}(\{|z|=r\})$. Notice that $A(x,y)$ is also an annulus separating $\{x,y\}$ and does not depend on the choice of ϕ_A.

2. TREES.

Let f be a rational function which has Herman rings. A *critical point* of f is a point at which f is not locally injective. Set

$\mathcal{A}_0$ = {connected components of (Herman rings - the closure of the orbits of critical points)},

$\mathcal{A}'$ = {connected components of $f^{-n}(A) \mid A \in \mathcal{A}_0,\ n \geq 0$},

and B = the union of the boundaries of Herman rings. Note that both $\mathcal{A}_0$ and $\mathcal{A}'$ consist of disjoint annuli, and that for $A \in \mathcal{A}'$, $f : A \to f(A)$ is a covering map.

An annulus $A \in \mathcal{A}'$ is *essential*, if $f^n(A)$ separates B for any $n \geq 0$. Finally, let $\mathcal{A} = \{A \in \mathcal{A}' \mid A$ is essential$\}$.

Let us define for $x,y \in \overline{\mathbf{C}}$

$$d(x,y) = \sum_{A \in \mathcal{A}} m(A(x,y)).$$

We have following lemmas.

LEMMA 1. For any $x,y \in \overline{\mathbf{C}}$, $d(x,y) < \infty$.

LEMMA 2. $d(x,y) = d(y,x)$,

$d(x,z) \leq d(x,y) + d(y,z)$.

Hence, d is a pseudo-metric on $\overline{\mathbf{C}}$.

Now, we can give the definition of our main object. Define

$$T_f = \overline{\mathbf{C}}/\sim,$$

where $x \sim y$ if and only if $d(x,y) = 0$. Let π denote the natural projection from $\overline{\mathbf{C}}$ to T_f. The original pseudo-metric d on $\overline{\mathbf{C}}$ is projected to a metric d on T_f.

LEMMA 3. T_f is a topologically finite tree.

Each annulus of $\mathcal{A}$ is mapped to an arc by π.

Let us define a map $f_* : T_f \to T_f$ by

$$f_*(x) = \pi \circ f(\partial\pi^{-1}(x)),$$

where $\partial\pi^{-1}(x)$ is the boundary of $\pi^{-1}(x)$ in $\overline{\mathbf{C}}$.

LEMMA 4. f_* is well-defined.

It is natural to consider the tree T_f together with the map f_* as a representation of the configuration of Herman rings and their inverse images. Moreover, (T_f, f_*) can be finitely presented and is easy to compute (see §7). So it fits to our aim.

3. PROPERTIES OF (T_f, f_*).

Here are some terminologies necessary to state the properties of (T_f, f_*). Let T be a tree. A *branch* at x is a component of $T-\{x\}$. Let $\mathcal{B}_x$ denote the collection of the branches at x. A point x of T is an *end point* if $\#\mathcal{B}_x = 1$, and a *branch point* if $\#\mathcal{B}_x \geq 3$.

A metric d on T is *linear*, if for any (simple) arc α joining x and y, and any $z \in \alpha$, the equality $d(x,y) = d(x,z) + d(z,y)$ holds.

THEOREM 1. Write $T = T_f$, $F = f_*$. Then (T,d,F) has following properties.

(a) (T,d) is a topologically finite tree with a linear metric.

(b) $F : T \to T$ is continuous.

(c) There exist a finite subset $\mathrm{Sing}(T,F)$ and a locally constant map $DF : T-\mathrm{Sing}(T,F) \to \mathbf{N}$ such that:

if T' is a connected component of $T-\mathrm{Sing}(T,F)$, then

$F|_{T'}: T' \to F(T')$ is a homeomorphism,

DF is equal to a constant n on T',

and $d(F(x),F(y)) = n\cdot d(x,y)$ for $x,y \in T'$.

(d) There exist arcs I_{ij} $(i=1,\ldots,\ell; j=0,\ldots,p_i)$ with disjoint interiors such that:

I_{ij} contain no branch point except at its end points;

$F(I_{ij}) = I_{ij+1}$, where $I_{ip_i} = I_{i0}$;

$F^{p_i}|_{I_{ij}} = \mathrm{id}$.

(e) $T = \overline{\bigcup_{i,j,n\geq 0} F^{-n}(I_{ij})}$.

(f) Every end point of T is an end point of an I_{ij}.

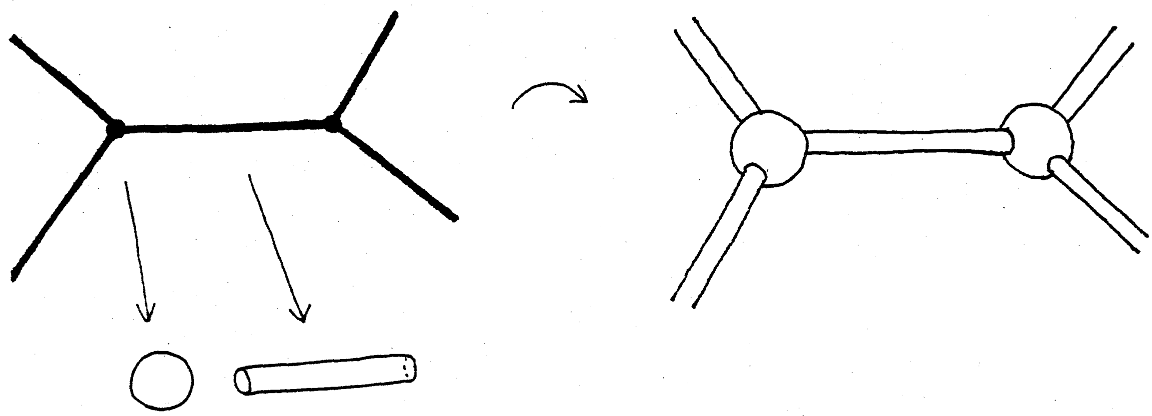
A point of $\mathrm{Sing}(T,F)$ is called a *singular point* and maximal arcs satisfying (d) *periodic intervals*. Let $T^{(n)} = \bigcup_{i,j} F^{-n}(I_{ij})$. For any x and $\beta \in \mathcal{B}_x$, we can define $DF(x,\beta) = \lim_{\beta\ni x'\to x} DF(x')$.

In the above Theorem, $\mathrm{Sing}(T_f,f_*) = \{\text{branch points of } T_f\} \cup \bigcup_{i,j} \partial I_{ij} \cup \pi(\{\text{critical points of } f\})$ and I_{ij} are the projections of Herman rings by π. Moreover if $A \in \mathcal{A}$, then Df_* on $\pi(A)$ is equal to the degree of the covering $f : A \to f(A)$, where f denotes the original rational function.

4. HOW TO CONSTRUCT A RATIONAL FUNCTION REALIZING A TREE.

Let us investigate the converse problem, i.e. under what condition a tree T and a map F can be those which are obtained from a rational function according to §2. In other words, we want to reproduce a rational function from a given tree. Of course, the conditions (a)-(f) are necessary.

Our plan is as follows:
First, thicken all the segments of the tree to tubes with a small common radius. Second, blow up all the singular points to balls. Glueing the tubes and the balls, we get a topological sphere. Next, define a mapping on the tubes so that if x, $F(x) \notin \mathrm{Sing}(T,F)$, it is a covering of degree $DF(x)$ from the circle corresponding to x to that corresponding to $F(x)$. Finally, find a suitable mapping on each ball, so that one can get, by the surgery in [S], a rational function with the desired configuration of Herman rings.



The result will be given in §6. But before that, we need to study how to define the mapping on the balls.

5. LOCAL MODEL FOR SINGULAR ORBITS.

Suppose (T,F) satisfies (a)-(f). Agree to add all the branch points and all the end points of I_{ij} to $\mathrm{Sing}(T,F)$. Let $X_1 = \mathrm{Sing}(T,F)$, $X = X_1 \cup F(X_1)$ and $X_* = \{x \in X_1 \mid x$ has a pre-periodic orbit in $X_1\}$. Consider $X \times \overline{\mathbb{C}}$ and define $\overline{\mathbb{C}}_x = \{x\} \times \overline{\mathbb{C}} \subset X \times \overline{\mathbb{C}}$ for $x \in X$.

A *local model* for singular orbits of (T,F) is $(g,\{p_\beta\})$ satisfying:

(g) $g : X_1 \times \overline{C} \to X \times \overline{C}$ is an analytic map such that $g(\overline{C}_x) \subset \overline{C}_{F(x)}$; For $x \in X$, p_β $(\beta \in \mathcal{B}_x)$ are distinct points of $\overline{C}_x$.

(h) $g(p_\beta) = p_{F(\beta)}$, where $F(\beta)$ is the branch at $F(x)$ containing $F(x')$ for $x' \in \beta$ sufficiently close to x.

(i) $\deg_{p_\beta} g = DF(x,\beta)$.

(j) If x is an end point of an I_{ij} and $\beta \in \mathcal{B}_x$ contains I_{ij}, then p_β is the center of a Siegel disk of g with rotation number θ_β.

(k) If $\partial I_{ij} = \{x,x'\}$ and β (resp. β') the branch at x (resp. x') containing x (resp. x'), then $\theta_\beta = -\theta_{\beta'}$.

(ℓ) If $x \in X_*$ and $z \in \overline{C}_x$ is a critical point of g in the stable set, then z is preperiodic with respect to g.

The definitions of the stable set, Siegel disk, etc. for $g|_{X_* \times \overline{C}}$ are similar to those for a single rational function on $\overline{C}$.

To examine these condition is, in general, not so easy. However if we restrict our attention to rational functions of low degree, then it becomes quite easy. See Example 3 in §7.

In [S], a rational function with Herman rings is decomposed into cyclic rational maps with Siegel disks. These cyclic maps are nothing but the local model for singular orbits of the tree obtained from the original function.

6. REALIZATION THEOREM.

We have a partial answer to Problem 2 in terms of our trees.

THEOREM 2. Suppose that (T,F) satisfies the conditions (a)-(f), and that there are a local model for singular orbits of (T,F). For any $n \geq 0$, there exist a rational function f with Herman rings and an isometry $h : T^{(n)} \to T_f^{(n)}$ satisfying

$$\begin{array}{ccc} T^{(n)} & \xrightarrow{F} & T^{(n)} \\ h\downarrow & \circlearrowright & \downarrow h \\ T_f^{(n)} & \xrightarrow{f_*} & T_f^{(n)} \end{array}.$$

Moreover if all the singular points of (T,F) are preperiodic, the conclusion holds with $T^{(n)}$, $T_f^{(n)}$ replaced by T, T_f.

The degree of f is given by

$2(\deg f - 1) = \#\{$critical points of g other than $p_\beta\}$.

This result is not completely satisfactory, but it is useful enough to study the "rough" configuration of Herman rings. See the next section. I hope that the theorem would be improved to conclude the complete realization in any case.

7. EXAMPLES.

Let us see some examples of trees and how the realization theorem is applied. The trees shown below subject to the following conventions:

$\xrightarrow{i}$: a periodic interval, where the number i indicates its cyclic order and the arrow its orientation;

On $—$ (resp. $=$, $\equiv$), DF = 1 (resp. 2, 3);

o = π(a critical point of f), where f is supposed to be the original rational function, and similarly

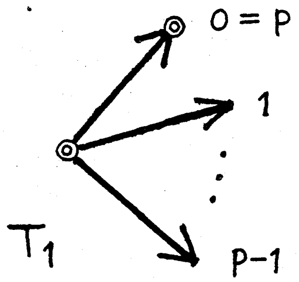
◎ = π(two critical points), etc.;

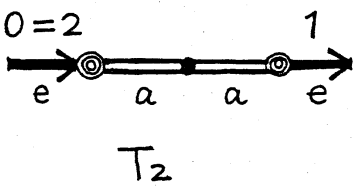
● = a fixed point;

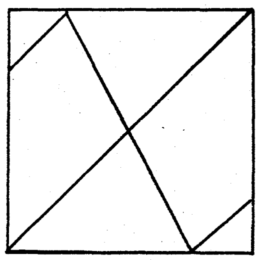
Alphabets are the lengths of respective segments;

The maps F on the trees are the simplest piecewise linear maps which send each periodic interval i to i+1.

EXAMPLE 1 and 2. The trees for the rational functions obtained in the theorem 5 A) and B) of [S].



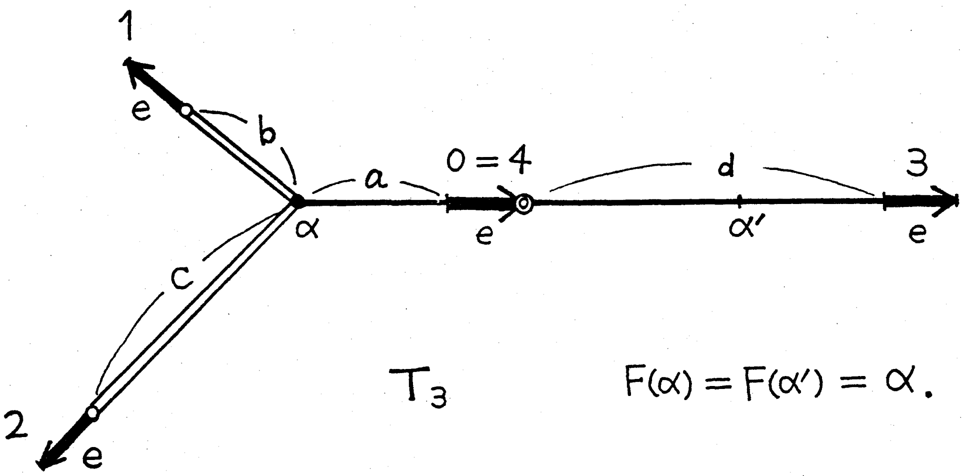




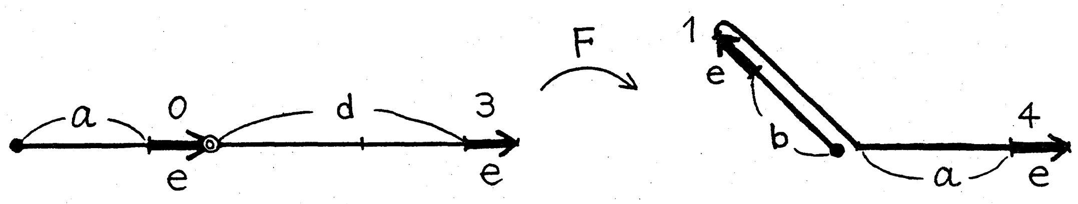
Graph of F on T_2

The tree T_1 is supposed to be the simplest tree with periodic intervals of period p. For T_2, from the graph of F, we have $2a = a + e$, hence $a = e$.

EXAMPLE 3.



Let us determine the lengths a,b,.... For example, from



we have $a = b$ and $d = e + b + a$. Similarly, $2b = c$ and $2c = a + e + d$. We immediately conclude that

$$a = b = 2e, \; c = 4e, \; d = 5e \quad \text{and} \quad e > 0 \quad \text{is arbitrary.}$$

It is easy to see that T_3 and F satisfy the conditions (a)-(f) under these relations.

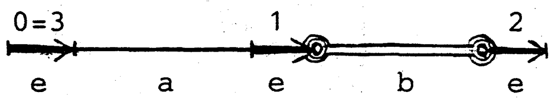
Let us construct a local model for singular orbits of T_3. Define the end points i_+ and i_- of a periodic interval i by $i_- \xrightarrow{i} i_+$. We need the model at 9 points- $0_+, 0_-, 1_+, \ldots$ and α. Here is an example of the local model, where "at x, $a \to \zeta$" means that $p_\beta = \zeta$, for the branch β at x containing the segment a.

Point x	$g = g\|_{\bar{C}_x} : \bar{C}_x \to \bar{C}_{F(x)}$	$a \to \zeta$
α	$g(z) = (z-1)^2/z^2$	$a \to \infty$, $b \to 1$, $c \to 0$
0_+	$g(z) = e^{2\pi i\theta} \cdot z(1-z)$	$e \to 0$, $d \to 1$
1_+, 2_+, 3_+	$g(z) = z$	$e \to 0$
1_-, 2_-	$g(z) = e^{-\pi i\theta} \cdot z(1-z)$	$e \to 0$, $b,c \to \infty$
0_-, 3_-	$g(z) = z$	$e \to 0$, $a,d \to \infty$

Here the θ is to be an irrational satisfying the Diophantine condition.

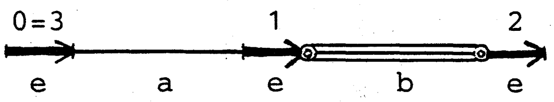
Check the conditions (g)-(ℓ). Then Theorem 2 gives us a rational function f whose tree is the T_3. Counting its critical points, we have $\deg f = 3$.

EXAMPLE 4.



This is impossible, because $a = b$ and $2b = e + a + e + b + e$, hence $e = 0$. However, we can make it realizable by setting DF = 3 on the segment b.

T_4



Then $a = b = 3e$. Constructing a local model, one can show that T_4 is realized by a rational function of degree 5.

EXAMPLE 5. See the tree in the next page.
Although it looks complicated, T_5 is proved to be realizable by a rational function of degree 3. Try to find the lengths.

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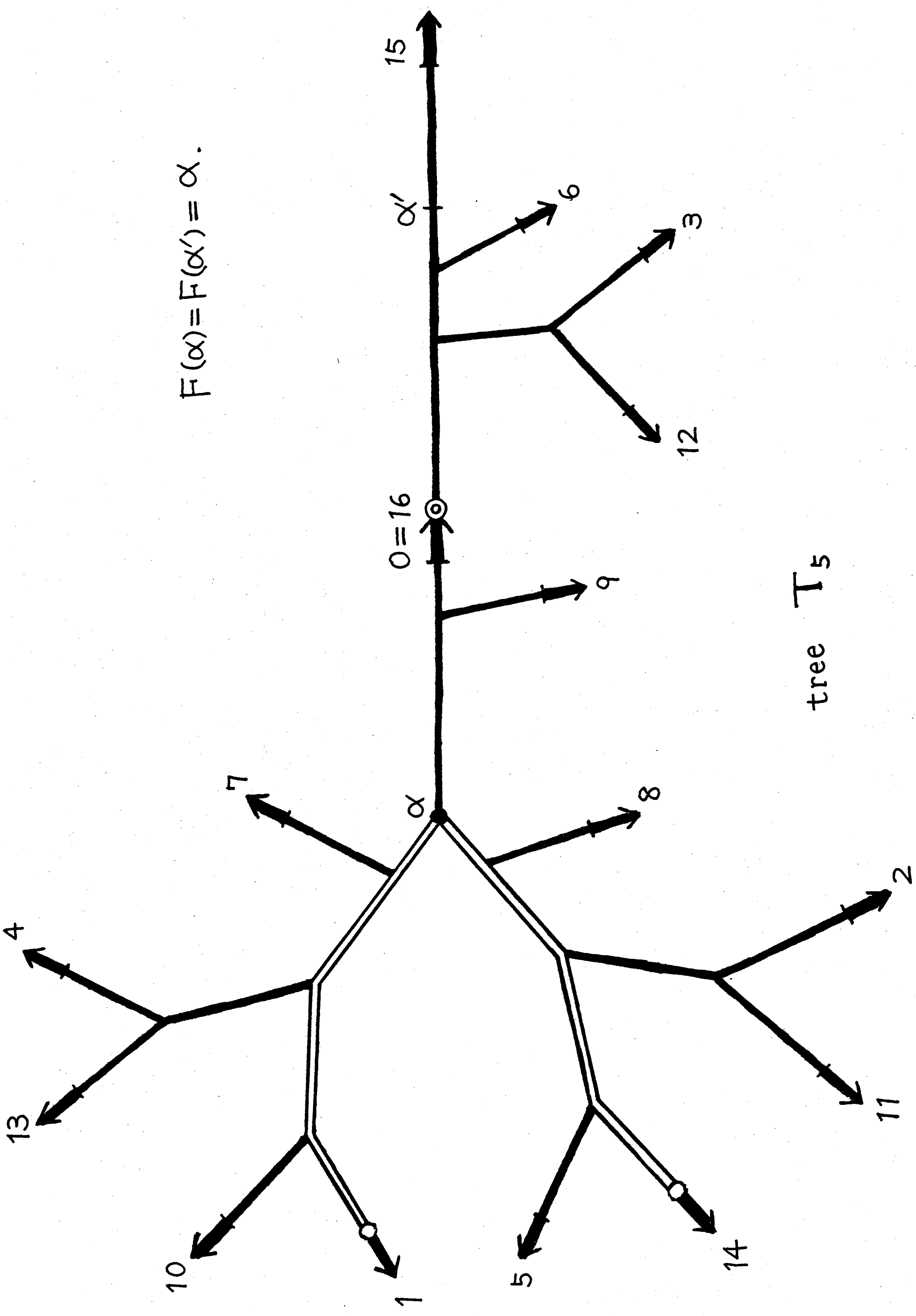
F(α)=F(α′)=α.
15
α′
6
3
12
O=16
9
tree T5
α
7
8
4
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13
11
10
1
5
14