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Spectral analysis of self-affine functions

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By definition, a <u>self-affine</u> function is a continuous function $f:[0,1]\to R$ with $f(0)=0$, $f(x)\not\equiv 0$ such that there exist a real number α with $0<\alpha\leqq 1$, an integer $r\geqq 2$ and finitely many functions $g_1,g_2,\ldots,g_k:[0,1]\to R$ satisfying that

(1) for any $N=0,1,2,\ldots$ and $j=0,1,\ldots,r^N-1$, there exists $h=1,2,\ldots,k$ such that

(#) $\qquad r^{\alpha N}(f((j+x)r^{-N})-f(jr^{-N}))=g_h(x) \qquad (\forall x\in[0,1])$, and

(2) for any $n=0,1,2,\ldots$, $\ell=0,1,\ldots,r^n-1$ and $h=1,2,\ldots,k$, there exist integers N and j with $N\geqq n$ and $\ell r^{N-n}\leqq j<(\ell+1)r^{N-n}$ such that (#) holds.

The above α and r are called an <u>order</u> and a <u>base</u> of f, respectively. The notion of self-affine function was introduced by N. Kôno [1] in a restrictive sense, which was generalized by the author [2] as above. It is known that

I. if f is a self-affine function of order α, then f satisfies the Hölder continuity of order exactly α at each point [1], hence, α is determined by f,

II. a self-affine function is a linear function if and only if it has an order 1,

III. if a self-affine function f has bases r and ℓ with $\log r/\log \ell$ irrational, then f is a linear function, and

IV. there exist exactly a countably infinite number of self-affine functions which are linearly independent.

A non-linear, self-affine function f has a remarkable spectral property, namely, the Fourier coefficient $\hat{f}(n)$ decreases in the order of n^{-1} as $n\to\infty$, where

$$\hat{f}(n):=\int_0^1 (f(x)-xf(1))e^{-2\pi inx}dx \qquad (n\in Z).$$

For a self-affine function f with order α and base r, define

$$\tau_{r,N}(j)=r^{\alpha N}(f((j+1)r^{-N})-f(jr^{-N})),$$

where $N=0,1,2,\ldots$ and $j=0,1,\ldots,r^N-1$. Note that

$$\{\tau_{r,N}(j);\ N=0,1,2,\ldots,\ j=0,1,\ldots r^N-1\}$$

is a finite set. A self-affine function f with base r is called r-*invertible* if

$$\tau_{r,N}(j)=\tau_{r,N}(j')$$

holds for any $N=0,1,2,\ldots$ and $j=0,1,\ldots,r^N-1$, where j' is an integer with $0\leq j'<r^N$ such that

$$[j'r^{-n}]-r[j'r^{-n-1}]=[jr^{N-n-1}]-r[jr^{N-n}]$$

for any $n=0,1,\ldots,N-1$.

THEOREM. *Let* f *be a self-affine function with order* α *and base* r. *Then,*

$$\hat{f}(n)=\frac{1}{2\pi in}\lim_{N\to\infty} r^{-\alpha N}\sum_{j=0}^{r^N-1}\tau_{r,N}(j)e^{-2\alpha ijnr^{-N}}$$

holds for any $n\in Z$ *with* $n\neq 0$. *Moreover, if* f *is* r-*invertible, then we have*

$$\hat{f}(rn)=r^{-1}\hat{f}(n)$$

for any $n\in Z$ *with* $n\neq 0$.

Example 1. Let

$$f(x)=\sum_{n=1}^{\infty} 3^{-n}\eta^{\sum_{j=1}^{n} x_{2j-1}}(x_{2n}),$$

where

$$x=\sum_{n=1}^{\infty} x_n 3^{-n}$$

with $x_n \varepsilon \{0,1,2\}$ and η is a permutation $\binom{0\ 1\ 2}{2\ 1\ 0}$. Then, f is a 9-invertible self-affine function of order 1/2, which shall be called Peano function. We have

$$\hat{f}(n)=\frac{1}{i\pi n}\prod_{k=1}^{\infty}\frac{1}{3}\frac{1+\omega_k^n+\omega_k^{2n}}{1-\omega_k^n+\omega_k^{2n}},$$

where $\omega_k=e^{2\pi i 9^{-k}}$ and $n\neq 0$.

<u>Example</u> 2. Let

$$f(x)=\sum_{n=1}^{\infty} 2^{-[n/2]}(-1)^{\sum_{j=1}^{n-2} x_j x_{j+1}} x_n,$$

where

$$x=\sum_{n=1}^{\infty} x_n 2^{-n}$$

with $x_n \varepsilon \{0,1\}$ Then, f is a 4-invertible self-affine function with order 1/2, which shall be called Rudin-Shapiro function. We have

$$\hat{f}(n)=\frac{1}{2\pi i n}\lim_{N\to\infty} 2^{-N}\sum_{j=0}^{4^N-1} a_j e^{-2\pi i j n 4^{-N}}$$

for any $n\neq 0$, where

$$a_j=(-1)^{\sum_{k=0}^{\infty} j_k j_{k+1}}$$

with

$$j=\sum_{k=0}^{\infty} j_k 2^k \qquad (j_k \varepsilon \{0,1\}).$$

REFERENCES

[1] Norio Kôno, On self-affine functions, Japan Journal of Applied Mathematics 3-2 (1986), pp. 259-269

[2] Teturo Kamae, A characterization of self-affine functions, ibid., pp. 271-280