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A survey of modified analytic trivializations

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1. Definition of modified analytic trivializations.

Let

$$F(x;t):(\mathbb{R}^n\times I,0\times I)\longrightarrow(\mathbb{R},0)$$

be a real analytic function-germ where I is a compact cube in $\mathbb{R}^m$. We call F(x;t) a real analytic family of real analytic function-germs $f_t(x):=F(x;t)$ $t\in I$.

Let $\pi:X\longrightarrow\mathbb{R}^n$ be an analytic modification.

(1.1) Definitions.

(1.1.1) A real analytic family F(x;t) admits a π-modified analytic trivialization (abbreviated to a π-MAT) along I via the modification π if there exist an analytic isomorphism

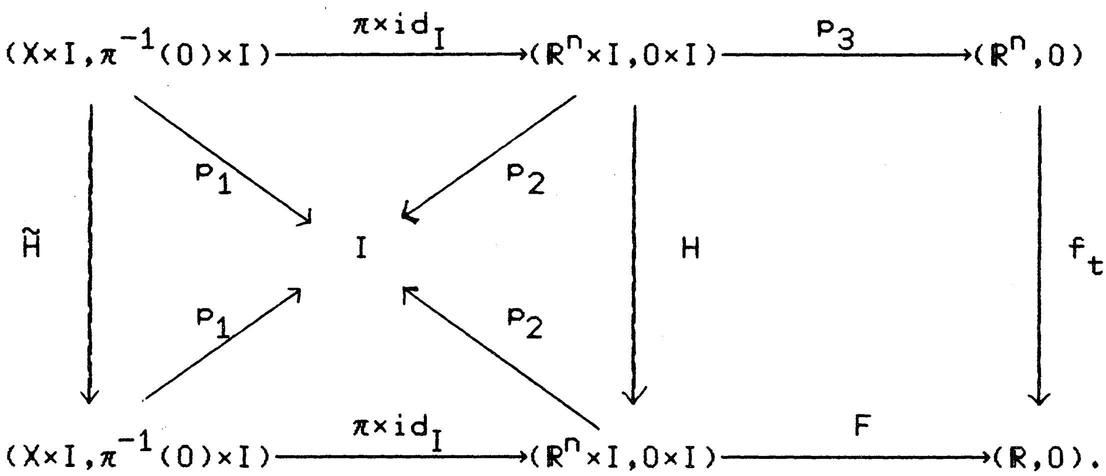
$$\tilde{H}:(X\times I,\pi^{-1}(0)\times I)\longrightarrow(X\times I,\pi^{-1}(0)\times I)$$

and a homeomorphism

$$H:(\mathbb{R}^n\times I,0\times I)\longrightarrow(\mathbb{R}^n\times I,0\times I)$$

such that the following commutative diagram holds :

(1.1.2) <u>Commutative diagram.</u>



Where p_i $1\leqq i\leqq 3$ are canonical projections and t_0 is an element of I.

(1.1.3) A real analytic family $F(x;t)$ admits an <u>almost π-modified analytic trivialization</u> (abbreviated to an <u>almost π-MAT</u>) along I via the modification π if there exists an analytic isomorphism $\tilde{H}$ such that the commutative diagram (1.1.2) except the arrow H holds.

2. Kuo's motivation and classification theorem.

In this section, we survey the results of the paper [4].

The (right) C^r equivalence relation, the natural equivalence relation in the set $C^r(n,1)$ of C^r function-germs $f:(\mathbb{R}^n,0)\longrightarrow(\mathbb{R},0)$, is defined as follows: $f \underset{C^r}{\sim} g$ for $f,g\in C^r(n,1)$ if there exists a local C^r diffeomorphism $h:(\mathbb{R}^n,0)\longrightarrow(\mathbb{R}^n,0)$ such that $f=g\circ h$. Then the following two results are well-known.

(2.1) Theorem (T. Fukuda). Let $P(n,k)$ be the set of polynomials of n variables whose degree is lower than or equal to k. Then the set $P(n,k)$ consists of finite C^0 equivalence classes.

(2.2) Whitney's example. The family $W_t(x,y):=y(y-x)(y-2x)(y-tx):(\mathbb{R}^2,0)\longrightarrow(\mathbb{R},0)$, consists of infinite C^1 equivalence classes.

Theorem (2.1) and (2.2) tell us that a C^0 equivalence classe consists of sufficiently many elements but a C^1 equivalence class does not. There is a big gap between the C^0 and C^1 equivalence relations.

T-C. Kuo is not pleased with this fact. He says in [4] that a fundamental problem in the theory of real singularities is to seach for a "nice and natural" equivalence relation in the set $A(\mathbb{R}^n)$ of germs of analytic functions : $(\mathbb{R}^n,0)\longrightarrow(\mathbb{R},0)$. He proposes the notion of blow-analytic equivalence relation as one of them. We summarize here his idea and results in [4].

(2.3) Kuo's definitions.

(2.3.1) Let X^* be a complex space. Then a modification of X^* is a proper surjective holomorphic map $\pi^*:\tilde{X}^*\longrightarrow X^*$, which is a biholomorphism outside $\pi^{*-1}(N)$, N being a thin set.

(2.3.2) Let X be a real space. Then a modification of X is a proper surjective analytic map $\pi:\tilde{X}\longrightarrow X$ whose complexification $\pi^*:\tilde{X}^*\longrightarrow X^*$ is a modification.

(2.3.3) A map $f:X\longrightarrow Y$ of real spaces is called

<u>blow-analytic</u> if there exists a modification $\pi:\tilde{X}\longrightarrow X$ such that $f\circ\pi:\tilde{X}\longrightarrow Y$ is analytic.

(2.3.4) A homeomorphism $\phi:X\longrightarrow Y$ of real spaces is called <u>blow-analytic</u> if ϕ and ϕ^{-1} are blow-analytic.

(2.3.5) Analytic function-germs $f,g\in A(\mathbb{R}^n)$, are <u>blow-analytically equivalent</u> if there exists a blow-analytic, local homeomorphism $\phi:(\mathbb{R}^n,0)\longrightarrow(\mathbb{R}^n,0)$ such that $f=g\circ\phi$. In fact, this is an equivalence relation (Proposition 2 in [4]).

(2.4) <u>Example</u>. Let $f(x,y):\mathbb{R}^2\longrightarrow\mathbb{R}$ be the function defined by

$$f(x,y):=\begin{cases}\dfrac{x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

The function $f(x,y)$ is blow-analytic but not C^1.

(2.5) <u>Note</u>. If a real analytic family $F(x;t)$ admits a π-MAT along a parameter space I via a modification π, then the function-germs parametrized by I, form a blow-analytic equivalence class.

(2.6) <u>Theorem ([4])</u>. <u>Let</u> $F(x;p):(\mathbb{R}^n\times P,0\times P)\longrightarrow(\mathbb{R},0)$ <u>be a real analytic family where the parameter space</u> P <u>is a subanalytic subset of some Euclidean space. Suppose for fixed</u> p, $f_p:=F(x;p)$: $(\mathbb{R}^n,0)\longrightarrow(\mathbb{R},0)$ <u>admits</u> $0\in\mathbb{R}^n$ <u>as an isolated singularity. Then there exists a finite filtration of</u> P <u>by subanalytic subsets</u> $P^{(i)}$

$$P=P^{(0)}\supset P^{(1)}\supset\cdots\supset P^{(\ell)}\supset P^{(\ell+1)}=\phi,$$

<u>with the following properties</u>:

(2.6.1) dim $P^{(i)}$>dim $P^{(i+1)}$, $P^{(i)}-P^{(i+1)}$ are smooth,

(2.6.2) For p, p′ in a same connected component of $P^{(i)}-P^{(i+1)}$, $f_p, f_{p'}$ are blow-analytically equivalent.

(2.7) Theorem ([3]). If a real analytic family F(x;t) admits a simultaneous resolution ϕ, then it admits a $\pi o \phi$ -MAT where π is a finite succession of blowing-ups of $\mathbb{R}^n$ with non-singular centers.

3. MAT via the blowing-up of $\mathbb{R}^n$ at the origin.

Now, the problem of what kind of singularities form a blow-analytic equivalence class via what kind of modifications, arises.

In this section, we survey the results in [2] by T-C. Kuo. The problem above is treated firstly in the following theorem.

(3.1) Theorem ([2]). Suppose the initial form $H_k(x;t)$ in x of F(x;t) has an isolated singularity at the origin of $\mathbb{R}^n$ for any t∈I. Let $\pi:M \longrightarrow \mathbb{R}^n$ be the blowing-up of $\mathbb{R}^n$ at the origin. Then F(x;t) admits a π-MAT along I.

(3.2) Examples.

(3.2.1) Whitney′s example. The family $W_t(x,y)$ defined in (2.2) admits a π-MAT along I, where I is a compact interval in $\mathbb{R}$ and $\{0,1,2\} \cap I = \phi$. So the family W_t t∈$\mathbb{R}$ consists of seven blow-analytic equivalence classes at most. On the other hand, it consists of infinite C^1 equivalence classes.

(3.2.2) Zeeman double cusp. The family $Z_t(x,y):=x^4+y^4+tx^2y^2$ admits a π-MAT along I, where I is a compact interval in $\mathbb{R}$ and $2\notin I$.

4. Skech of the proof of (3.1).

Let $F:(\mathbb{R}^n\times I, 0\times I)\longrightarrow(\mathbb{R},0)$ be a real analytic family. Define the vectors $\vec{a}$ and $\vec{b}$ as follows:

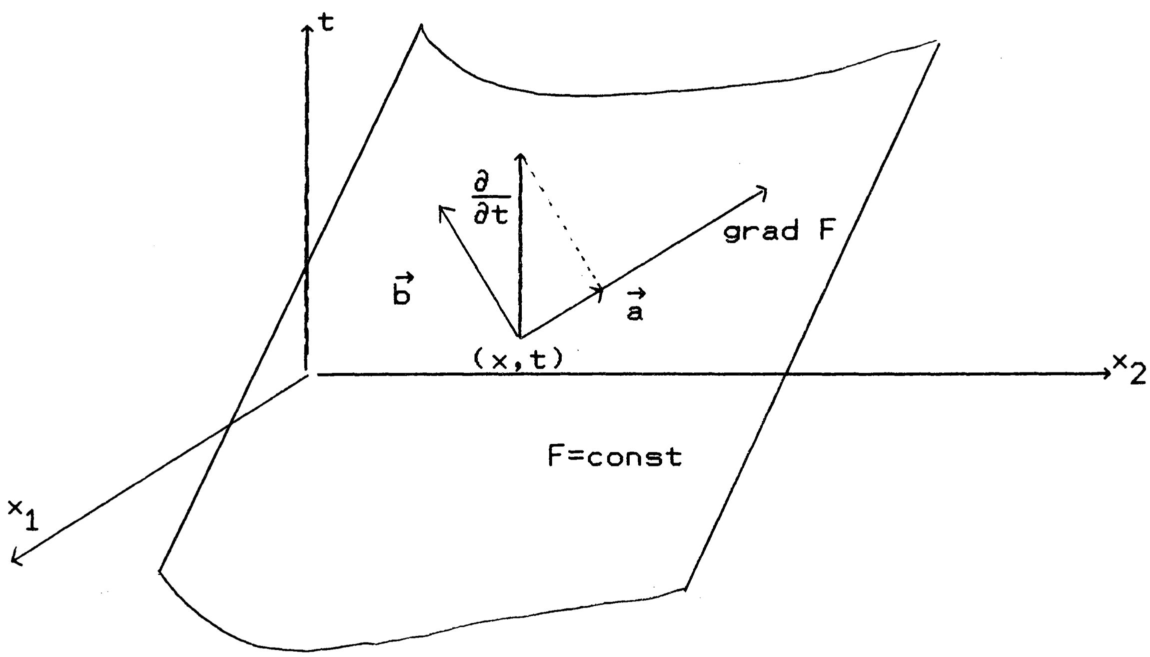
$$\vec{a} := \langle\frac{\partial}{\partial t}, \text{grad } F/|\text{grad } F|\rangle \text{grad } F/|\text{grad } F|$$

$$= (\frac{\partial F}{\partial t}/|\text{grad } F|^2)\text{grad } F$$

and

$$\vec{b} := \frac{\partial}{\partial t} - \vec{a}$$

$$=(-\frac{\partial F}{\partial t}/|\text{grad } F|^2)\text{grad}_x F + ((|\text{grad } F|^2 - (\frac{\partial F}{\partial t})^2/|\text{grad } F|^2)\frac{\partial}{\partial t}.$$



(4.1) Kuo vector field.

$$V(x;t) := -(\frac{\partial F}{\partial t}/|\text{grad } F|^2)\text{grad}_x F + \frac{\partial}{\partial t}.$$

This is the vector normalizing $\vec{b}$ so that the coefficient of $\frac{\partial}{\partial t}$ is equal to 1. The vector field V(x;t) is tangent to the level surface of F(x;t) at its regular point.

Then we can show that the vector field $d(\pi|M-\pi^{-1}(0))^{-1}(V)$ has an analytic extension $\tilde{V}$ to a neighbourhood of $\pi^{-1}(0)$ in M. The trajectory of the vector field $\tilde{V}$ defines the analytic isomorphism $\tilde{H}$, which satisfies the commutative diagram (1.1.2). Hence the family F(x;t) admits a π-MAT along I.

5. MAT via toroidal embeddings.

In this section, we summarize the results in [5].

(5.1) Definitions. Let K=$\mathbb{R}$ or $\mathbb{C}$.

(5.1.1) Let $f(x)= \sum_k a_k x^k$ be the Taylor expansion of f(x) at the origin. The convex hull of the set

$$\cup\{k\in\mathbb{Z}_+{}^n \mid a_k\neq 0\}$$

in $\mathbb{R}^n$ is denoted by $\Gamma_+(f)$, a so-called Newton polygon of f.

(5.1.2) A function-germ f is K-non-degenerate if

$$\{x\in K^n \mid \frac{\partial f_\gamma}{\partial x_1} =\cdots= \frac{\partial f_\gamma}{\partial x_1} = 0\} \subset \{x\in K^n \mid x_1x_2\cdots x_n=0\}$$

for any compact face γ of $\Gamma_+(f)$, where

$$f_\gamma(x) := \sum_{k\in\gamma} a_k x^k.$$

In this case, we say also that a function-germ f has a <u>K-non-degenerate Newton principal part</u>.

(5.1.3) An analytic family F(x;t) is called a family with a <u>fixed Newton polygon</u> Γ_+ if all Newton polygon $\Gamma_+(f_t)$ are equal to Γ_+. A family F(x;t) is <u>K-non-degenerate</u> if F is a family with a fixed Newton polygon and f_t $t\in I$ is K-non-degenerate.

It is well-known that there is a so-called toroidal embedding $\pi:X\longrightarrow\mathbb{R}^n$ associated with a Newton polygon $\Gamma_+=\Gamma_+(F)$.

(5.2) <u>Theorem ([5])</u>. <u>An $\mathbb{R}$-non-degenerate analytic family</u> F(x;t) $t\in I$, <u>where I is a compact cube in $\mathbb{R}^m$, admits an almost</u> π-MAT <u>along I via the toroidal embedding π associated with the</u> <u>Newton polygon</u> $\Gamma_+(F)$.

(5.3) <u>Theorem ([5])</u>. <u>Suppose that the hypothesis of (5.2)</u> <u>holds. If</u> $F_\gamma(x;t)$ <u>is independent of t for any non-compact,</u> <u>non-coordinate face</u> γ <u>of</u> $\Gamma_+(F)$, <u>then F(x;t) admits a π-MAT along</u> I.

These theorems are ones of generalizations of the theorem (3.1) in a sense. It should be noted that function-germs of real analytic family treated here, are not necessarily isolated sigularities.

6. MAT via blowing-ups and subblowing-ups.

In this section, we survey the results of the forthcoming paper [9].

Let $F(x;t)$ be a real analytic family with fixed Newton polygon Γ_+. We can choose finite monomials $x^{i_j}=x_1^{i_{j1}}x_2^{i_{j2}}\cdots x_n^{i_{jn}}$ $0\leq j\leq k$ and analytic functions $c_j(x;t)$ with $c_j(0;t)\neq 0$ so that

$$F(x;t) = \sum_{j=0}^{k} c_j(x;t)x^{i_j}.$$

We let

$K:=\mathbb{R}$ or $\mathbb{C}$,

$KW:=\{x\in K^n \mid x^{i_j}=0,\ 0\leq j\leq k\}$,

$KX_W^*:=\{(x,\zeta)\in(K^n-KW)\times KP^k \mid \zeta_0:\zeta_1:\cdots:\zeta_k=x^{i_0}:x^{i_1}:\cdots:x^{i_k}\}$,

$KX:=KX_W:=$the topological closure of KX_W^* in the Hausdorff space $K^n\times KP^k$,

and

$\mathbb{C}X|\mathbb{R}:=\mathbb{C}X\cap(\mathbb{R}^n\times\mathbb{R}P^k)$.

(6.1) Definitions.

(6.1.1) We call the canonical projection

$$\mathbb{R}\pi:\mathbb{R}X\longrightarrow\mathbb{R}^n$$

(or simply, $\mathbb{R}X$) the subblowing-up of $\mathbb{R}^n$ with center $\mathbb{R}W$.

(6.1.2) We call the canonical projection

$$\mathbb{C}\pi:\mathbb{C}X\longrightarrow\mathbb{C}^n$$

(or simply, $\mathbb{C}X$) the blowing-up of $\mathbb{C}^n$ with center $\mathbb{C}W$.

(6.1.3) We call the restriction map $\mathbb{C}\pi|\mathbb{R}$ of $\mathbb{C}\pi$ to the real part $\mathbb{C}X|\mathbb{R}:=\mathbb{C}X\cap(\mathbb{R}^n\times\mathbb{R}P^k)$

$$\mathbb{C}\pi|\mathbb{R}:\mathbb{C}X|\mathbb{R}\longrightarrow\mathbb{R}^n$$

(or simply, $\mathbb{C}X|\mathbb{R}$) <u>the blowing-up of $\mathbb{R}^n$ with center $\mathbb{R}W$.</u>

It is well-known that the subblowing-up $\mathbb{R}X$ is a semi-algebraic set and the blowing-up $\mathbb{C}X$ (resp. $\mathbb{C}X|\mathbb{R}$) is the Zariski closure of $\mathbb{C}X_W^*$ (resp. $\mathbb{R}X_W^*$) in $\mathbb{C}^n\times\mathbb{C}P^k$ (resp. $\mathbb{R}^n\times\mathbb{R}P^k$).

The subblowing-up $\mathbb{R}\pi:\mathbb{R}X\longrightarrow\mathbb{R}^n$ is a proper surjective modification of $\mathbb{R}^n$ in the sense that the map $\mathbb{R}\pi$ is a proper surjective analytic map and the restriction

$$\mathbb{R}\pi|\mathbb{R}X_W^*:\mathbb{R}X_W^*\longrightarrow\mathbb{R}-\mathbb{R}W$$

is an analytic isomorphism of real analytic manifolds.

The blowing-up $\mathbb{C}\pi$ (resp. $\mathbb{C}\pi|\mathbb{R}$) is, of course, a proper surjective modification of $\mathbb{C}^n$ (resp. $\mathbb{R}^n$).

(6.2) <u>Definition</u>. Given a subset J of $\{1,2,\cdots,n\}$, an analytic function-germ f is <u>K-J-non-degenerate</u> if

$$\{x\in K^n \mid x_p \frac{\partial f_\gamma}{\partial x_p}=0 \ p\in J\} \subset \{x_1x_2\cdots x_n=0\}$$

for any compact face γ of $\Gamma_+(f)$.

(6.3) <u>Example</u>. An $f(x_1,x_2,x_3):=x_1^4+x_1^2x_2^3+x_2^8+x_2^6x_3^2$ is $\mathbb{R}$-$\{1,2\}$-non-degenerate.

(6.4) Theorem. A germ $f(x)$ is K-J-non-degenerate if and only if the linear equation

(6.4.1) $i_{0p}c_0(0)\zeta_0+i_{1p}c_1(0)\zeta_1+\cdots+i_{kp}c_k(0)\zeta_k=0,\ p\in J$

has no solutions in $K\pi^{-1}(0)$.

(6.5) Definition. For $K=\mathbb{R}$ or $\mathbb{C}$, we call $F(x;t)$ a K-J-non-degenerate family if F is a family with a fixed Newton polygon and each function-germ $f_t(x):=F(x;t), t\in I$, is K-J-non-degenerate.

Let $\gamma_p, 1\leq p\leq M$, be the all faces of Γ_+ (which contain the face of dimension n) and construct a new family $\tilde{F}(x;T), T=(t_{q,p})\in\tilde{I}$, changing the parameter $t=(t_1,t_2,\cdots,t_m)$ of $F(x;t)$ as follows. Substitute a new parameter $t_{q,p}$ for the parameter t_q (if there exists) in the coefficient $c_j(x;t)$ of term x^{i_j}, $i_j\in\mathrm{Int}(\gamma_p)$. Then $\tilde{F}(x;T)$ is a real analytic family with the fixed Newton polygon Γ_+ and $\dim\tilde{I}\leq mM$. We call $\tilde{F}(x;T)$ the corresponding family to $F(x;t)$. We denote the boundary of γ by $\partial\gamma$.

(6.6) Definitions.

(6.6.1) For a face γ of the Newton polygon Γ_+, a subset J_γ of $\{1,2,\cdots,n\}$ is called the transversal direction of the face γ if J_γ is the subset of all indexes p of x_p-axes, each of which is transversal to the face γ, namely there is no parallel translation τ of $\mathbb{R}^n$ such that the affine space determined by γ contains $\tau(x_p\text{-axis})$.

(6.6.2) We call a K-non-degenerate family $F(x;t)$ a strongly K-non-degenerate family if the corresponding family $\tilde{F}(x;T)$ is K-J_γ-non-degenerate and $F_{\partial\gamma}(x;t)$ is independent of t for any non-compact, non-coordinate face γ of Γ_+.

(6.7) Theorem. Let $F(x;t)$ be an $\mathbb{R}$ (resp. $\mathbb{C}$)-non-degenerate real analytic family. Then the family $F(x;t)$ admits an almost $\mathbb{R}\pi$-MAT (resp. an almost $\mathbb{C}\pi|\mathbb{R}$-MAT) along I.

(6.8) Theorem. Let $F(x;t)$ be an $\mathbb{R}$ (resp. $\mathbb{C}$)-non-degenerate real analytic family. Suppose that that $F_\gamma(x;t)$ is independent of t for any non-compact, non-coordinate face γ of Γ_+. Then the family $F(x;t)$ admits an $\mathbb{R}\pi$-MAT (resp. a $\mathbb{C}\pi|\mathbb{R}$-MAT) along I.

(6.9) Theorem. Let $F(x;t)$ be a strongly $\mathbb{R}$ (resp. $\mathbb{C}$)-non-degenerate real analytic family. Then the family $F(x;t)$ admits an $\mathbb{R}\pi$-MAT (resp. a $\mathbb{C}\pi|\mathbb{R}$-MAT) along I.

I think that these theorems are ones of direct generalizations of the theorem (3.1) formulated by making use of blowing-ups.

Assuming that the theorem (6.4) is valid, we show the lines of the proof of the theorems (6.7), (6.8) and (6.9).

Let us introduce a singular Riemannian metric on $\mathbb{R}^n$. Let $g_p(x), 1 \leq p \leq n$, be analytic functions on $\mathbb{R}^n$. Define the inner product by

$$\langle g_p \frac{\partial}{\partial x_p} , g_q \frac{\partial}{\partial x_q} \rangle := \delta_{pq}$$

where δ_{pq}, $1\leq p,q\leq n$, are the Kronecker's symbols. This inner product induces a Riemannian metric on $\mathbb{R}^n-\{x\in\mathbb{R}^n \mid g_1g_2\cdots g_n=0\}$, which we call a <u>singular Riemannian metric</u> on $\mathbb{R}^n$.

Using this singular metric, we have the following representation of the Kuo vector field $V(x;t)$.

(6.10) <u>Lemma</u>.

$$V(x;t)=-\frac{\partial F}{\partial t}\sum_{p=1}^{n} g_p^{\,2}\frac{\partial F}{\partial x_p}\frac{\partial}{\partial x_p}\Big/\sum_{p=1}^{n}\Big(g_p\frac{\partial F}{\partial x_p}\Big)^2+\frac{\partial}{\partial t}.$$

Define

$$E_p:=\begin{cases} x_1x_2\cdots x_n/x_p & \text{if } p\notin J,\\ 1 & \text{otherwise.}\end{cases}$$

(6.11) <u>Lemma</u>. <u>Suppose that $F(x;t)$ be a real analytic family of germs of $\mathbb{R}$-J-non-degenerate (resp. $\mathbb{C}$-J-non-degenerate) functions $f_t(x):=F(x;t)$ with a fixed Newton polygon Γ_+. Then, the vector field</u>

$\tilde{V}(x,\zeta;t):=$

$$\frac{-\Big(\sum_{j=0}^{k}\frac{\partial c_j}{\partial t}\zeta_j\Big)\sum_{p=1}^{n}E_p^{\,2}\Big\{\sum_{j=0}^{k} i_{jp}c_j(x;t)\zeta_j+x_p\sum_{j=0}^{k}\frac{\partial c_j}{\partial x_p}\zeta_j\Big\}x_p\frac{\partial}{\partial x_p}}{\sum_{p=1}^{n}E_p^{\,2}\Big\{\sum_{j=0}^{k} i_{jp}c_j(x;t)\zeta_j+x_p\sum_{j=0}^{k}\frac{\partial c_j}{\partial x_p}\zeta_j\Big\}^2}+\frac{\partial}{\partial t}$$

<u>is analytic in a neighbourhood of</u> $\mathbb{R}\pi^{-1}(0)\times I$ (<u>resp</u>. $(\mathbb{C}\pi|\mathbb{R})^{-1}(0)\times I$)

<u>in</u> $\mathbb{R}^n \times \mathbb{R}P^k \times \mathbb{R}$ <u>and</u>

$$d(\mathbb{R}\pi)(\tilde{V}(x,\zeta;t))=d(\mathbb{C}\pi|\mathbb{R})(\tilde{V}(x,\zeta;t))=V(x;t).$$

The analyticity of $\tilde{V}$ follows (6.4) and the last equality is proved by the application of (6.10) in which put $g_p:=E_p x_p$, $1 \leqq p \leqq n$.

Defining an anlytic isomorphism $\tilde{H}$ making use of trajectories of $\tilde{V}_q$, we can prove the theorems (6.7), (6.8) and (6.9). The vector field $\tilde{V}_q$ is defined by $\tilde{V}$ of (6.11), in which put $\frac{\partial}{\partial t}:=\frac{\partial}{\partial t_q}$. See [9], in detailes.

7. Necessary conditions for MAT.

In previous sections, we surveyed sufficient conditions for MAT. In this section, we study necessary conditions.

T-C. Kuo [2] shows that the so-called kite singularity $K(x,y;t):=y^2-t^2x^3-x^5$ does not admit a π-MAT along any interval I, $0 \in I$ via the blowing-up π of $\mathbb{R}^2$ at the origin. To show this, he proves the following

(7.1) <u>Theorem</u>. <u>If a real analytic family</u> $F(x;t)$ <u>admits a</u> π-MAT <u>along</u> I <u>via the blowing-up</u> π <u>of</u> $\mathbb{R}^n$, <u>then</u>

$$O(\lambda,\mu)=O(H_t(\lambda),H_t(\mu))$$

<u>for any analytic arc</u> λ, μ <u>started from the origin of</u> $\mathbb{R}^n$. <u>Where</u> $O(\lambda,\mu)$ <u>denotes their order of contact at the origin</u>.

M. Suzuki [8] studies also a necessary condition for π-MAT and proves the following

(7.2) Theorem. If a real analytic family $F(x,y;t)$ admits a π-MAT along I via the blowing-up of $\mathbb{R}^2$ at the origin, then $F(x,y;t)$ is a family with a fixed Newton polygon.

I think that these are very interesting themselves. I hope that the theorem in these directions is proved via more general modifications.

8. Problems.

We summarize the problems associated with MAT.

(8.1) Study the problem of what kind of degenerate families admits a π-MAT via what kind of modifications.

I know only a very simple degenerate family which admits a bug π-MAT. A bug π-MAT means, by definition, the fact that there exists an analytic isomorphism $\tilde{H}$ of $X \times I - N$ in (1.1.2), where N (resp. $N \cap X \times t$) is a thin set of $X \times I$ (resp. $\pi^{-1}(0) \times t$). Can we study more general cases in this line?

(8.2) Seach necessary conditions for MAT. As I mentioned in the section 7, do similar results to (7.1) and (7.2) hold via another modifications?

(8.3) Construct a catastrophe theory making use of blow-analytic equivalence classes. At the same time, it would be a problem to determine the normal forms of functions in a some category. T-C. Kuo referes also this subject in his address at Tokyo (1979).

(8.4) It would be an important subject to study a map version of MAT. T. Fukui [6] make a study of this subject and prove interesting results.

(8.5) Construct a blow-analytic theory. T-C. Kuo proposes a notion of blow-analyticity of mappings and blow-analytic equivalence relation in the set of analytic function-germs. There are some results associated with blow-analytic equivalence relations as we saw in this short survey. But there are no results associated with blow-analyticity itself. For example, families of blow-analytic functions should be studied in the future. I am sure that there would be a big theory named blow-analytic theory.

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