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QUADRATIC CONTROL OF LINEAR STOCHASTIC PERIODIC SYSTEMS

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1. INTRODUCTION

Recently periodic systems and optimal control problems of periodic systems have been studied by several authors for example [3], [6], [9], [15], [16].

In this paper we consider linear stochastic periodic systems and obtain sufficient conditions for the existence of periodic solutions. We then introduce control to our systems and consider average quadratic cost control problems. Under the assumption of stochastic stabilizability and detectability there exists a unique periodic solution to the Riccati equation associated with this problem. We show that the optimal control is a periodic feedback control involving this periodic solution. The optimal closed loop system has a unique periodic solution which is globally exponentially asymptotically stable. As a corollary we obtain optimal stationary controls for time invariant systems. An example is given to illustrate the theory.

2. PERIODIC SOLUTIONS OF STOCHASTIC SYSTEMS

Let Y be a real separable Hilbert space Y and let $A(t)$ be a possibly unbounded linear operator on Y with $A(t+T)=A(t)$, $t \in R$. We assume that A generates a strongly continuous evolution operator $U(t,s)$, $t \geq s$ [17], [18]. We also assume that there exists a family of linear operators $A_n(t)$ generating evolution operators $U_n(t,s)$ such that $U_n(t,s)$ is differentiable on Y_0 dense in Y and converges strongly in Y to $U(t,s)$ uniformly on $0 \leq s \leq t \leq T$.

Let (Ω,F,F_t,P) be a stochastic basis. Consider

$$(2.1) \qquad dy = [A(t)y + f(t)]dt + G_i(t)y dw_i + G(t)dw, \quad y(0) = y_0$$

where f is T-periodic and is in $L_2(0,T;Y)$, G_i, $G \in L(Y)$ are T-periodic and strongly continuous, (w_i) is a k-dimensional Wiener process, $w(t)$ is a Wiener process in a real Hilbert space H, $\mathrm{cov}[w(t)]=tW$, W nuclear, w_i, w are independent and the repeated i denotes the summation from $i=1$ to k. We assume that F_0 is rich enough that there exist Gaussian random vectors [11] and Wiener processes in (Ω,F_0,P). For each F_0-measurable y_0 with $E|y_0|^2 < +\infty$ we define the mild solution of (2.1) by the unique solution in $C([0,L];L^2(\Omega,Y))$

$$(2.2) \qquad y(t) = U(t,0)y_0 + \int_0^t U(t,s)f(s)ds + \int_0^t U(t,s)G_i(s)y(s)dw_i(s) + \int_0^t U(t,s)G(s)dw(s)$$

The existence and uniqueness of a mild solution on arbitrary $[0,L]$ is well-known [4].

We recall that a stochastic process $y(t)$ in Y is called T-periodic if the joint distribution of $y(t_1+T)$, $y(t_2+T)$, $\cdots$, $y(t_n+T)$, for any

$t_1,\cdots, t_n$ is independent of T [16]. y(t) is called weakly T-periodic if it has T-periodic mean and covariance. Note that if y(t) is Gaussian, then y(t) is T-periodic if and only if it is weakly T-periodic.

We say that (2.1) has a T-periodic (weakly T-periodic) solution if there exists a y_0 such that the mild solution (2.2) is T-periodic (weakly T-periodic).

First we consider the special case $G_i=0$. Then (2.1) is now

$$dy = [A(t)y + f(t)]dt + G(t)dw \tag{2.3}$$

and its mild solution is given explicitly by

$$y(t) = U(t,0)y_0 + \int_0^t U(t,s)f(s)ds + \int_0^t U(t,s)G(s)dw(s). \tag{2.4}$$

Proposition 2.1. (i) The system (2.3) has a T-periodic solution if and only if there exists $\overline{y}_0 \in Y$ and $0 \leq P_0 \in L(Y)$ such that

$$\text{(2.5)} \quad \begin{aligned} &\text{a)} \quad [I - U(T,0)]\overline{y}_0 = \int_0^T U(T,s)f(s)ds \\ &\text{b)} \quad P_0 = U(T,0)P_0U^*(T,0) + \int_0^T U(T,s)G(s)WG^*(s)U^*(T,s)ds \,. \end{aligned}$$

If there exist $\overline{y}_0$ and P_0 satisfying (2.5), then we can choose y_0 Gaussian with mean $\overline{y}_0$ and covariance P_0.

(ii) If U(t,s) is exponentially stable, then there exists a unique Gaussian T-periodic solution of (2.3). Moreover $\overline{y}_0$ and P_0 are given by

$$\text{(2.6)} \quad \begin{aligned} &\text{a)} \quad \bar{y}_0 = \int_{-\infty}^{0} U(0,s)f(s)ds \\ &\text{b)} \quad P_0 = \int_{-\infty}^{0} U(0,s)G(s)WG^*(s)U^*(0,s)ds \ . \end{aligned}$$

Proof. The first part follows from Morozen [16]. To prove (ii), we need to show that $(\bar{y}_0, P_0)$ given by (2.6) is the solution of (2.5). We shall only consider P_0. Note that (2.5b) is well defined since $U(t,s)$ is exponentially stable. Note also $U(T,0)=U(nT,(n-1)T)$ for any $n=1,2,\cdots$, $U(t,s)U(s,v)=U(t,v)$, $t \geq s \geq v$ and $U(T+t,T+s)=U(t,s)$, $t \geq s$. Now

$$\begin{aligned} &U(T,0)P_0U^*(T,0) + \int_0^T U(T,s)G(s)WG^*(s)U^*(T,s)ds \\ &= \int_{-\infty}^{0} U(T,s)G(s)WG^*(s)U^*(T,s)ds + \int_0^T U(T,s)G(s)WG^*(s)U^*(T,s)ds \\ &= \int_{-\infty}^{T} U(T,s)G(s)WG^*(s)U^*(T,s)ds \\ &= \int_{-\infty}^{0} U(0,r)G(r)WG^*(r)U(0,r)dr \\ &= P_0 \ . \end{aligned}$$

Hence P_0 satisfies (2.5b). Now we shall show that P_0 is the only solution of (2.5b). Let P_1 be another solution and set $P = P_0 - P_1$. Then

$$P = U(T,0)PU^*(T,0) = U(2T,T)U(T,0)PU^*(T,0)U^*(2T,T)$$
$$= U(2T,0)PU^*(2T,0) = U(nT,0)PU^*(nT,0)$$

for any positive integer n. Letting $n \to \infty$ we get $P=0$ by exponential stability of $U(t,s)$.

Remark 2.1. If $U(t,s)$ is exponentially stable, then

$$(2.7) \qquad y(t) = \int_{-\infty}^{t} U(t,s)f(s)ds + \int_{-\infty}^{t} U(t,s)G(s)dw(s)$$

is the unique T-periodic solution of (2.3), where w is now extended to $(-\infty,\infty)$ keeping the covariance operator W unchanged. (2.6) is obtained from (2.7) by setting $t=0$.

From now on we assume that w_i and w are defined on $(-\infty,\infty)$. Now we go back to our original system (2.1). To show the existence of a T-periodic solution to (2.1) we assume that the homogeneous part of (2.1) is exponentially stable. To be more precise consider

$$(2.8) \qquad dy = A(t)ydt + G_i(t)ydw_i, \quad y(s) = y_0 \in Y.$$

Let $V(t,s)$ be the stochastic fundamental solution [7], [15] of (2.8) so that $y(t,s;y_0)=V(t,s)y_0$. We assume that

$$(2.9) \qquad E|V(t,s)y_0|^2 \le M_1 e^{-a(t-s)} E|y_0|^2 ; \quad t \ge s$$

for some $M_1 \ge 0$, $a > 0$. We shall say for simplicity that (A,G_i) is exponentially stable.

Theorem 2.1. Suppose that the homogeneous system (2.8) is exponentially stable, namely, (2.9) holds. Then there exists a unique T-periodic solution y to (2.1). It is given by

$$y(t) = \int_{-\infty}^{t} V(t,s)f(s)ds + \int_{-\infty}^{t} V(t,s)G(s)dw(s) \ . \tag{2.10}$$

Proof. Note first that $V(t,s)$ and $w(s)$ are independent and hence the stochastic integral in (2.10) is well-defined. We first prove existence. Let y be given by (2.10), then we have

$$y(t) = \int_{0}^{t} V(t,s)f(s)ds + \int_{0}^{t} V(t,s)G(s)dw(s) + V(t,0)y(0)$$

so that y is a mild solution of (2.1) with $y(0)=y_0$. This is proved in Arnold [1] via Ito's formula when Y is finite dimensional. The general case then follows by taking approximating system with $A(t)$ replaced by $A_n(t)$ and passing to the limit as $n\to\infty$. Now as $V(T+t,T+s)$ has the same distribution as $V(t,s)$ (by [15]) we can easily check that y is T-periodic. It remains to prove uniqueness. Let y be a T-periodic solution to (2.1). Then we have

$$y(t) = V(t,b)y(b) + \int_{b}^{t} V(t,s)f(s)ds + \int_{b}^{t} V(t,s)G(s)dw(s).$$

Letting $b\to-\infty$, we find (2.10).

3. QUADRATIC CONTROL

Now we consider the optimal control problem :

(3.1) $$dy = [A(t)y + B(t)u + f(t)]dt + G_i(t)y\,dw_i + G(t)dw, \quad y(0) = y_0 ,$$

(3.2) $$J(u) = \overline{\lim_{L\to\infty}} \frac{1}{L} E\int_0^L [|M(t)y|^2 + \langle N(t)u,u\rangle]\, dt,$$

where u is a control in a real separable Hilbert space U, $B \in L(U,Y)$, $N \in L(U)$, $M \in L(Y)$ are T-periodic and strongly continuous and $N(t) \geq cI$ for some $c > 0$. We wish to minimize (3.2) over

(3.3) $$U_{ad} = \{u : u \text{ is } F_t\text{-measurable}, \ \overline{\lim_{L\to\infty}} \frac{1}{L} E\int_0^L |u(t)|^2 dt < \infty$$
$$\text{such that} \ \sup_{t\geq 0} E|y(t)|^2 < \infty\} .$$

To make our problem non trivial we need some assumptions. Let $D : [0,\infty) \to L(Y)$ be strongly continuous.

Following [8] we say that

(i) $(A,B;G_i)$ is stabilizable if there exists $K : [0,\infty) \to L(Y,U)$ bounded, strongly continuous such that $(A-BK,G_i)$ is exponentially stable, i.e., the stochastic evolution operator generated by $(A-BK,G_i)$ is exponentially stable.

(ii) $(A,D;G_i)$ is detectable if there exists $J : [0,\infty) \to L(Y,U)$ bounded, strongly continuous such that $(A-JD,G_i)$ is exponentially stable.

The feedback control

$$(3.4) \qquad u = -K(t)y + h(t)$$

is admissible if K, h are T-periodic and strongly continuous and if $(A-BK,G_i)$ is exponentially stable.

In order to solve our control problem we consider the Riccati equation

$$(3.5) \qquad Q' + A^*(t)Q + QA(t) + M^*(t)M(t) + G_i^*(t)QG_i(t) - QB(t)N^{-1}(t)B^*(t)Q = 0 .$$

<u>Theorem 3.1.</u> (i) Suppose that $(A,B;G_i)$ is stabilizable. Then there exists a T-periodic nonnegative solution to (3.5).
(ii) If $(A,M;G_i)$ is detectable, then there exists at most one solution to (3.5) which is T-periodic and nonnegative.
Moreover, if Q is the solution of (3.5), then $(A-BN^{-1}B^*Q,G_i)$ is stable.

<u>Proof.</u> The proof of (i) is very similar to the deterministic case [6]. In fact we consider (3.5) with terminal condition $Q(n)=0$; the solution Q_n is monotone increasing in n and is uniformly bounded in t and n since $(A,B;G_i)$ is stabilizable. Then there exists a limit $Q(t) = \lim_{n\to\infty} Q_n(t)$ in the strong sense and it is the required periodic solution. The second part can be shown as in [8, Theorem 4.1].

Now our control problem can be solved as in the deterministic case.

<u>Theorem 3.2.</u> Suppose that $(A,B;G_i)$ is stabilizable and $(A,M;G_i)$ detectable. Then the feedback law

$$(3.6) \qquad \overline{u} = -N^{-1}B^*(Qy+r)$$

is optimal and

$$(3.7) \qquad J(\bar{u}) = \frac{1}{T}\int_0^T [2 < r,f > - < BN^{-1}B^*r,r > + \text{tr}\, GWG^*Q]\, dt$$

where Q is the T-periodic solution of (3.5) and r is the unique T-periodic solution of

$$(3.8) \qquad r' + (A^* - QBN^{-1}B^*)r + Qf = 0$$

given by

$$r(t) = \int_t^\infty U_Q^*(s,t)Q(s)f(s)ds,$$

U_Q is the evolution operator generated by $A - BN^{-1}B^*Q$ and tr denotes the trace of nuclear operators.

Proof. Let $u \in U_{ad}$ and let y be its reponse. We note that the formal application of Ito's formula to $< Qy,y > + 2< r,f >$ can be justified by using approximating systems involving $A_n(t)$ (see [2], [5], [6]). Thus we have

$$d[<Qy,y> +2 <r,y >] = \{|N^{1/2}[u + N^{-1}B^*(Qy+r)]|^2 - |My|^2$$
$$- <Nu,u> + 2<r,f> - <BN^{-1}B^*r,r> + \text{tr}\, GWG^*Q\}\, dt$$
$$+ 2<Qy+r , G_i y dw_i> + 2<Qy+r,Gdw> .$$

Integrating this from 0 to L and taking expectations, we have

$$E[\langle Q(L)y(L),y(L)\rangle + 2\langle r(L),y(L)\rangle - \langle Q(0)y_0,y_0\rangle - 2\langle r(0),y_0\rangle]$$

$$+ E\int_0^L [|My|^2 + \langle Nu,u\rangle]dt$$

$$= E\int_0^L |N^{1/2}[u + N^{-1}B^*(Qy+r)]|^2 dt$$

$$+ \int_0^L [2\langle r,f\rangle - \langle BN^{-1}B^*r,r\rangle + \mathrm{tr}\, GWG^*Q]\, dt .$$

Now dividing this by L and taking limit supremum as $L\to\infty$, we obtain

$$(3.9) \qquad J(u) = \overline{\lim_{L\to\infty}} \frac{1}{L} E\int_0^L |N^{1/2}[u + N^{-1}B^*(Qy+r)]|^2 dt$$

$$+ \frac{1}{T}\int_0^T [2\langle r,f\rangle - \langle BN^{-1}B^*r,r\rangle + \mathrm{tr}\, GWG^*Q]\, dt .$$

The optimality of (3.6) and (3.7) follows immediately.

The closed loop system corresponding to the optimal control $\bar{u}$ is

$$(3.10) \qquad dy = [(A-BN^{-1}B^*Q)y + f - BN^{-1}B^*r]dt + G_i(t)ydw_i + G(t)dw .$$

Since $(A-BN^{-1}B^*Q,G_i)$ is exponentially stable, (3.10) has a unique T-periodic solution by Theorem 3.1. More precisely we have :

<u>Theorem 3.3.</u> Under the hypotheses of Theorem 3.2, there exists a unique T-periodic solution to (3.10) given by :

$$(3.11)\ y_p(t) = \int_{-\infty}^t V_Q(t,s)[f(s)-B(s)N^{-1}(s)B^*(s)r(s)]ds + \int_{-\infty}^t V_Q(t,s)G(s)dw(s) ,$$

where $V_Q(t,s)$ is the fundamental solution of the homogeneous system obtained from (3.10).

Moreover y_p is exponentially asymptotically stable in mean square.

Proof. Note that y_p, given by (3.11), is the unique T-periodic solution to (3.10). The last assertion of the theorem follows from the identity:

$$y(t,y_0) - y_p(t) = V_Q(t,0)(y_0 + y_p(0))$$

where $y(t,y_0)$ is the mild solution of (3.10) with $y(0) = y_0$.

Suppose now $G_i = 0$. Then the Riccati equation (3.5) is equal to the deterministic one. Thus if (A,B) is stabilizable and (A,M) detectable [19], then

(3.12) $$dy = [(A-BN^{-1}B^*Q)y + f - BN^{-1}B^*r]dt + G(t)dw$$

has a unique Gaussian T-periodic solution y_p. In fact by Theorem 3.1 we have :

Corollary 3.1. Assume $G_i = 0$ and that (A,B) is stabilizable and (A,M) detectable. Then the system (3.12) has a unique Gaussian T-periodic solutions given by

$$y_p(t) = \int_{-\infty}^{t} U_Q(t,s)[f(s)-B(s)N^{-1}(s)B^*(s)r(s)]ds + \int_{-\infty}^{t} U_Q(t,s)G(s)dw(s) ,$$

where $U_Q(t,s)$ is the evolution operator generated by $A-BN^{-1}B^*Q$.

Moreover

(3.13)
$$Ey_p(0) = \int_{-\infty}^{0} U_Q(0,s)[f(s)-B(s)N^{-1}(s)B^*(s)r(s)]ds$$
$$Cov[y_p(0)] = \int_{-\infty}^{0} U_Q(0,s)G(s)WG^*(s)U_Q^*(0,s)ds \ .$$

Now we consider special cases. First we take a class of T-periodic admissible controls

(3.14) $$U_{pad} = \{u: u \text{ is } F_t\text{-measurable }, (3.1) \text{ has a T-periodic solution for some } y_0, F_0\text{-measurable}\}$$

and calculate the cost along periodic solutions. Thus we take

(3.15) $$J_p(u) = \frac{1}{T} E\int_0^T [|My|^2 + \langle Nu,u\rangle]dt .$$

Under the assumptions of Theorem 3.2 U_{pad} is not empty. Note that the optimal cost in (3.7) does not depend on y_0. Thus we have :

<u>Corollary 3.2.</u> The feedback control (3.6) is optimal for the periodic control problem (3.1), (3.15) and $J_p(\bar{u}) = J(\bar{u})$. Moreover $\bar{u}(t)$ is also T-periodic.

Now we consider time-invariant case. The Riccati equation (3.5) becomes an algebraic equation

(3.16) $$A^*Q + QA + M^*M + G_i^*QG_i - QBN^{-1}B^*Q = 0 .$$

The stabilizability of $(A,B;G_i)$ and the detectability of $(A,M;G_i)$ are the same as in [8].

Corollary 3.3. Let the operators in (3.1), (3.2) and f are all constant. Suppose that $(A,B;G_i)$ is stabilizable and $(A,M;G_i)$ detectable. Then the feedback control

$$(3.17)\qquad \bar{u} = -N^{-1}B^*(Qy + r)$$

is optimal and

$$(3.18)\qquad J(\bar{u}) = 2\langle r,f\rangle - \langle BN^{-1}B^*r,r\rangle + \operatorname{tr} GWG^*Q ,$$

where Q is the unique nonnegative solution of (3.16) and r is the unique solution of the algebraic equation

$$(3.19)\qquad (A^* - QBN^{-1}B^*)r + Qf = 0$$

given by

$$(3.20)\qquad r = -(A^* - QBN^{-1}B^*)^{-1}Qf = \int_t^\infty e^{(s-t)(A^* - QBN^{-1}B^*)}\, Qf\, ds .$$

Moreover the closed loop system corresponding to the feedback control (3.17) has a unique stationary solution (i.e., T-periodic solution for arbitrary T).

The stochastic version of a stationary problem may be the following.

We define

(3.21) $\quad U_{sad} = \{ u: u$ is F_t-measurable and stationary with $E|u(0)|^2 < +\infty$, there exists a stationary solution y of (3.1) with $E|y(0)|^2 < +\infty \}$.

We take

(3.22) $\quad J_s(u) = E[|My(0)|^2 + \langle Nu(0), u(0) \rangle]$

and minimize J_s over U_{sad}. Then we have :

Corollary 3.4. Assume the condition in Corollary 3.3. Then the feedback control (3.17) is optimal and $J_s(\bar{u}) = J(\bar{u})$ given by (3.18).

Similar results to Corollaries 3.3 and 3.4 can be found in [13] and [14].

Filtering problem and optimal control under partial observation can be found in [10]. Tracking problems can be also considered.

4. AN EXAMPLE

Consider the stochastic wave equation

$$
(4.1) \quad \begin{aligned} d(\partial y/\partial t) &= [-2\,\partial y/\partial t + \partial^2 y/\partial x^2]dt + \sin t \sin x \, dw\,, \quad 0 < x < \pi\,, \\ y(t,0) &= y(t,\pi) = 0\,. \end{aligned}
$$

We take $Y = H_0^1(o,\pi) \times L^2(0,\pi)$ and

$$A = \begin{bmatrix} 0 & I \\ -A_0 & -2I \end{bmatrix}, \quad D(A) = D(A_0) \times H_0^1(0,\pi)$$

(4.2)

$$A_0 = -d^2/dx^2 , \qquad D(A_0) = H^2(0,\pi) \cap H_0^1(0,\pi) .$$

Then A generates a C_0-semigroup

$$(4.3)\quad S(t) = e^{-t}\begin{bmatrix} \cos\sqrt{A_0-I}\,t + \sqrt{A_0-I}^{-1}\sin\sqrt{A_0-I}\,t , & \sqrt{A_0-I}^{-1}\sin\sqrt{A_0-I}\,t \\ -(\sqrt{A_0-I}^{-1} + \sqrt{A_0-I})\sin\sqrt{A_0-I}\,t , & \cos\sqrt{A_0-I}\,t - \sqrt{A_0-I}^{-1}\sin\sqrt{A_0-I}\,t \end{bmatrix}$$

where $\cos\sqrt{A_0-I}\,t\,y = \sum_{n=1}^{\infty} (2/\pi)\cos\sqrt{n^2-1}\,t \langle y,\sin nx\rangle \sin nx$ and $\sin\sqrt{A_0-I}\,t$ is defined in a similar manner with exception $\sqrt{n^2-1}^{-1}\sin\sqrt{n^2-1}\,t\big|_{n=1} \equiv t$.

The unique 2π-periodic solution of (4.1) is

$$(4.4)\qquad \begin{bmatrix} y(t) \\ \partial y(t)/\partial t \end{bmatrix} = \int_{-\infty}^{t} S(t-s) \begin{bmatrix} 0 \\ \sin s \sin x \end{bmatrix} dw(s) .$$

Now consider the controlled system

$$(4.5)\qquad \begin{aligned} d(\partial y/\partial t) &= [\partial^2 y/\partial x^2 + u + \sin t \sin x\,]dt + (\partial y/\partial t)dw \\ y(t,0) &= y(t,\pi) = 0 \end{aligned}$$

$$(4.6)\qquad J(u) = \overline{\lim_{L\to\infty}} \frac{1}{L} E\int_0^L [2|\partial y/\partial t|_Z^2 + |u|_Z^2\,]\,dt, \qquad Z = L^2(0,\pi) .$$

We take Y as before but A is replaced by

$$\overline{A} = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \qquad D(\overline{A}) = D(A) .$$

We choose $U = L^2(0,\pi)$, $Bu = \begin{bmatrix} 0 \\ u \end{bmatrix}$, $N = I$ and $G = M_0 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$.

Then $M^*M = 2M_0$, $BB^* = M_0$ and $\overline{A}^* = -\overline{A}$. Thus the algebraic Riccati equation (3.16) is

$$-\overline{A}Q + Q\overline{A} - QM_0Q + 2M_0 + M_0^*QM_0 = 0 ,$$

whose nonnegative solution is $Q = 2I$. Hence the generator of the optimal closed system is

$$\overline{A} - BN^{-1}B^*Q = \overline{A} - 2M_0 = A .$$

Note that

$$S^*(t) = e^{-t} \begin{bmatrix} \cos\sqrt{A_0-1}\,t + \sqrt{A_0-1}^{\,-1}\sin\sqrt{A_0-1}\,t \,, & A_0^{-1}\sqrt{A_0-1}^{\,-1}\sin\sqrt{A_0-1}\,t \\ -A_0(\sqrt{A_0-1}^{\,-1} + \sqrt{A_0-1})\sin\sqrt{A_0-1}\,t \,, & \cos\sqrt{A_0-1}\,t - \sqrt{A_0-1}^{\,-1}\sin\sqrt{A_0-1}\,t \end{bmatrix} .$$

The optimal control is

$$\overline{u} = -B^* \{ Q \begin{bmatrix} y \\ \partial y/\partial t \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \} = -2\,\partial y/\partial t - r_2 ,$$

where

$$r(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} = \int_t^\infty S^*(s-t) \begin{bmatrix} 0 \\ \sin s \sin x \end{bmatrix} ds .$$

Hence

$$r_2(t) = \int_0^\infty e^{-s}(1-s)\sin(s+t)\sin x \, ds$$

$$= \frac{1}{2}(\sin t - \cos t)\sin x \ .$$

The minimal cost is

$$J(\bar{u}) = \frac{1}{2\pi}\int_0^{2\pi} [2\langle r(s), \begin{bmatrix} 0 \\ \sin s \quad \sin x \end{bmatrix} \rangle - |r_2(s)|^2] ds$$

$$= \frac{1}{16}\int_0^{2\pi} (3\sin^2 s - \cos^2 s - 2\sin s \cos s)\, ds$$

$$= \frac{\pi}{8} \ .$$

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