A NOTE ON BMO-MARTINGALES

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Let \((\Omega, \mathcal{F}, P; (\mathcal{F}_t))\) be a probability system where the filtration \((\mathcal{F}_t)\) satisfies the usual conditions. We now consider two important subclasses of BMO, namely the class \(L^\infty\) of all bounded martingales and the class \(H^\infty\) of all martingales with bounded quadratic variation. There is not always an inclusion relation between these classes. In fact, if \(B=(B_t)\) is a one dimensional Brownian motion starting at 0, then \((B_{t \wedge 1}) \in H^\infty \setminus L^\infty\) clearly. On the other hand, the process \(B\) stopped at \(\tau\), where \(\tau = \min(\{t: |B_t| = 1\})\), belongs to \(L^\infty \setminus H^\infty\) (see [2]).

Our aim is to prove the following.

THEOREM. Suppose the sample continuity of any martingale adapted to \((\mathcal{F}_t)\). Then the following properties are equivalent:

(a) \(\text{BMO} = L^\infty\).
(b) \(\text{BMO} = H^\infty\).
(c) \(\mathcal{F}_t = \mathcal{F}_0\) for every \(t\).

Furthermore, excepting such a trivial case, \(H^\infty U L^\infty\) is not dense in BMO.

Recall that BMO is the class of those uniformly integrable martingales \(M\) which satisfy \(\|M\|_{\text{BMO}} = \sup_T \|E[(M_{\infty} - M_{T-})^2 |\mathcal{F}_T]\|^{1/2} \leq \infty\).
where the supremum is taken over all stopping times $T$. It is well-known that BMO is Banach space with the norm $\|M\|_{\text{BMO}}$.

PROOF. Firstly, we shall establish the implication (a)$\implies$(c). In order to see (c), it suffices to prove that for any martingale $M$, almost all sample functions of $M$ are constant, and by using the usual stopping argument we may assume $M \in \text{BMO}$.

Suppose now $\text{BMO}=L^\infty$. Then the two norms $\|M\|_{\text{BMO}}$ and $\|M\|_\infty$ on BMO are equivalent by the Closed Graph Theorem. So there exists a constant $C>0$, depending only on $M$, such that $\|K \cdot M\|_\infty<C$ for any predictable process $K=(K_t,\mathcal{F}_t)$ with $|K| \leq 1$. Here $K \cdot M$ denotes the stochastic integral of $K$ relative to $M$. We show below that the negation of (c) causes a contradiction. If we deny (c), then there exist $t>0$ and a partition $\Delta:0=t(0)<t(1)<\cdots<t(n)=t$ of $[0,t]$ such that $P(A)>0$ where $A=\{\sum_{j=1}^n |M_{t(j)}-M_{t(j-1)}|>2C\}$. Let now

$$B_{j,\varepsilon(j)}=\begin{cases} (M_{t(j)}-M_{t(j-1)}) \geq 0 & \text{if } \varepsilon(j)=1, \\ (M_{t(j)}-M_{t(j-1)}) < 0 & \text{if } \varepsilon(j)=-1 \end{cases}$$

for $j=1,2,\cdots,n$. Since $A=\bigcup_{1 \leq j \leq n, \varepsilon(j)=\pm 1} A \cap B_{1,\varepsilon(1)} \cap \cdots \cap B_{n,\varepsilon(n)}$, we have for some $\varepsilon^*(j)\ (1 \leq j \leq n)$

$$P(A \cap B_{1,\varepsilon^*(1)} \cap \cdots \cap B_{n,\varepsilon^*(n)})>0.$$ Then the process $K$ defined by $K_s=\sum_{j=1}^n \varepsilon^*(j) I_{[t(j-1),t(j))]}(s)$ is a predictable process with $|K| \leq 1$, so that $\|K \cdot M\|_\infty \leq C$ must follow. On the contrary, we find

$$\sum_{j=1}^n |M_{t(j)}-M_{t(j-1)}|>2C$$
on the set $A \cap B_1 \cap \ldots \cap B_n \cap \mathcal{E}^*(1) \cap \mathcal{E}^*(n)$. Thus (a) implies (c). The implication (c) $\Rightarrow$ (b) is trivial. Finally, to prove the implication (b) $\Rightarrow$ (a), let us suppose $\text{BMO} \neq \text{L}^\infty$. Then by the result of Dellacherie, Meyer and Yor $\text{L}^\infty$ is not dense in $\text{BMO}$, and further Kazamaki and Shiota have recently shown in [2] that the $\text{BMO}$-closure of $\text{L}^\infty$ contains $\text{H}^\infty$. Therefore, combining these results, we have $\text{BMO} \neq \text{H}^\infty$. That is, the contraposition of the implication (b) $\Rightarrow$ (a) is established. The latter half of the theorem follows at the same time. This completes the proof.

We now exemplify that $\text{H}^\infty$ as well as $\text{L}^\infty$ is not always closed in $\text{BMO}$ under the same assumption as in the theorem. For that, consider the identity mapping $S$ of $R_+$ onto $R_+$. Let $\mu$ be the probability measure on $R_+$ defined by $\mu(S \varepsilon dx) = \sqrt{2/(\pi e^x^2/2) dx}$ and $\mathcal{G}_t$ be the $\mu$-completion of the Borel field generated by $S \vee t$. Then $(R_+, \mathcal{G}, \mu: (\mathcal{G}_t))$, where $\mathcal{G} = \mathcal{G}_t$, is a probability system. Clearly, $S$ is a stopping time over $(\mathcal{G}_t)$. We next consider in the usual way a probability system $(\Omega, \mathcal{F}, P: (\mathcal{F}_t))$ by taking the product of the system $(R_+, \mathcal{G}, \mu: (\mathcal{G}_t))$ with another system $(\Omega', \mathcal{F}', P': (\mathcal{F}'_t))$ which carries a one dimensional Brownian motion $B = (B_t)$ starting at 0. Then the filtration $(\mathcal{F}_t)$ satisfies the usual conditions and $S$ is also a stopping time over this filtration. Let $M$ denote the process $B$ stopped at $S$. It is a continuous martingale over $(\mathcal{F}_t)$ such that $\langle M \rangle_t = t \wedge S$ where $\langle M \rangle$ denotes the continuous increasing process associated with $M$. We first verify $M \in \text{BMO}$. Since $(S \vee t)$ is an $\mathcal{F}_t$-atom, we have
\[
E[\langle M\rangle_\infty - \langle M\rangle_t | \mathcal{F}_t] = E[S-t | \mathcal{F}_t] 1_{t<S}
\leq (\int_t^\infty e^{-x^2/2} dx)^{-1} (\int_t^\infty (x-t) e^{-x^2/2} dx),
\]
which converges to 0 as \( t \to \infty \). That is, there is a constant \( C > 0 \) such that \( E[\langle M\rangle_\infty - \langle M\rangle_t | \mathcal{F}_t] \leq C \) for every \( t \). In our setting, this yields that \( E[\langle M\rangle_\infty - \langle M\rangle_T | \mathcal{F}_T] \leq C \) for every stopping time \( T \). Thus \( M \in BMO \). As a matter of course, we have \( M \in H^\infty \). Secondly, let \( M^{(n)} = B^{n \wedge S} \) (\( n = 1, 2, \ldots \)). Then \( M^{(n)} \in H^\infty \). Since \( \langle M^{(n)} \rangle_t = t \wedge S - t \wedge n \wedge S \), we find
\[
E[\langle M^{(n)} \rangle_\infty - \langle M^{(n)} \rangle_t | \mathcal{F}_t] 
\leq (\int_{t \wedge n}^\infty e^{-x^2/2} dx)^{-1} (\int_{t \wedge n}^\infty (x-t \wedge n) e^{-x^2/2} dx),
\]
from which \( M^{(n)} \) converges in BMO to \( M \) as \( n \to \infty \). Consequently, \( M \in H^\infty \setminus H^\infty \).

REFERENCES


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