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A NOTE ON BMO-MARTINGALES

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Let $(\Omega,\mathfrak{F},P;(\mathfrak{F}_t))$ be a probability system where the filtration $(\mathfrak{F}_t)$ satisfies the usual conditions. We now consider two important subclasses of BMO, namely the class L^∞ of all bounded martingales and the class H^∞ of all martingales with bounded quadratic variation. There is not always an inclusion relation between these classes. In fact, if $B=(B_t)$ is a one dimensional Brownian motion starting at 0, then $(B_{t\wedge 1})\in H^\infty\setminus L^\infty$ clearly. On the other hand, the process B stopped at τ, where $\tau=\min\{t:|B_t|=1\}$, belongs to $L^\infty\setminus H^\infty$ (see [2]).

Our aim is to prove the following.

THEOREM. Suppose the sample continuity of any martingale adapted to $(\mathfrak{F}_t)$. Then the following properties are equivalent:

(a) $BMO=L^\infty$.

(b) $BMO=H^\infty$.

(c) $\mathfrak{F}_t=\mathfrak{F}_0$ for every t.

Furtheremore, excepting such a trivial case, $H^\infty\cup L^\infty$ is not dense in BMO.

Recall that BMO is the class of those uniformly integrable martingales M which satisfy $\|M\|_{BMO}=\sup_T\|E[(M_\infty-M_{T-})^2|\mathfrak{F}_T]^{1/2}\|_\infty<\infty$

where the supremum is taken over all stopping times T. It is well-known that BMO is Banach space with the norm $\|M\|_{BMO}$.

PROOF. Firstly, we shall establish the implication (a)→(c). In order to see (c), it suffices to prove that for any martingale M, almost all sample functions of M are constant, and by using the usual stopping argument we may assume $M \in BMO$. Suppose now $BMO=L^{\infty}$. Then the two norms $\|M\|_{BMO}$ and $\|M\|_{\infty}$ on BMO are equivalent by the Closed Graph Theorem. So there exists a constant $C>0$, depending only on M, such that $\|K \circ M\|_{\infty} < C$ for any predictable process $K=(K_t, \mathfrak{F}_t)$ with $|K| \leq 1$. Here $K \circ M$ denotes the stochastic integral of K relative to M. We show below that the negation of (c) causes a contradiction. If we deny (c), then there exist $t>0$ and a partition $\Delta: 0=t(0)<t(1)<\cdots<t(n)=t$ of $[0,t]$ such that $P(A)>0$ where $A=\{\sum_{j=1}^{n} |M_{t(j)}-M_{t(j-1)}|>2C\}$. Let now

$$B_{j,\varepsilon(j)}=\begin{cases} \{M_{t(j)}-M_{t(j-1)} \geq 0\} & \text{if } \varepsilon(j)=1, \\ \{M_{t(j)}-M_{t(j-1)} < 0\} & \text{if } \varepsilon(j)=-1 \end{cases}$$

for $j=1,2,\cdots,n$. Since $A=\bigcup_{1\leq j\leq n, \varepsilon(j)=\pm 1} A \cap B_{1,\varepsilon(1)} \cap \cdots \cap B_{n,\varepsilon(n)}$, we have for some $\varepsilon^*(j) (1 \leq j \leq n)$

$$P(A \cap B_{1,\varepsilon^*(1)} \cap \cdots \cap B_{n,\varepsilon^*(n)})>0.$$

Then the process K defined by $K_s=\sum_{j=1}^{n} \varepsilon^*(j) I_{]t(j-1),t(j)]}(s)$ is a predictable process with $|K| \leq 1$, so that $\|K \circ M\|_{\infty} \leq C$ must follow. On the contrary , we find

$$(K \circ M)_t=\sum_{j=1}^{n} |M_{t(j)}-M_{t(j-1)}|>2C$$

on the set $A\cap B_{1,\varepsilon^*(1)}\cap\cdots\cap B_{n,\varepsilon^*(n)}$. Thus (a) implies (c). The implication (c)→(b) is trivial. Finally, to prove the implication (b)→(a), let us suppose $BMO\neq L^\infty$. Then by the result of Dellacherie, Meyer and Yor L^∞ is not dense in BMO, and further Kazamaki and Shiota have recently shown in [2] that the BMO-closure of L^∞ contains H^∞. Therefore, combining these results, we have $BMO\neq H^\infty$. That is, the contraposition of the implication (b)→(a) is established. The latter half of the theorem follows at the same time. This completes the proof.

We now exemplify that H^∞ as well as L^∞ is not alwayas closed in BMO under the same assumption as in the theorem. For that,consider the identity mapping S of R_+ onto R_+. Let μ be the probability measure on R_+ defined by $\mu(S\in dx)=\sqrt{2/\pi}e^{-x^2/2}dx$ and $\mathfrak{G}_t$ be the μ-completion of the Borel field generated by $S\wedge t$. Then $(R_+,\mathfrak{G},\mu:(\mathfrak{G}_t))$, where $\mathfrak{G}=\bigvee_{t\geq 0}\mathfrak{G}_t$, is a probability system. Clearly, S is a stopping time over $(\mathfrak{G}_t)$. We next consider in the usual way a probability system $(\Omega,\mathfrak{F},P:(\mathfrak{F}_t))$ by taking the product of the system $(R_+,\mathfrak{G},\mu:(\mathfrak{G}_t))$ with another system $(\Omega',\mathfrak{F}',P':(\mathfrak{F}'_t))$ which carries a one dimensional Brownian motion $B=(B_t)$ starting at 0. Then the filtration $(\mathfrak{F}_t)$ satisfies the usual conditions and S is also a stopping time over this filtration. Let M denote the process B stopped at S. It is a continuous martingale over $(\mathfrak{F}_t)$ such that $\langle M\rangle_t=t\wedge S$ where $\langle M\rangle$ denotes the continuous increasing process associated with M. We first verfy $M\in BMO$. Since $\{S>t\}$ is an $\mathfrak{F}_t$-atom, we have

$$E[\langle M\rangle_\infty-\langle M\rangle_t|\mathfrak{F}_t]=E[S-t|\mathfrak{F}_t]I_{\{t<S\}}$$
$$\le(\int_t^\infty e^{-x^2/2}dx)^{-1}(\int_t^\infty (x-t)e^{-x^2/2}dx),$$

which converges to 0 as $t\to\infty$. That is, there is a constant $C>0$ such that $E[\langle M\rangle_\infty-\langle M\rangle_t|\mathfrak{F}_t]\le C$ for every t. In our setting, this yields that $E[\langle M\rangle_\infty-\langle M\rangle_T|\mathfrak{F}_T]\le C$ for every stopping time T. Thus $M\in BMO$. As a matter of course, we have $M\in H^\infty$. Secondly, let $M^{(n)}=B^{n\wedge S}$ $(n=1,2,\cdots)$. Then $M^{(n)}\in H^\infty$. Since $\langle M^{(n)}-M\rangle_t=t\wedge S-t\wedge n\wedge S$, we find

$$E[\langle M^{(n)}-M\rangle_\infty-\langle M^{(n)}-M\rangle_t|\mathfrak{F}_t]$$
$$\le(\int_{t\vee n}^\infty e^{-x^2/2}dx)^{-1}(\int_{t\vee n}^\infty (x-t\vee n)e^{-x^2/2}dx),$$

from which $M^{(n)}$ converges in BMO to M as $n\to\infty$. Consequently, $M\in\overline{H^\infty}\setminus H^\infty$.

REFERENCES

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