

DERIVATIONS IN MATRIX SUBRINGS

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This note is an abstract of the author's papers [1], [2] and [3]. Let R be a ring with identity and let $M_n(R)$ denotes the ring of $n \times n$ matrices over R . We say that a subring P of $M_n(R)$ is special with the relation ω if P is of the form

$$P = \{ A \in M_n(R); A_{ij} = 0, \text{ for } (i,j) \notin \omega \},$$

where ω is a relation (reflexive and transitive) on the set $\{1, \dots, n\}$.

We describe in this note all derivations, R -derivations and higher R -derivations of the ring P .

I. DERIVATIONS AND HIGHER DERIVATIONS IN A RING

1. Derivations. Let P be a ring with identity. An additive mapping $d: P \rightarrow P$ is called a derivation (or an ordinary derivation) of P if $d(xy) = d(x)y + xd(y)$, for any $x, y \in P$. We denote by $D(P)$ the set of all derivations of P . If d and d' are derivations of P then the mapping $d + d'$ is also a derivation of P , so $D(P)$ is an abelian group.

Let $a \in P$ and let $d_a: P \rightarrow P$ be a mapping defined by $d_a(x) = ax - xa$, for any $x \in P$. Then d_a is a derivation of P .

Let $d \in D(P)$. If there exists an element $a \in P$ such that $d = d_a$ then d is called an inner derivation (with respect to a) of P . We denote by $ID(P)$ the set of all inner derivations of P . $ID(P)$ is a (normal) subgroup of $D(P)$.

We shall say that P is an NS-ring if $ID(P) = D(P)$, that is, a ring P is an NS-ring if and only if every derivation of P is inner.

2. Higher derivations. Let P be a ring with identity and let S be a segment of $N = \{0, 1, \dots\}$, that is, $S = N$ or $S = \{0, 1, \dots, s\}$ for some $s \geq 0$. A family $d = (d_m)_{m \in S}$ of mappings $d_m: P \rightarrow P$ is called a derivation of order s of P (where $s = \sup(S) \leq \infty$) if the following properties are satisfied:

$$(1) \quad d_m(x + y) = d_m(x) + d_m(y),$$

$$(2) \quad d_m(xy) = \sum_{i+j=m} d_i(x)d_j(y),$$

$$(3) \quad d_0 = \text{id}_P,$$

for any $x, y \in P$, $m \in S$.

The set of derivations of order s of P , denoted by $D_s(P)$, is the group under the multiplication $*$ defined by the formula

$$(d * d')_m = \sum_{i+j=m} d_i \circ d'_j,$$

where $d, d' \in D_s(P)$ and $m \in S$.

Let $\delta: P \rightarrow P$ be a mapping. Then δ is an ordinary derivation of P if and only if (id_P, δ) is a derivation of order 1 of P . Therefore we may identify: $D(P) = D_1(P)$.

3. Examples of higher derivations. Let P and S be as in Section 2.

Example 3.1. Let $a \in P$, $d_0 = \text{id}_P$, and $d_m(x) = a^m x - a^{m-1} xa$, for $m \geq 1$, $x \in P$. Then $d = (d_m)_{m \in S}$ belongs to $D_s(P)$.

Example 3.2. Let $d \in D_s(P)$, $k \in S - \{0\}$ and let $\delta = (\delta_m)_{m \in S}$ be the family of mappings from P to P defined by

$$\delta = \begin{cases} 0, & \text{if } k \nmid m \\ d_r, & \text{if } m = rk. \end{cases}$$

Then $\delta \in D_s(P)$.

The derivation d (of order s) from Example 3.1 will be denoted by $[a, 1]$. The derivation δ (of order s) from Example 3.2, for $d = [a, 1]$

will be denoted by $[a, k]$.

4. Inner derivations. Let P and S be as in Section 2 and let $\underline{a} = (a_m)$ (where $m \in S$) be a sequence in P . Denote by $\Delta(\underline{a})$ the element in $D_S(P)$ defined by

$$\Delta(\underline{a})_m = ([a_1, 1] * [a_2, 2] * \dots * [a_m, m])_m,$$

for any $m \in S$.

Definition 4.1. Let $d \in D_S(P)$. If there exists a sequence $\underline{a} = (a_m)$ of elements of P such that $d = \Delta(\underline{a})$ then d is called an inner derivation of order s of P .

Denote by $ID_S(P)$ the set of inner derivations of order s of P .

Proposition 4.2. $ID_S(P)$ is a normal subgroup of $D_S(P)$.

Proposition 4.3. The following properties are equivalent

- (1) P is an NS-ring,
- (2) $ID_S(P) = D_S(P)$, for any $0 < s \leq \infty$,
- (3) $ID_S(P) = D_S(P)$, for some $0 < s \leq \infty$.

5. R-derivations. Let $R \subset P$ be rings with identity and let S be a segment of N . If a derivation (of order s) $d \in D_S(P)$ satisfies the condition

$$d_m(r) = 0,$$

for all $m \in S - \{0\}$, $r \in R$, then d is called R-derivation of order s of P , and the set of all such derivations is denoted by $D_S^R(P)$.

We define similarly an ordinary R-derivation, an inner R-derivation, an inner R-derivation of order s and also, we define similarly the groups $D^R(P)$, $ID^R(P)$ and $ID_S^R(P)$.

The group $D_S^R(P)$ is a subgroup of $D_S(P)$ and the group $ID_S^R(P)$ is a normal subgroup of $D_S^R(P)$.

We shall say that P is an NS-ring over R if $ID^R(P) = D^R(P)$.

Proposition 5.1. The following properties are equivalent

- (1) P is an NS-ring over R ,
- (2) $ID_s^R(P) = D_s^R(P)$, for any $0 < s \leq \infty$,
- (3) $ID_s^R(P) = D_s^R(P)$, for some $0 < s \leq \infty$.

II. SPECIAL SUBRINGS OF MATRIX RINGS

6. Notices. Let R be a ring with identity, n a fixed natural number and ω a reflexive and transitive relation on the set $I_n = \{1, \dots, n\}$. We denote by $M_n(R)$ the ring of $n \times n$ matrices over R and by $Z(R)$ the center of R . Moreover, we use the following conventios:

$F(R)$ = the set of mappings from R to R ,

$\bar{\omega}$ = the smallest equivalence relation on I_n containing ω ,

T_ω = a fixed set of representatives of equivalence classes of $\bar{\omega}$,

A_{ij} = ij -coefficient of a matrix A ,

E^{ij} = the element of the standard basis of $M_n(R)$,

$M_n(R)_\omega = \{A \in M_n(R); A_{ij} = 0, \text{ for } (i,j) \notin \omega\}$.

The set $P = M_n(R)_\omega$ is a subring of $M_n(R)$ called a special subring with the relation ω . Every special subring contains the ring R (via injection $r \mapsto \bar{r}$, where \bar{r} is the diagonal matrix whose all coefficients on the diagonal are equal to $r \in R$).

7. Transitive mappings and regular relations. Let G be an abelian group. A mapping $f: \omega \rightarrow G$ will be called transitive iff

$$f(a,c) = f(a,b) + f(b,c),$$

for any $a_\omega b$ and $b_\omega c$.

If $f: \omega \rightarrow G$ is a transitive mapping then we denote by $[f, _]$

(in the case $G = R$) the mapping from ω to $F(R)$ defined by

$$[f, _](a,b)(r) = f(a,b)r - rf(a,b),$$

for $a\omega b$ and $r \in R$. Clearly, $[f, _]$ is transitive too.

We shall say that f is trivial if there exists a mapping $\sigma: I_n \rightarrow G$ such that

$$f(a,b) = \sigma(a) - \sigma(b),$$

for any $a\omega b$. Moreover, we shall say that f is quasi-trivial (in the case $G = R$) if $[f, _]$ is trivial.

Every trivial transitive mapping from ω to R is quasi-trivial, but the converse is not necessarily true.

Proposition 7.1. Let $f: \omega \rightarrow R$ be a quasi-trivial transitive mapping. Then there exists a unique mapping $\tau: I_n \rightarrow F(R)$ such that

$$(1) \quad [f, _](i,j) = \tau(i) - \tau(j), \text{ for all } i\omega j,$$

$$(2) \quad \tau(t) = 0, \text{ for } t \in T_\omega.$$

Moreover, $\tau(1), \dots, \tau(n)$ are inner derivations of R .

Definition 7.2. The relation ω is called regular over an abelian group G if every transitive mapping from ω to G is trivial.

8. The graph $\Gamma(\omega)$ and homology groups. Let \div be the equivalence relation on I_n defined by: $x \div y$ iff $x\omega y$ and $y\omega x$. Denote by $[x]$ the equivalence class of $x \in I_n$ with respect to \div and let I'_n be the set of all equivalence classes. We define a relation ω' of partial order on I'_n as follows:

$$[x]\omega'[y] \text{ iff } x\omega y.$$

We will denote the pair (I'_n, ω') by $\Gamma(\omega)$ and call it the graph of ω . Elements of I'_n we call vertices of $\Gamma(\omega)$ and pairs (a,b) , where $a\omega'b$ and $a \neq b$, arrows of $\Gamma(\omega)$.

Let us imbed the set of the vertices of $\Gamma(\omega)$ in an Euclidean space of a sufficiently high dimension so that the vertices will be

linearly independent.

If a_0, a_1, \dots, a_k are elements of Γ'_n such that $a_i \omega' a_{i+1}$ and $a_i \neq a_{i+1}$ for $i=0, 1, \dots, k-1$, then by (a_0, a_1, \dots, a_k) we denote the k -dimensional simplex with vertices a_0, \dots, a_k . The union of all 0, 1, 2 or 3-dimensional such simplicies we will denote also by $\Gamma(\omega)$. Therefore, $\Gamma(\omega)$ is a simplicial complex of dimension ≤ 3 .

Let $C_k(\omega)$, for $k=0, 1, 2, 3$, be the free abelian group whose free generators are k -dimensional simplicies of $\Gamma(\omega)$. We have the following standard complex of abelian groups:

$$0 \longrightarrow C_3(\omega) \xrightarrow{\partial_3} C_2(\omega) \xrightarrow{\partial_2} C_1(\omega) \xrightarrow{\partial_1} C_0(\omega) \longrightarrow 0$$

where

$$\partial_1(a, b) = (b) - (a),$$

$$\partial_2(a, b, c) = (b, c) - (a, c) + (a, b),$$

$$\partial_3(a, b, c, d) = (b, c, d) - (a, c, d) + (a, b, d) - (a, b, c).$$

Then $H_1(\Gamma(\omega)) = \text{Ker} \partial_1 / \text{Im} \partial_2$, $H_2(\Gamma(\omega)) = \text{Ker} \partial_2 / \text{Im} \partial_3$ and (by the Kunneth formulas)

$$H^1(\Gamma(\omega), G) = \text{Hom}(H_1(\Gamma(\omega)), G),$$

for an arbitrary abelian group G .

III DERIVATIONS IN SPECIAL SUBRINGS

9. Examples of derivations. Let $P = M_n(\mathbb{R})$ be a special subring of $M_n(\mathbb{R})$.

Example 9.1. Assume that $f: \omega \longrightarrow R$ is a quasi-trivial transitive mapping and denote by Δ^f the mapping from P to P defined by

$$\Delta^f(B)_{pq} = B_{pq} f(p, q) + \tau_f(p)(B_{pq}),$$

for $B \in P$, $p\omega q$, where τ_f is the mapping τ from Proposition 7.1.

Then Δ^f is a derivation of P . Moreover Δ^f is inner if and only if f is trivial.

Example 9.2. Let $\delta = \{\delta_t; t \in T_\omega\}$ be a set of derivations of R . Denote by θ_δ the mapping from P to P defined by

$$\theta_\delta(B)_{pq} = \delta_t(B_{pq}),$$

for $B \in P, pq$, where $t \in T_\omega$ such that $p\omega t$.

Then θ_δ is a derivation of P . Moreover, θ_δ is inner if and only if δ_t is inner for any $t \in T_\omega$.

10. A description of $D(P)$. Let $P = M_n(R)_\omega$ be a special subring of $M_n(R)$. The following theorem describes all derivations of P .

Theorem 10.1. Every derivation d of P has a unique representation:

$$d = d_A + \Delta^f + \theta_\delta,$$

where (1) d_A is an inner derivation of P with respect to a matrix $A \in P$ such that $A_{pp} = 0$, for $p=1, \dots, n$,

(2) $f: \omega \rightarrow R$ is a quasi-trivial transitive mapping and Δ^f is the derivation from Example 9.1,

(3) $\delta = \{\delta_t; t \in T_\omega\}$ is a set of derivations of R and θ_δ is the derivation from Example 9.2.

The next theorem describes special subrings which are NS-rings.

Theorem 10.2. The following conditions are equivalent

- (1) P is an NS-ring,
- (2) R is an SN-ring and the relation ω is regular over $Z(R)$.

11. R-derivations of $M_n(R)_\omega$. Let $P = M_n(R)_\omega$ be a special subring of $M_n(R)$.

Example 11.1. Let $f: \omega \rightarrow Z(R)$ be a transitive mapping and denote by Δ^f the mapping from P to P defined by $\Delta^f(B)_{pq} = f(p,q)B_{pq}$, for $B \in P$ and $p\omega q$. Then Δ^f is an R-derivation of P . Moreover Δ^f is

inner if and only if f is trivial.

Theorem 11.2. Any R -derivation d of P has a unique representation

$$d = d_A + \Delta^f,$$

where (1) d_A is an inner derivation of P with respect to a matrix $A \in P$ such that $A_{ij} \in Z(R)$ for $i, j=1, \dots, n$, and $A_{ii} = 0$ for $i=1, \dots, n$,

(2) $f: \omega \rightarrow Z(R)$ is a transitive mapping and Δ^f is the derivation from Example 11.1.

Theorem 11.3. The following conditions are equivalent

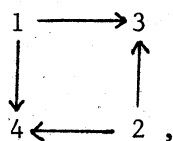
- (1) P is an NS-ring over R ,
- (2) ω is regular over $Z(R)$.

Corollary 11.4. If d and δ are R -derivations of R then the derivation $d\delta - \delta d$ is inner.

Corollary 11.5. If d is an R -derivation of P then $d(Z(R)) = 0$.

Corollary 11.6. If d is an R -derivation of P and U is an ideal of P then $D(U) \subseteq U$.

12. An example of non-inner R -derivation. For $n \leq 3$ every relation ω (reflexive and transitive) on I_n is regular over any group. Therefore (by Theorem 11.3), in this case any special subring of $M_n(R)$ has only inner R -derivations. For $n=4$ it is not true. Let ω_0 be the relation on $I_4 = \{1, 2, 3, 4\}$ defined by the graph



that is, $\omega_0 = \{(1,1), (2,2), (3,3), (4,4), (1,3), (1,4), (2,3), (2,4)\}$.

Denote by $S_4(R)$ the special subring of $M_4(R)$ with the relation ω_0

Then we have

$$S_4(R) = \begin{bmatrix} R & 0 & R & R \\ 0 & R & R & R \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix}.$$

Consider the mapping $d: S_4(R) \rightarrow S_4(R)$ defined by

$$d\left(\begin{bmatrix} x_{11} & 0 & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{44} \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then d is an R -derivation of $S_4(R)$ and d is not inner.

In [1] there is a description of the group $D^R(S_4(R))$. Note one of the properties of R -derivations of the ring $S_4(R)$.

Corollary 12.1. If d_1 and d_2 are R -derivations of $S_4(R)$ then the composition $d_1 d_2$ is also R -derivation of $S_4(R)$.

13. A description of regular relations. Let $P = M_n(R)_\omega$ be a special subring of $M_n(R)$.

We know, by Theorem 10.2, that P is an NS-ring if and only if R is an NS-ring and the relation ω is regular over $Z(R)$. We know also, by Theorem 11.3, that P is an NS-ring over R if and only if the relation ω is regular over $Z(R)$.

In this section we give some sufficient and necessary conditions for the relation ω to be regular over an abelian group.

We may reduce our consideration to the case where ω is connected (that is, for any $a, b \in I_n$ there exist elements $a_1, \dots, a_r \in I_n$ such that $a = a_1$, $b = a_r$ and $a_i \omega a_{i+1}$ or $a_{i+1} \omega a_i$, for $i=1, \dots, r-1$), because it is easy to show the following

Proposition 13.1. Let G be an abelian group. The relation ω is regular over G if and only if every connected component of ω is regular over G .

The next proposition says that we may also reduce our consideration to the case where ω is a partial order.

Proposition 13.2. ω is regular over G if and only if ω' (see Section 8) is regular over G .

Now we may give a description of regular relations.

Theorem 13.3. Assume that ω is a connected partial order. The following properties are equivalent:

- (1) ω is regular over some non-zero group,
- (2) ω is regular over every torsion-free group,
- (3) ω is regular over some torsion-free group,
- (4) ω is regular over Z ,
- (5) $H_1(\Gamma(\omega))$ is finite,
- (6) $H^1(\Gamma(\omega), G) = 0$, for any torsion-free group G .

Theorem 13.4. Assume that ω is connected partial order. The following properties are equivalent:

- (1) ω is regular over any group,
- (2) ω is regular over Q/Z ,
- (3) $H_1(\Gamma(\omega)) = 0$,
- (4) $H^1(\Gamma(\omega), G) = 0$, for any group G .

Theorem 13.5. Assume that ω is connected partial order, such that the order of the group $H_1(\Gamma(\omega))$ is equal to $m > 1$. Let G be an abelian group. The following properties are equivalent:

- (1) ω is regular over G ,
- (2) G is an m -torsion-free group,
- (3) $H^1(\Gamma(\omega), G) = 0$.

Corollary 13.6. Let $P = M_n(\mathbb{R})_\omega$ be a special subring of $M_n(\mathbb{R})$. The

following properties are equivalent

- (1) Every R-derivation of P is inner,
- (2) The relation ω is regular over $Z(R)$,
- (3) $H^1(\Gamma(\omega), Z(R)) = 0$.

14. Examples. Let $P = M_n(R)_\omega$ where

- a) $n \leq 3$, or
- b) the graph $\Gamma(\omega)$ is a tree, or
- c) the graph $\Gamma(\omega)$ is a cone (that is, there exists $b \in I_n$

such that $b\omega a$ or $a\omega b$ for any $a \in I_n$), in particular $P = M_n(R)$ is the ring of $n \times n$ matrices over R, or P is the ring of triangular $n \times n$ matrices over R.

Then every R-derivation (or every derivation, if every derivation of R is inner) of P is inner.

By Theorem 13.5 it follows that there exist relations ω which are regular over some groups and which are not regular over another groups. In the paper [1] there is an example of such a relation ω (for $n=17$) that if R is 2-torsion-free ring then $P = M_n(R)_\omega$ is an NS-ring over R, and if $\text{char}(R) = 2$ then $P = M_n(R)_\omega$ is not an NS-ring over R.

IV HIGHER DERIVATIONS IN SPECIAL SUBRINGS

15. An example of higher derivations. Let $P = M_n(R)_\omega$ be a special subring of $M_n(R)$, S a segment of N, and let $\underline{d} = \{d^{(t)}; t \in T_\omega\}$ be a family of derivations of order s (where $s = \text{sup}(S)$) of the ring R.

Denote by $\theta(\underline{d})$ the sequence $(d_m)_{m \in S}$ of mappings from P to P defined by

$$d_m(A)_{ij} = d_m^{(\nu(i))}(A_{ij}),$$

for $m \in S$, $A \in P$, where $\nu: I_n \rightarrow T_\omega$ is the mapping: $\nu(p) = t$ iff $p\omega t$.

Then $\theta(\underline{d})$ is a derivation of order s of P . If $\underline{d} \neq 0$ then the derivation $\theta(\underline{d})$ is not an R -derivation.

In the next sections of this note we shall be interested only in R -derivations of order s of P .

16. Transitive mappings of order s . A sequence $f = (f_m)_{m \in S}$ of mappings $f_m: \omega \rightarrow Z(R)$ is called a transitive mapping of order s (from ω to R) if the following properties are satisfied:

- (1) $f_0(p, q) = 1$, for all $p \omega q$,
- (2) $f_m(p, r) = \sum_{i+j=m} f_i(p, q) f_j(q, r)$, for all $m \in S$ and $p \omega q$ and $q \omega r$.

If $f = (f_m)_{m \in S}$ is a transitive mapping of order s then

$$f_1(p, r) = f_1(p, q) + f_1(q, r),$$

for any $p \omega q \omega r$ so, $f_1: \omega \rightarrow Z(R)$ is a transitive mapping in the sense of Section 7.

17. R -derivations of order s . In this section we give a description of the group $D_S^R(P)$.

Example 17.1. Let $f = (f_m)_{m \in S}$ be a transitive mapping of order s from ω to $Z(R)$. Denote by Δ^f the sequence $(\Delta_m^f)_{m \in S}$ of mappings $\Delta_m^f: P \rightarrow P$ defined by the following formula:

$$\Delta_m^f(A)_{pq} = f_m(p, q) A_{pq},$$

for all $A \in P$ and $p \omega q$.

Then Δ^f is an R -derivation of order s of P .

Theorem 17.2. Every R -derivation d of order s of P has a unique representation:

$$d = \Delta(A) * \Delta^f,$$

where

(1) $\underline{A} = (A^{(m)})_{m \in S - \{0\}}$ is a sequence of matrices $A^{(m)} \in P \cap M_n(Z(R))$ such that $A_{ii}^{(m)} = 0$, for $i=1, \dots, n$, and $\Delta(\underline{A})$ is the inner derivation of order s with respect to \underline{A} ;

(2) f is a transitive mapping of order s from ω to R and Δ^f is the R -derivation from Example 17.1.

Corollary 17.3. If $d \in D_S^R(P)$ and U is an ideal of P then $d_m(U) \subset U$, for all $m \in S$.

Corollary 17.4. If $d \in D_S^R(P)$, then $d_m(Z(R)) = 0$, for all $m \in S - \{0\}$.

Corollary 17.5. Assume that there do not exist three different elements $a, b, c \in I_n$ such that $awbwc$. Let $d = (d_m)_{m \in S}$ be a sequence of mappings from P to P such that $d_0 = id_P$. Then d is an R -derivation of order s of P if and only if every mapping d_m (for $m \in S - \{0\}$) is an ordinary R -derivation of P .

18. Integrable derivations. Let $S = \{0, 1, \dots, s\}$, where $s < \infty$. Assume that S' is a segment of N such that $S \not\subset S'$. We say that an R -derivation $d \in D_S^R(P)$ is s' -integrable (where $s' = \sup(S') \leq \infty$) if there exists an R -derivation $d' = (d'_m)_{m \in S'}$ of order s' of P such that $d'_m = d_m$, for all $m \in S$.

In the paper [3] there are some necessary conditions for any R -derivation of order s of P to be s' -integrable, and there is an example of non-integrable R -derivation (In this example $n=17$ and $R = Z_2$). In this paper there are also proofs of the following two partial results:

Theorem 18.1. Let $s < s' \leq \beta$. If $H_2(\Gamma(\omega)) = 0$ and $H_1(\Gamma(\omega))$ is a free abelian group then every R -derivation of order s of P is s' -integrable.

Theorem 18.2. Assume that the homology group $H_1(\Gamma(\omega))$ is free abelian. Then

- (1) Every R-derivation of order $s < 3$ of P is 3-integrable.
- (2) If R is 2-torsion-free then every R-derivation of order < 5 of P is 5-integrable.
- (3) If R is 6-torsion-free then every R-derivation of order < 7 of P is 7-integrable.

R e f e r e n c e s

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