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Long Cycles through Specified Vertices in a Graph

Akira Saito
(斎藤 明)
Department of Electrical Communications
Tohoku University
Sendai, Miyagi 980
JAPAN

ABSTRACT

In this paper, we consider the length of the longest cycle through specified vertices. We show the following two results. (1) Let G be a k-connected graph of order at least $2k$ and circumference l. Suppose $m < k$. Then for any m vertices of G, G has a cycle which contains all of them and has length at least $\frac{k-m}{k}l + 2m$. (2) Let G be a 3-connected planar graph with circumference l. Then for any three vertices of G, there exists a cycle which contains all of them and has length at least $\frac{1}{4}l + 3$.

Here, we consider finite simple graphs. Let G be a graph. By Dirac's theorem[3] G has a cycle through specified k vertices. In [2] Dirac also showed that a 2-connected graph of order n and minimum degree at least d has a cycle of length at least $\min\{n, 2d\}$. Locke[4] and Voss[7] generalized his result by showing that under the same conditions the graph has a cycle of length at least $\min\{n, 2d\}$ which contains specified two vertices.

These results lead us to the following question: Does a k-connected graph have a long cycle through specified m vertices ($m \leq k$)? In this paper we investigate this question.

For basic graph-theoretic terminology, we refer the reader to [1]. Let G be a graph. The *circumference* of G, denoted by $\mathrm{cir}(G)$, is the length of the longest cycle of G. We denote by $w(G)$ the number of components of G. For $k \geq 0$ and $S \subset V(G)$, we call S a k-cutset if $w(G - S) \geq 2$ and $|S| = k$. We often identify a subgraph H of G with its vertex set $V(H)$. Especially, when x is a vertex of H, we write $x \in H$ instead of $x \in V(H)$. Furthermore, we write $|H|$ instead of $|V(H)|$. When we consider a cycle, we always give it an orientation. Let C^+ be the orientation of a cycle C and C^- be its reverse orientation. Let $C^+ = x_0, x_1, \ldots, x_{n-1}, x_n$ be a cycle. For x_i, $x_j \in C$, we define a subpaths $C^+[x_i, x_j]$ and $C^-[x_i, x_j]$ of C by

$$C^+[x_i, x_j] = x_i, x_{i+1}, \ldots, x_{j-1}, x_j,$$

and

$$C^-[x_i, x_j] = x_i, x_{i-1}, \ldots x_{j+1}, x_j.$$

We also define $C^+(x_i, x_j)$ and $C^-(x_i, x_j)$ by

$$C^+(x_i, x_j) = C^+[x_i, x_j] - \{x_i, x_j\},$$

and

$$C^-(x_i, x_j) = C^-[x_i, x_j] - \{x_i, x_j\}.$$

Furthermore, $C^+[x_i, x_j) = C^+[x_i, x_j] - \{x_j\}$. Subpaths $C^-[x_i, x_j)$, $C^+(x_i, x_j]$, $C^-(x_i, x_j]$ are defined similarly. Let $x_1, x_2, \ldots, x_s$ be a path. We denote by $\mathrm{end}(P)$ the set of endvertices of P; $\mathrm{end}(P) = \{x_1, x_s\}$. Let $P = x_1, x_2, \ldots, x_s$ and $Q = y_1, y_2, \ldots, y_t$ be paths such that $x_s = y_1$. We denote by $P \cdot Q$ the walk $x_1, x_2, \ldots, x_s = y_1, y_2, \ldots, y_t$.

Let $z \in V(G)$ and $S \subset V(G) - \{z\}$. A subgraph F of G is called a (z, S)-fan if F has the following decomposition $F = \cup_{i=1}^{k} P_i$, where
(1) each P_i is a path between z and $a_i \in S$, and
(2) $P_i \cap S = \{a_i\}$, and $P_i \cap P_j = \{z\}$ if $i \neq j$.
We call k the *size* of the fan F. The vertices $a_1, \ldots, a_k$ are called endvertices of F and the set of its endvertices is denoted by $\mathrm{end}(F)$. Since F is a tree, for any two vertices x, $y \in F$ the path in F which joins x and y is unique. We denote this path by $F[x, y]$. We define $F[x, y)$ by $F[x, y) = F[x, y] - \{y\}$. Paths $F(x, y]$ and $F(x, y)$ are defined similarly.

The following theorem is well-known, called the generalized Menger's theorem.

THEOREM A ([1, Theorem 6.7]). *Let G be a k-connected graph, $z \in V(G)$, and $S \subset V(G) - \{z\}$. Then G has a (z, S)-fan of size* $\min\{|S|, k\}$. ■

The following theorem was proved by Perfect[5].

THEOREM B (Perfect[5]). *Let G be a graph, $z \in V(G)$, and $S \subset V(G) - \{z\}$. Suppose G has two (z, S)-fans F_1 and F_2 of size k_1 and k_2, respectively. If $k_1 \leq k_2$, then G has a (z, S)-fan F' of size k_2 such that* $\mathrm{end}(F_1) \subset \mathrm{end}(F')$. ■

We use these two theorems in the proofs our results.

First, we show that the existence of long cycles through specified m vertices in a k-connected graph is assured if $m < k$. Note that a k-connected graph is hamiltonian if its order is at most $2k$, by Dirac's theorem.

THEOREM 1. *Let $k \geq 2$, $0 \leq m \leq k$ and G be a k-connected graph of order at least $2k$. For any m vertices $x_1, \ldots, x_m$ of G, there exists a cycle such that*
(1) $x_1, \ldots, x_m \in V(C)$, and
(2) $|C| \geq \frac{k-m}{k} \operatorname{cir}(G) + 2m$.

Recently, Seymour and Truemper sent me a proof which is simpler than the original one. We show their proof.

Proof (due to Seymour and Truemper). The proof is by induction on m. For $m = 1$, let $x \in V(G)$, and let C be a longest cycle in G. Since $|C| \geq 2k$,

$$\frac{k-1}{k}\operatorname{cir}(G) + 2 = |C| - \frac{|C|}{k} + 2 \leq |C|.$$

So we may assume $x \notin V(C)$. Now G has an (x, C)-fan of size k. The endvertices of F divide C into k paths, and any shortest one P of these paths, say $P = C^+[u, v]$ has length at most $\frac{1}{k}\operatorname{cir}(G)$. So $C^+[v, u] \cdot F[u, v]$ is a cycle which contains x and has length at least

$$|C| - \frac{\operatorname{cir}(G)}{k} + 2 = \frac{k-1}{k}\operatorname{cir}(G) + 2$$

as desired.

Suppose $m > 1$, and let C be a longest cycle containing at least $m-1$ members of S. By the induction hypothesis,

$$\begin{aligned} |C| &\geq \frac{k-m+1}{k}\operatorname{cir}(G) + 2(m-1) \\ &= \frac{k-m}{k}\operatorname{cir}(G) + 2m + \frac{\operatorname{cir}(G)}{k} - 2 \\ &\geq \frac{k-m}{k}\operatorname{cir}(G) + 2m. \end{aligned} \qquad (*)$$

So we may assume that exactly one member x of S does not lie on C. Since $\operatorname{cir}(G) \geq 2k$, $|C| \geq 2k$. So G has an (x, C)-fan of size k. The endvertices of F divide C into k paths. We call such a path *bad* if it contains some member of S internally, and we call it *good* if it is not bad. Let b represent the number of bad paths, and let L be the sum of lengths of the bad paths. Then some good path $P = C^+[u, v]$ has length at most

$$\frac{|C| - L}{k - b}$$

(, where $|C| \geq 2k$ and $k \geq m-1$). Keeping $|C|$ and k fixed, and under the conditions $L \geq 2b$ and $b \leq m-1$, this is maximized when $L = 2b$ and $b = m-1$. Hence,

$$|P| \leq \frac{|C| - 2(m-1)}{k-m+1}.$$

A cycle $C^+[v,u] \cdot F[u,v]$ contains S, and from (*) it has length at least

$$|C| - \frac{|C| - 2(m-1)}{k-m+1} + 2 \geq \frac{k-m}{k}\mathrm{cir}(G) + 2m$$

as desired. ■

Theorem 1 is sharp. Let, $k \geq 2$, $s \geq 1$, and $0 \leq m \leq k$. Let $H_0, H_1, \ldots, H_k$ and H_0' be graphs such that $H_1 \simeq \cdots \simeq H_k \simeq K_s$, $H_0 \simeq \overline{K_m}$ and $H_0' \simeq \overline{K_k}$. Suppose vertex sets $V(H_0), \ldots, V(H_k)$ and $V(H_0')$ are disjoint. Define $G(k,m,s)$ by $G(k,m,s) = (H_1 \cup \cdots \cup H_k \cup H_0) + H_0'$. Then $G(k,m,s)$ is k-connected, $|G(k,m,s)| = ks + k + m \geq 2k$, and $\mathrm{cir}(G(k,m,s)) = ks + k$. On the other hand, the length of the longest cycle through $V(H_0)$ is $(k-m)s + k + m$. The above example shows that large circumference does not assure the existence of long cycles through specified k vertices in k-connected graphs.

Next, we confine ourselves to planar graphs. Even if we consider only planar graphs, the length of the longest cycle through specified two vertices in a 2-connected graph is independent of its circumference. Let $C = x_0, x_1, \ldots, x_m = x_0$ be a cycle of length m ($m \geq 4$). Add a new vertex y and join yx_1 and yx_{m-1}. Then this graph has circumference m, but the unique cycle through y and x_0 has length four. On the other hand, by Tutte's theorem[6] 4-connected planar graphs are hamiltonian, and hence the length of the longest cycle through four specified vertices in a 4-connected planar graph is equal to its circumference. On a planar graph of connectivity three, we show the following theorem.

Theorem 2. *Let G be a 3-connected planar graph. Then any three vertices of G lie on a cycle of length at least $\frac{1}{4}\mathrm{cir}(G) + 3$.*

The proof of Theorem 2 is given by the following two lemmas.

Lemma 1. *Let G be a 3-connected planar graph. Then for any two vertices x, y, there exists a cycle C such that*
(1) $x, y \in V(C)$.
(2) $|C| \geq \frac{1}{2}\mathrm{cir}(G) + 2$.

Lemma 2. *Let G be a 3-connected planar graph, $x, y, z \in V(G)$ and C be a cycle of G such that $x, y \in V(C)$. Then there exists a cycle C' such that*
(1) $x, y, z \in V(C')$.
(2) $|C'| \geq \frac{1}{2}|C| + 2$.

Proof of Lemma 1. If G is hamiltonian, then the lemma clearly holds. So we may assume that G is not hamiltonian, which implies $|G| \geq 7$ and $\mathrm{cir}(G) \geq 6$. Let C be a longest cycle of G. We consider three cases.

Case 1. $\{x, y\} \subset V(C)$.

This case is trivial.

Case 2. $|\{x, y\} \cap V(C)| = 1$.

We may assume that $x \in V(C)$ and $y \notin V(C)$. Consider a (y, C)-fan F of size three. Let $\mathrm{end}(F) = \{y_1, y_2, y_3\}$. If $x \in \{y_1, y_2, y_3\}$, say $x = y_1$, then we have two cycles $C^+[x, y_2] \cdot F[y_2, x]$ and $C^-[x, y_2] \cdot F[y_2, x]$, one of which has length at least $\frac{1}{2}|C| + 2 = \frac{1}{2}\mathrm{cir}(G) + 2$ and contains both x and y. Next, assume $x \notin \{y_1, y_2, y_3\}$. We may assume $x \in C^+(y_3, y_1)$. Then one of the two cycles $C^+[y_3, y_2] \cdot F[y_2, y_3]$ and $C^-[y_1, y_2] \cdot F[y_2, y_1]$ has the desired properties.

Case 3. $\{x, y\} \cap V(C) = \emptyset$.

First, we show the following claims.

Claim 1. Suppose there exists a path P in G such that
(1) P joins two distinct vertices of C and P intersects C only at its endvertices.
(2) $x, y \in V(P)$.
Then the Lemma follows.

Proof. Let a and b be endvertices of P. Then one of the two cycles $P[a, b] \cdot C^+[b, a]$ and $P[a, b] \cdot C^-[b, a]$ satisfies the desired properties.

Claim 2. Suppose there exist two paths P and Q such that
(1) $V(P) \cap V(Q) = \emptyset$.
(2) Both P and Q join two vertices of C.
(3) $V(P) \cap V(C) = \mathrm{end}(P)$ and $V(Q) \cap V(C) = \mathrm{end}(Q)$.
(4) Vertices of $\mathrm{end}(P)$ and vertices of $\mathrm{end}(Q)$ appear alternately around C^+.
(5) $x \in V(P)$ and $y \in V(Q)$.
Then the lemma follows.

Proof. Let $\mathrm{end}(P) = \{x_1, x_2\}$ and $\mathrm{end}(Q) = \{y_1, y_2\}$. We may assume x_1, y_1, x_2 and y_2 appear in this order around C^+. Then one of the two cycles

$$C^+[x_1, y_1] \cdot Q[y_1, y_2] \cdot C^-[y_2, x_2] \cdot P[x_2, x_1]$$

and

$$C^-[x_1, y_2] \cdot Q[y_2, y_1] \cdot C^+[y_1, x_2] \cdot P[x_2, x_1]$$

has the desired properties.

Let $\mathrm{end}(F_1) = \{x_1, x_2, x_3\}$. We may assume that x_1, x_2, x_3 appear in this order around C^+. If $y \in V(F_1)$, then the theorem follows by Claim 1. Suppose $y \notin V(F_1)$. Let $D = C \cup F_1$. Let F_2 be a (y, D)-fan of size three. Let $\mathrm{end}(F_2) = \{y_1, y_2, y_3\}$. If $\mathrm{end}(F_2) \cap (F_1 - \{x_1, x_2, x_3\}) \neq \emptyset$, then the lemma follows by Claim 1. So we may assume $\mathrm{end}(F_2) \subset V(C)$.

Claim 3. If $\{y_1, y_2, y_3\} \subset C^+[x_i, x_{i+1}]$ (If $i = 3$, we consider $x_4 = x_1$), then the lemma follows.

Proof. We may assume $y_1, y_2, y_3 \in C^+[x_1, x_2]$ and y_1, y_2 and y_3 appear in this order around C^+. Then

$$C^+[x_3, y_1] \cdot F_2[y_1, y_2] \cdot C^+[y_2, x_2] \cdot F_1[x_2, x_3]$$

or

$$C^+[x_1, y_2] \cdot F_2[y_2, y_3] \cdot C^+[y_3, x_3] \cdot F_1[x_3, x_1]$$

has the desired properties.

By Claims 1, 2, 3, the only possible case in which the lemma would not hold is $\{x_1, x_2, x_3\} = \{y_1, y_2, y_3\}$. We may assume $x_i = y_i$ $(i = 1, 2, 3)$. Let $D' = D \cup F_2$. Since C is a longest cycle, $C^+(x_1, x_2) \neq \emptyset$. Since G is 3-connected, there exists a path P joining $C^+(x_1, x_2)$ and $D' - C^+[x_1, x_2]$ in $G - \{x_1, x_2\}$. Let $\mathrm{end}(P) = \{u, v\}$, $u \in C^+(x_1, x_2)$ and $v \in D' - C^+[x_1, x_2]$. If $v \in V(F_1) \cup V(F_2)$, then the lemma follows by Claim 2. So we may assume $v \in C^+(x_2, x_3]$. Then F_1, F_2, $C^+[x_1, x_2]$ and $P[u, v] \cdot C^+[v, x_3]$ form a subdivision of $K_{3,3}$. This contradicts the planarity of G. Therefore, the lemma follows. ■

Proof of Lemma 2. Let C_0 be a longest cycle which contains x and y. Then $|C_0| \geq |C|$. If G is hamiltonian, then C_0 is a hamiltonian cycle, and $|C_0| \geq 4$. Hence the result follows. Threfore, we may assume G is not hamiltonian, and $|G| \geq 7$. By Lemma 1, $|C_0| \geq \frac{1}{2} \cdot 7 + 2 \geq 5$. So $|C_0| \geq \frac{1}{2}|C_0| + 2 \geq \frac{1}{2}|C| + 2$. Hence we may assume $z \notin C_0$. Consider a (z, C_0)-fan F_1. Let $\mathrm{end}(F_1) = \{z_1, z_2, z_3\}$. We may assume that z_1, z_2, z_3 appear in this order around C^+. We consider three cases.

Case 1. $\mathrm{end}(F_1) \subset C_0^+[x, y]$ *or* $\mathrm{end}(F_1) \subset C_0^+[y, x]$.

We may assume $\{z_1, z_2, z_3\} \subset C_0^+[x, y]$. Then one of the two cycles $C_0^+[z_2, z_1] \cdot F_1[z_1, z_2]$ and $C_0^+[z_3, z_2] \cdot F_1[z_2, z_3]$ has the desired properties.

Case 2. One of $\mathrm{end}(F_1)$ *lies on* $C_0^+(y,x)$ *and the other two lie on* $C_0^+(x,y)$.

We may assume $z_1, z_2 \in C_0^+(x,y)$ and $z_3 \in C_0^+(y,x)$. Let $C_1 = C_0^+[z_2,z_1] \cdot F_1[z_1,z_2]$. Then $C_0 - C_1 = C_0^+(z_1,z_2)$. Let $D = C_0 \cup F_1$. By Theorem B, there exists an $(x, D - C_0^+(z_3,z_1))$-fan F_2 of size three, such that z_1, $z_3 \in \mathrm{end}(F_2)$. Let $\mathrm{end}(F_2) = \{z_1, z_3, a\}$. If $a \in F_1[z,z_1)$ or $a \in F_1[z,z_2)$, let

$$C_2 = C_0^+[z_1,z_3] \cdot F_1[z_3,a] \cdot F_2[a,z_1].$$

If $a \in F_1[z,z_3)$, let

$$C_2 = C_0^+[z_1,z_3] \cdot F_2[z_3,a] \cdot F_1[a,z_1].$$

If $a \in C_0^+(z_2,y]$, let

$$C_2 = C_0^+[a,z_3] \cdot F_1[z_3,z_2] \cdot C_0^-[z_2,z_1] \cdot F_2[z_1,a].$$

If $a \in C_0^+(y,z_3)$, let

$$C_2 = C_0^-[a,z_1] \cdot F_1[z_1,z_3] \cdot F_2[z_3,a].$$

Then in either case, $C_0^+(z_1,z_2) \subset C_2$ and either C_1 and C_2 satisfies the desired properties. So the only remaining case is $a \in C_0^+(z_1,z_2]$. Let $D' = D \cup F_2$.

Next, consider a $(y, D' - C_0^+(z_2,z_3))$-fan F_3 such that $\{z_2,z_3\} \subset \mathrm{end}(F_3)$. Let $\mathrm{end}(F_3) = \{z_2,z_3,b\}$. If $b \in (F_1 - \mathrm{end}(F_1)) \cup C_0^+(z_3,z_1)$, then the lemma follows by the same argument. If $b \in F_2(x,a) \cup F_2(x,z_1)$, let

$$C_3 = F_3[b,z_2] \cdot C_0^-[z_2,z_1] \cdot F_1[z_1,z_3] \cdot F_2[z_3,b].$$

If $b \in F_2(x,z_3)$, let

$$C_3 = F_3[b,z_3] \cdot F_1[z_3,z_2] \cdot C_0^-[z_2,z_1] \cdot F_2[z_1,b].$$

Then in either case $C_0^+(z_1,z_2) \subset C_3$ and hence either C_1 or C_3 satisfies the desired properties. So the lemma follows unless $b \in C_0^+[z_1,z_2)$. (Possibly $a = b$.)

Now we consider the case $a \in C_0^+(z_1,z_2)$ and $b \in C_0^+(z_1,z_2)$. If z_1, b, a, z_2 appear in this order around C_0^+, let

$$C_4 = F_3[z_3,b] \cdot C_0^+[b,z_2] \cdot F_1[z_2,z_1] \cdot C_0^-[z_1,z_3]$$

and

$$C_5 = F_2[z_3,a] \cdot C_0^-[a,z_1] \cdot F_1[z_1,z_2] \cdot C_0^+[z_2,z_3].$$

If z_1, a, b, z_2 appear in this order around C^+, let

$$C_4 = F_3[z_2, b] \cdot C_0^-[b, z_3] \cdot F_1[z_3, z_2]$$

and

$$C_5 = F_2[z_1, a] \cdot C_0^+[a, z_3] \cdot F_1[z_3, z_1].$$

Then in either case we have $\{x, y, z\} \subset C_4 \cap C_5$, $C_0 \subset C_4 \cup C_5$, and hence $|C_4| \geq \frac{1}{2}|C_0| + 2$ or $|C_5| \geq \frac{1}{2}|C_0| + 2$. So the lemma follows.

Now, we may assume that $a = z_2$ or $b = z_1$. If $a = z_2$, then F_1, F_2, F_3 and $C_0^-[b, z_1]$ form a subdivision of $K_{3,3}$. If $b = z_1$, then F_1, F_2, F_3 and $C_0^+[a, z_2]$ form a subdivision of $K_{3,3}$. Hence both contradicts the planarity of G. Therefore, the proof in this case is complete.

Case 3. $|\{x, y\} \cap \mathrm{end}(F_1)\}| = |C_0^+(x, y) \cap \mathrm{end}(F_1)| = |C_0^+(y, x) \cap \mathrm{end}(F_1)| = 1$.

We may assume $z_1 = x$, $z_2 \in C_0^+(x, y)$ and $z_3 \in C_0^+(y, x)$. Then either

$$C_6 = F_1[z_1, z_2] \cdot C_0^+[z_2, z_1], \quad \text{or}$$
$$C_7 = F_1[z_1, z_3] \cdot C_0^-[z_3, z_1]$$

satisfies the desired properties.

Therefore, in each case, G has a cycle through x, y and z of length at least $\frac{1}{2}|C_0| + 2$. ■

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