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EXAMPLES OF SEMI-STABLE DEGENERATIONS OF KUNEV SURFACES

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0. This article consists of some examples of semi-stable degenerations of Kunev surfaces together with some remarks concerning about a compactification of their moduli space and the Torelli problem.

A *Kunev surface* X *is* defined as a canonical surface with $\chi(\mathcal{O}_X) = 2$ and $(\omega_X)^2 = 1$ which has an involution σ such that X/σ is a K3 surface with rational double points (RDP, for short). Let $\hat{X}$ be the minimal model of a Kunev surface. Then it is known that $\hat{X}$ is a simply connected surface of general type with $p_g = c_1^2 = 1$ and its bicanonical map is a Galois cover of $\mathbb{P}^2$ with $\mathcal{G}al \simeq (\mathbb{Z}/2)^{\oplus 2}$ whose branch locus consists of two cubics and a line on $\mathbb{P}^2$, and that $\hat{X}$ has the following numerical invariants:

(0.1)
$$h^{2,0}(\hat{X}) = h^{0,2}(\hat{X}) = 1, \quad h^{1,1}_{prim}(\hat{X}) = 18.$$
$$H^0(T_{\hat{X}}) = H^2(T_{\hat{X}}) = 0, \quad h^1(T_{\hat{X}}) = 18.$$
$$h^{2,0}(\hat{X})^\sigma = h^{0,2}(\hat{X})^\sigma = 1, \quad h^{1,1}_{prim}(\hat{X})^\sigma = 10.$$
$$h^1(T_{\hat{X}})^\sigma = 12.$$

It is also known that the moduli space $\mathfrak{M}$ (resp. $\mathfrak{K}$) of surfaces with $p_g = c_1^2 = 1$ (resp. Kunev surfaces) is irreducible and rational (resp. irreducible). (For the above facts, see [Ca.1], [Ca.2], [U.2], [T.2], [SSU], [M].)

On the Hodge theoretic view-point, surfaces with $p_g = c_1^2 = 1$ or, as its subfamily, Kunev surfaces are interesting materials. After Kunev constructed an example of a Kunev surface as a counterexample to the infinitesimal Torelli theorem, the following results are known:

(0.2) *The generic infinitesimal Torelli theorem holds for surfaces in* $\mathfrak{M}$ ([Ca.1]).

(0.3) *The period map* Φ_2 *of surfaces in* $\mathfrak{M}$ *has positive dimensional fibers* ([T.1], [U.1], [U.2]; [T.1] treats only Kunev surfaces).

(0.4) $\mathfrak{K}$ *in* $\mathfrak{M}$ *is characterized by* $\dim \Phi_2^{-1}\Phi_2([X]) = 2$, *which is the maximal dimension of the fibers of* Φ_2 ([U.1]).

(0.5) *The infinitesimal mixed Torelli theorem holds for pairs* (X, C) *of surfaces* X *in* $\mathfrak{M}$ *and their smooth canonical curves* C ([U.3]).

(0.6) *The generic mixed Torelli theorem holds for Kunev surfaces* ([L], [SSU]; there is a point about monodromy which is not clear in [L]).

(0.7) *There exists a Zariski open subset* U *of* $\mathfrak{K}$ *such that* $\Phi^{-1}\Phi(U) = U$, *where* $\Phi : \mathfrak{M} \longrightarrow \Gamma\backslash D$ *is the mixed period map* ([SSU]). Hence, in order to solve the mixed Torelli problem for surfaces in $\mathfrak{M}$ via Kunev locus $\mathfrak{K}$, it is necessary to study the following:

(0.8) *A compactification of the mixed period map* $\Phi : \mathfrak{M} \longrightarrow \Gamma\backslash D$.

(0.9) *The monodromy* Γ *in (0.7), where we used a geometric one.* (For a general reference of the above as well as for the terminology such as *mixed* period map, *mixed* Torelli etc., see [SSU].)

We shall report here some results in an experiment concerning about the prblem (0.8). We shall construct some examples of semi-stable degenerations of pairs of Kunev surfaces and their canonical curves in the sequel. The examples in **1**, **2** and **3** are of *type I* with respect to the local monodromy of the pure second cohomology. The examples in 4 and 5 are of *type II* and *type III* respectively.

As for the problem of a compactification of the moduli space $\mathfrak{K}$ of Kunev surfaces, Horikawa and Shah constructed a compactification of the moduli space of K3 surfaces of degree 2 as one of the moduli space of sexetic curves on $\mathbb{P}^2$ by the geometric invariant theory ([H], [Sh]). The latter contains a 10-dimensional subspace $\bar{\mathfrak{N}}$ which is a compactification of

$\mathfrak{N} = \{\Sigma C_j \in \mathrm{Sym}^2|\mathcal{O}_{\mathbb{P}^2}(3)| \mid \Sigma C_j$ has only simple singularities$\}/SL_3$.

A "compactification" of $\mathfrak{K}$ sits over $\bar{\mathfrak{N}}$.

We use the terminology a *numerical K3 surface* for a minimal surface with $p_g = 1$, $q = 0$ and $c_1^2 = 0$ (for this terminology, cf. also [K]). Numerical K3 surfaces appeared in this article have an elliptic fibration with one double fiber.

1. Let C_1 and C_2 be general cubics on $\mathbb{P}^2$. Denote by $\check{C}_j \subset \check{\mathbb{P}}^2$ the dual curve of $C_j \subset \mathbb{P}^2$, i.e., the image of the Gauss map. Then each $\check{C}_j$ has nine cusps corresponding to nine inflexes on C_j, $\Sigma \check{C}_j$ has nine bitangents $\check{D}_i$ with tangent points P_{i1} and P_{i2} $(1 \le i \le 9)$ subjected to nine nodes of ΣC_j, and we have two stratifications of $\check{\mathbb{P}}^2$ determined by $\Sigma \check{C}_j$ and $\Sigma \check{D}_i$:

$$(1.1)\quad \check{\mathbb{P}}^2 = (\check{\mathbb{P}}^2 - \Sigma \check{C}_j) \amalg (\Sigma \check{C}_j - (\Sigma \check{P}_{ji} + \mathrm{Sing}(\Sigma \check{C}_j)) \amalg (\Sigma\, \mathrm{Sing}(\check{C}_j)) \amalg (\cap \check{C}_j) \amalg (\Sigma \check{P}_{ji}) =: R_0 \amalg R_1 \amalg R_1' \amalg R_2 \amalg R_0'.$$

$$(1.2)\quad \check{\mathbb{P}}^2 = (\check{\mathbb{P}}^2 - \Sigma \check{D}_i) \amalg (\Sigma \check{D}_i - \mathrm{Sing}(\Sigma \check{D}_i)) \amalg \mathrm{Sing}(\Sigma \check{D}_i)$$

$=: S_0 \amalg S_1 \amalg S_2.$

We denote by Y the minimal K3 surface which is obtained as the minimal resolution of the double cover of $\mathbb{P}^2$ branched along ΣC_j. Let $\alpha_1 : Y \longrightarrow \mathbb{P}^2$ be the projection and E_i $(1 \le i \le 9)$ be the exceptional curves for α_1, i.e., (-2)-curves.

(1.3) *Case* $t_0 \in S_1 \cap R_0$: We may assume $t_0 \in \check{D}_1$. Let Δ be a small disc with center $0 = t_0$ intersecting transversely with $\check{D}_1$ such that $\Delta^* := \Delta - \{0\} \subset S_0$. We denote by $\mathcal{L} \subset \Delta \times \mathbb{P}^2$ the total space of the family of lines $\{L_t\}_{t\in\Delta}$. We can construct a semi-stable degeneration of pairs of Kunev surfaces and the canonical curves over Δ in the follwing way: **(0)** Set $\alpha = 1 \times \alpha_1 : \Delta \times Y \longrightarrow \Delta \times \mathbb{P}^2$ and $\mathcal{E}_i = \Delta \times E_i$ $(1 \le i \le 9)$. **(i)** Let $\beta : \mathcal{Y} \longrightarrow \Delta \times Y$ be the blowing-up along $\alpha^{-1}\mathcal{L} \cap \mathcal{E}_1$. Denote by $W_{\mathcal{Y}}$ the exceptional divisor. **(ii)** Take the double cover $\gamma : \mathfrak{X}_1 \longrightarrow \mathcal{Y}$ branched along $(\alpha\beta)^{-1}\mathcal{L} + \beta^{-1}(\Sigma \mathcal{E}_i)$. **(iii)** Let $\delta : \mathfrak{X}_1 \longrightarrow \mathfrak{X}$ be the contraction of $(\beta\gamma)^{-1}(\Sigma \mathcal{E}_i)$. **(iv)** Let $\mathfrak{X} \longrightarrow \bar{\mathfrak{X}}$ be the contraction of $\delta(\beta\gamma)^{-1}W_{\mathcal{Y}}$. (In the above notation, we use $\alpha^{-1}\mathcal{L}$ etc. as the proper transforms.)

Set $\mathcal{L}_{\mathfrak{X}} = (\delta(\alpha\beta\gamma)^{-1}\mathcal{L}$ with reduced structure) and $W_{\mathfrak{X}} = \delta\gamma^{-1}W_{\mathcal{Y}}$. Then we can prove the following: **(a)** *The projection* $f : (\mathfrak{X}, \mathcal{L}_{\mathfrak{X}}) \longrightarrow \Delta$ *is a semi-stable degeneration of pairs* (for the terminology, see [SSU]). $K_{\mathfrak{X}} = \mathcal{L}_{\mathfrak{X}} + W_{\mathfrak{X}}$. **(b)** *The pair* $(X_t, L_{X,t}) := f^{-1}(t)$ *consists of a minimal Kunev surfaces and its canonical divisor which is an ample, smooth curve of genus* 2 *for* $t \in \Delta^*$. *The central fiber of* f *is as Figure 1.* **(c)** $\bar{\mathfrak{X}}$ *has an isolated singular point, raised from* $C_1 \cap C_2 \cap L_0$, *which is analytically isomorphic to the cone over the Veronese embedding of* $\mathbb{P}^2 \subset \mathbb{P}^5$ *by* $|\mathcal{O}_{\mathbb{P}^2}(2)|$. $K_{\bar{\mathfrak{X}}}$ is *nef.*

(1.4) *Case* $t_0 \in R_0'$: We may assume $t_0 = \check{P}_{11}$. We use the notation Δ, $\mathcal{L}$ etc. in the same sense as in (1.3). The construction (0)-(iv) in (1.3) yields a family of pairs $f : (\mathfrak{X}, \mathcal{L}_{\mathfrak{X}}) \longrightarrow \Delta$ and $\bar{\mathfrak{X}}$. We can prove the following: **(a)** *The central fiber* X_0 *of* f *is not a divisor with normal crossings (see Figure 2). The total space* $\mathfrak{X}$ *is smooth and* $K_{\mathfrak{X}} = \mathcal{L}_{\mathfrak{X}} + W_{\mathfrak{X}}$. **(b)** *After a base extension* $\Delta_2 \longrightarrow \Delta$, $s \longmapsto t = s^2$, *we can get a semi-stable reduction* $f' : (\mathfrak{X}', \mathcal{L}_{\mathfrak{X}'}) \longrightarrow \Delta_2$ *of* f *whose central fiber is as Figure 3.* (Because of the limit of pages, we omit the details of the process of reduction and contraction.) **(c)** *The same statement as (c) in (1.3) holds.*

(1.5) *Case* $t_0 \in S_2$: We may assume $t_0 \in \check{D}_1 \cap \check{D}_2$. We use the notation Δ, $\mathcal{L}$ etc. in a similar sense as in (1.3). An analoguous construction (0)-(iv) as in (1.3) yields a family of pairs $f : (\mathfrak{X}, \mathcal{L}_{\mathfrak{X}}) \longrightarrow \Delta$ and $\bar{\mathfrak{X}}$ with the following properties: **(a)** *The same statement as (a) in (1.3) holds.* **(b)** *The central fiber of* f *is as Figure 4.* **(c)** $\bar{\mathfrak{X}}$ *has two Veronese cone singularities raised from two points* $C_1 \cap C_2 \cap L_0$. $K_{\bar{\mathfrak{X}}}$ *is nef.*

Remark. For fixed general cubics C_1 and C_2, we constructed a complete family of degenerations of Kunev surfaces over $\check{\mathbb{P}}^2$ and described their fibers X_t, $t \in \check{\mathbb{P}}^2$, in [U.4,I], [U.5]. We point out here only that the stratification (1.1) controls RDP on the main components V_t of X_t, whereas the stratification (1.2) controls their Kodaira dimensions. In fact, we know that *the main component* V_t *is a (singular) Kunev surface, numerical K3 surface with one double fiber, or K3 surface according to* $t \in S_0$, S_1, *or* S_2 (for more general statement, see Remark in 3).

2. Next we consider the case that both cubics C_1 and C_2 degenerate into two pairs of cocurrent three lines. Denote by Q_j the triple point of C_j $(j = 1, 2)$. We deal with the general situation in this case, i.e., we assume moreover that ΣC_j consists of six different lines and that $Q_1 \neq Q_2$. Denote by $\check{Q}_j$ the line on $\check{\mathbb{P}}^2$ corresponding to the pencil of lines through Q_j on $\mathbb{P}^2$ $(j = 1, 2)$. Then we have a stratification of $\check{\mathbb{P}}^2$:

(2.1) $$\check{\mathbb{P}}^2 = (\check{\mathbb{P}}^2 - \Sigma \check{Q}_j) \amalg (\Sigma \check{Q}_j - \check{Q}_1 \cap \check{Q}_2) \amalg (\check{Q}_1 \cap \check{Q}_2)$$
$$=: T_0 \amalg T_1 \amalg T_2.$$

Let Y be the minimal K3 surface which is obtained as the minimal resolution of the double cover of $\mathbb{P}^2$ branched along ΣC_j and let $\alpha_1 : Y \longrightarrow \mathbb{P}^2$ be the projection. We denote by E_i $(1 \leq i \leq 9)$ the (-2)-curves on Y which are mapped by α_1 to the nine points $C_1 \cap C_2$. Notice that, beside the E_i, Y carries two D_4-configurations of (-2)-curves over the triple points Q_j of C_j.

(2.2) *Case* $t_0 \in T_1$ *and the line* $L_{t_0} \not\subset \Sigma C_j$: In this case, the line L_{t_0} passes one and only one of the two triple points, say Q_1. Let Δ be a small disc with center $0 = t_0$ intersecting transeversely with $\check{Q}_1$ such that $\Delta^* \subset T_0 - (\Sigma C_j)^{\vee}$. Let $\mathcal{L} \subset \Delta \times \mathbb{P}^2$ be the total space of the family of lines as before. Similarly as in (1.3), we can construct a family of pairs $f : (\mathfrak{X}, \mathcal{L}_{\mathfrak{X}}) \longrightarrow \Delta$ which is a degeneration of pairs of Kunev surfaces and the canonical curves in the following way: **(0)** Set $\alpha = 1 \times \alpha_1 : \Delta \times Y \longrightarrow \Delta \times \mathbb{P}^2$, $\mathcal{E}_i = \Delta \times E_i$ $(1 \leq i \leq 9)$ and $\mathcal{D}_j = \Delta \times \alpha_1^{-1}(Q_j)$ $(j = 1, 2)$.

(i) Take the double cover $\beta : \mathfrak{X}_1 \longrightarrow \Delta \times Y$ branched along $\alpha^{-1}\mathcal{L} + \Sigma\, \mathcal{E}_i$. **(ii)** Let $\gamma : \mathfrak{X}_1 \longrightarrow \mathfrak{X}$ be the contraction of $\beta^{-1}(\Sigma\, \mathcal{E}_i)$. **(iii)** Let $\delta : \mathfrak{X} \longrightarrow \bar{\mathfrak{X}}$ be the contraction of $\gamma\beta^{-1}(\Sigma\, \mathcal{D}_j)$.

Set $\mathcal{L}_{\mathfrak{X}} = (\gamma(\alpha\beta)^{-1}\mathcal{L}$ with reduced structure) and $f : (\mathfrak{X}, \mathcal{L}_{\mathfrak{X}}) \longrightarrow \Delta$, the porojection. Then we can prove: **(a)** *X_0 of the central fiber $f^{-1}(0) = (X_0, L_{X,0})$ is irreducible. Rising from the cocurrent four lines $C_1 + L$ on $\mathbb{P}^2$, X_0 has the singular locus which is a rational curve consisting of double points, on which there are four cuspidal points. The total space $\mathfrak{X}$ is smooth and $K_{\mathfrak{X}} = \mathcal{L}_{\mathfrak{X}}$ which is nef.* **(b)** *After a base extension $\Delta_2 \longrightarrow \Delta$, $s \longmapsto t = s^2$, we get a semi-stable degeneration $f' : (\mathfrak{X}', \mathcal{L}_{\mathfrak{X}'}) \longrightarrow \Delta_2$ of pairs of Kunev surfaces and the canonical curves, whose central fiber is as Figure 6.* (We omit the details of the process.) **(c)** *Rising from the two triple points Q_1 and Q_2, $\bar{\mathfrak{X}}$ has four compounds R.D.P. of type D_4, two of which coming from Q_1 clash to make up a simple elliptic singularity of type $\tilde{E}_8$ in the sense of K. Saito on the central fiber $\bar{X}_0$ with a local equation*

$$z^2 + y(x^4 + y^2) = 0.$$

(2.3) *Case $t_0 \in T_2$:* We use the notation Δ, $\mathcal{L}$, $\mathcal{E}_i$, $\mathcal{D}_j$ etc in (2.2). An analogous construction (0)-(iii) in (2.2) yields a family of pairs $f : (\mathfrak{X}, \mathcal{L}_{\mathfrak{X}}) \longrightarrow \Delta$ and $\bar{\mathfrak{X}}$ with the follwing properties: **(a)** *A similar statement of (a) in (2.2) holds but now the singular locus of X_0 consists of two copies of the rational curves as before.* **(b)** *The same statement as (b) in (2.2) holds and the central fiber of the semi-stable degeneration $f' : (\mathfrak{X}', \mathcal{L}_{\mathfrak{X}'}) \longrightarrow \Delta_2$ is as Figure 6.* **(c)** *A similar statement of (c) in (2.2) holds but now each pair of compounds R.D.P. of type D_4 on $\bar{\mathfrak{X}}$, coming from*

Q_j $(j = 1, 2)$, *clashes to make up a simple elliptic singularity of type* $\tilde{E}_8$.

3. Consider now the case that C_1 is degenerating into a smooth conic Q and a line L. Assume that Q, L and a smooth cubic C_2 are in general position on $\mathbb{P}^2$. Let $\{C_t\}$ be a family of cubics on $\mathbb{P}^2$ over a disc Δ such that $C_0 = Q + L$ and that C_t is smooth and intersects transversely with C_2 for $t \in \Delta^*$. Denote by $\mathscr{C}_1$ the total space of the family $\{C_t\}_{t\in\Delta}$, $\mathscr{C}_2 = \Delta \times C_2$ and $\mathscr{L} = \Delta \times L$. Assume that $\mathscr{C}_1$ is smooth. Then we can construct a degeneration of pairs $f : (\mathfrak{X}, \mathscr{L}_{\mathfrak{X}}) \longrightarrow \Delta$ of Kunev surfaces and their canonical curves in an analoguous way as (0)-(iv) in (1.3). On this stage, X_0 of the central fiber $f^{-1}(0)$ is not reduced, hence we need a semi-stable reduction by extending the base $\Delta_2 \longrightarrow \Delta$, $s \longmapsto t = s^2$. Figure 7 illustrates the central fiber $(X'_0, L_{X',0}) = f'^{-1}(0)$ of the resulting family $f' : (\mathfrak{X}', \mathscr{L}_{\mathfrak{X}'}) \longrightarrow \Delta_2$. Contracting W'_i $(3 \le i \le 5,\ 9 \le i \le 11)$, W'_2 and W'_i $(6 \le i \le 8)$ in this order, we get a simpler semi-stable degeneration $f'' : (\mathfrak{X}'', \mathscr{L}_{\mathfrak{X}''}) \longrightarrow \Delta_2$ of Kunev surfaces. The central fiber $(X''_0, L_{X'',0}) = f''^{-1}(0)$ is as Figure 8. $K_{\mathfrak{X}''}$ is not nef. Let $\mathfrak{X}'' \longrightarrow \bar{\mathfrak{X}}$ be the contraction of W''_i $(12 \le i \le 17)$. Then $K_{\bar{\mathfrak{X}}}$ becomes nef and $\bar{\mathfrak{X}}$ has six Veronese cone singularities.

Remark (cf.[U.4,II], [U.5]). Recall the notation $\mathfrak{N}$ in **0**. For any fixed $[\Sigma C_j] \in \mathfrak{N}$, we define functions in $t \in \check{\mathbb{P}}^2$ by

$$m(t) = \Sigma_{P\in\mathbb{P}^2} \min\{I(P, L_t \cap C_j) \mid j = 1, 2\}, \text{ and}$$

$$n(t) = \#\{\text{triple points of } C_j \text{ on } L_t,\ j = 1, 2\}.$$

Notice that if C_j has a triple point then C_j consists of three distinct lines with a common point. These functions define two stratifications of $\check{\mathbb{P}}^2$:

$\check{\mathbb{P}}^2 = S_0 \amalg S_1 \amalg S_2$, where $S_m = \{t \in \check{\mathbb{P}}^2 \mid m = \min\{2, m(t)\}\}$.

$\check{\mathbb{P}}^2 = T_0 \amalg T_1 \amalg T_2$, where $T_n = \{t \in \check{\mathbb{P}}^2 \mid n = n(t)\}$.

Notice that codim $S_m = m$, codim $T_0 = 0$, and codim $T_n = n$ if T_n is non-empty $(n = 1, 2)$. We can construct a complete family of degenerations of Kunev surfaces $\mathcal{F} : \mathfrak{X} \longrightarrow \check{\mathbb{P}}^2$ and we can prove:

(3.1) *The main component* V_t *of the fiber* $\mathcal{F}^{-1}(t)$ *is a (singular) Kunev surface, numerical K3 surface with one double fiber, K3 surface, elliptic surface with* $p_g = q = 1$, *or abelian surface according to* $t \in S_0 \cap T_0$, S_1, S_2, $S_0 \cap T_1$, *or* T_2.

(3.2) *If* $t \notin S_0 \cap T_0$, *the main component* V_t *is an elliptic surface.* V_t *has constant J-invariant if and only if* $t \in T_1 \cap T_2$. *If this is the case, the K3 surface* Y *is a Kummer surface associated to a decomposable abelian surface* $D_1 \times D_2$, *where* D_j *is an elliptic curve* $(j = 1, 2)$.

Combining with the Clemens-Schmid exact sequence ([Cl]), (3.1) yields a uniform explanation of the appearance of positive dimensional fibers of the period map of Kunev surfaces and elliptic surfaces with $p_g = 1$ and $q = 0, 1$:

(3.3) $S_0 \cap T_0$, S_1, *and* $S_0 \cap T_1$ *appear as positive dimensional fibers of the period map of the pure second cohomology of Kunev surfaces, numerical K3 surfaces with one double fiber, and elliptic surfaces with* $p_g = q = 1$ *respectively.* (For Kunev surfaces and elliptic surfaces with $p_g = q = 1$, these phenomea were pointed out separately in [T.1], [U.1], [U.2] and [Sa].)

4. Next we consider the case that C_1 is approaching C_2. Let C_2 be a smooth cubic and L a line on $\mathbb{P}^2$. Assume that C_2 and L intersect transversely. Let $\{C_t\}$ be a smooth family of cubics on $\mathbb{P}^2$ over a disc Δ such that $C_0 = C_2$ and that C_t intersects transversely with C_2 for $t \in \Delta^*$. Denote by $\mathscr{C}_1$ the total space of the family $\{C_t\}_{t\in\Delta}$, $\mathscr{C}_2 = \Delta \times C_2$ and $\mathscr{L} = \Delta \times L$. Then, as before, we can construct a degeneration of pairs $f : (\mathfrak{X}, \mathscr{L}_{\mathfrak{X}}) \longrightarrow \Delta$ of Kunev surfaces and their canonical curves. On this stage, X_0 of the central fiber $f^{-1}(0)$ is not reduced, hence we need a semi-stable reduction by extending the base $\Delta_2 \longrightarrow \Delta$, $s \longmapsto t = s^2$. The resulting family $f' : (\mathfrak{X}', \mathscr{L}_{\mathfrak{X}'}) \longrightarrow \Delta_2$ has the central fiber illustrated as Figure 9. Contracting W'_i $(2 \le i \le 7)$ we get a nef terminal model $\bar{\mathfrak{X}}$ which has six Veronese cone singularities.

5. We consider here the case that C_1 and C_2 are degenerating into the same set of three distinct lines ΣM_i. Let L be a line on $\mathbb{P}^2$. Assume that $\Sigma M_i + L$ has no triple points. Let $\{C_{1,t}\}$ and $\{C_{2,t}\}$ be two families of cubics on $\mathbb{P}^2$ over a disc Δ such that $C_{1,0} = C_{2,0} = \Sigma M_i$ and that $C_{1,t}$ and $C_{2,t}$ are smooth and intersect transversely for $t \in \Delta^*$. Denote by $\mathscr{C}_1$ and by $\mathscr{C}_2$ the total space of the families $\{C_{1,t}\}_{t\in\Delta}$ and $\{C_{2,t}\}_{t\in\Delta}$ respectively and $\mathscr{L} = \Delta \times L$. We also assume that $\mathscr{C}_j$ $(j = 1, 2)$ are smooth. Then, as before, we can construct a degeneration of pairs $f : (\mathfrak{X}, \mathscr{L}_{\mathfrak{X}}) \longrightarrow \Delta$ of Kunev surfaces and their canonical curves. Since X_0 of the central fiber $f^{-1}(0)$ is not

reduced, we should perform a semi-stable reduction by extending the base $\Delta_2 \longrightarrow \Delta$, $s \longmapsto t = s^2$. The resulting family $f' : (\mathfrak{X}', \mathcal{L}_{\mathfrak{X}'}) \longrightarrow \Delta_2$ has the central fiber illustrated as Figure 10. Let $\mathfrak{X}' \longrightarrow \bar{\mathfrak{X}}$ be the contraction of W_i' $(19 \le i \le 24)$. Then $K_{\bar{\mathfrak{X}}} = \mathcal{L}_{\bar{\mathfrak{X}}}$, which is nef, and $\bar{\mathfrak{X}}$ has six Veronese cone singularities.

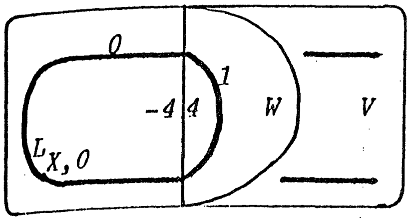


Figure 1

$f^{-1}(0) = (X_0, L_{X,0})$, $X_0 = V + W_{\mathfrak{X}}$

$K_{\mathfrak{X}} = \mathcal{L}_{\mathfrak{X}} + W_{\mathfrak{X}}$

V : *a homotopic K3 surface*

$K_V = \mathcal{L}_{\mathfrak{X}}|V$: *an elliptic curve*

$W := W_{\mathfrak{X}} \simeq \mathbb{P}^2$, $\mathcal{L}_{\mathfrak{X}}|W_{\mathfrak{X}}$: *a line*

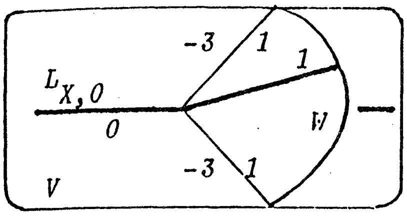


Figure 2

$f^{-1}(0) = (X_0, L_{X,0})$, $X_0 = V + W_{\mathfrak{X}}$

$K_{\mathfrak{X}} = \mathcal{L}_{\mathfrak{X}} + W_{\mathfrak{X}}$

V : *a homotopic K3 surface*

$W := W_{\mathfrak{X}} \simeq \mathbb{P}^2$

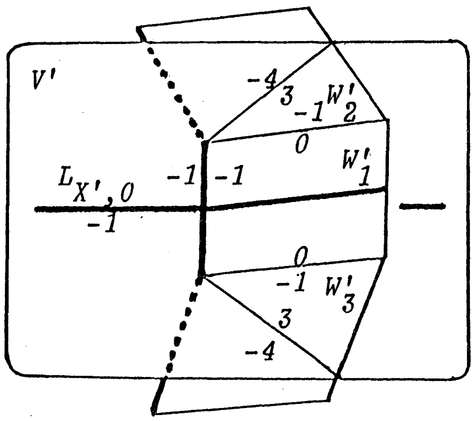


Figure 3

$f'^{-1}(0) = (X'_0, L_{X',0})$, $X'_0 = V' + \Sigma_1^3 W'_k$

$K_{\mathcal{X}'} = \mathcal{L}_{\mathcal{X}'} + \Sigma W'_k$

V' : *a numerical K3 surface with one double fiber*

$W'_k \simeq F_1$: *a rational ruled surface* $(1 \leq k \leq 3)$

$\mathcal{L}_{\mathcal{X}'} \cap V'$: *an elliptic curve*

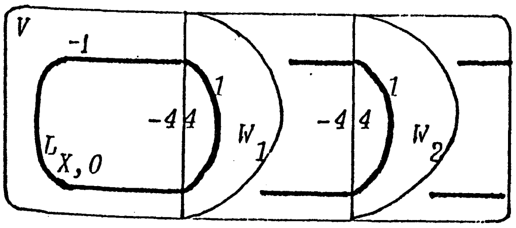


Figure 4

$f^{-1}(0) = (X_0, L_{X,0})$, $X_0 = V + \Sigma W_i$

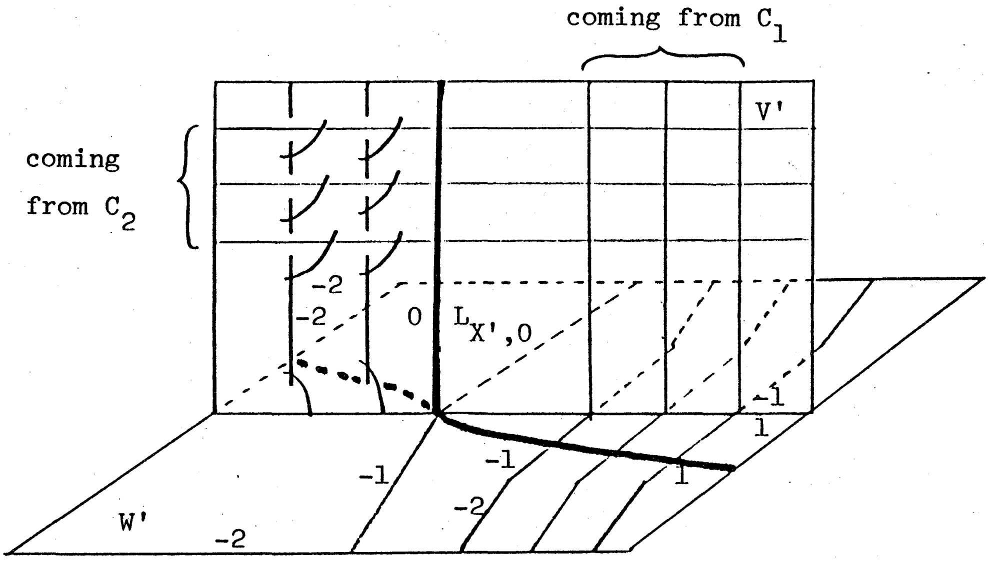
$K_{\mathfrak{X}} = \mathcal{L}_{\mathfrak{X}} + \Sigma W_i$

V : *a K3 surface*

$K_V = \mathcal{L}_{\mathfrak{X}}|V$: *a (-1)-curve*

$W_i \simeq \mathbb{P}^2$ $(i = 1, 2)$

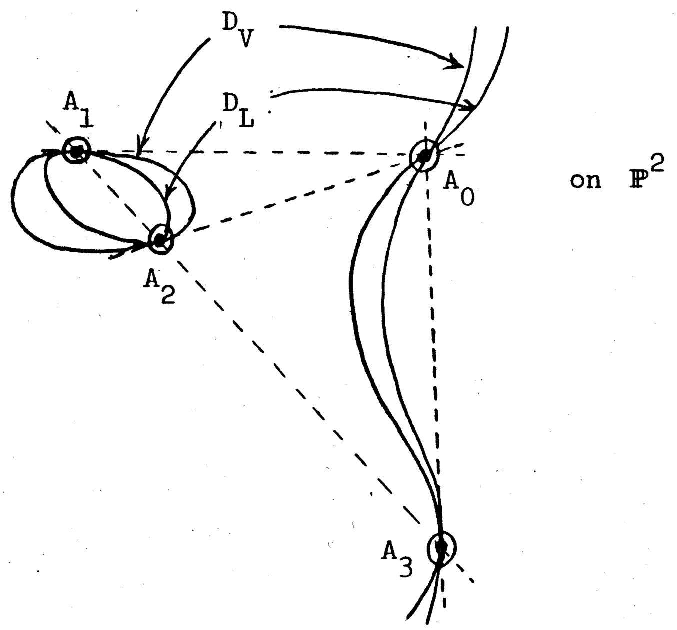
Figure 5



$f'^{-1}(0) = (X_0', L_{X',0})$, $X_0' = V' + W'$, $K_{\mathcal{X}'} = \mathcal{L}_{\mathcal{X}'} + W'$

V' : *an elliptic surface with* $p_g = q = 1$, *with two singular fibers of type* I_0^* *and with four sections each of which is an elliptic curve with self-intersection* -1

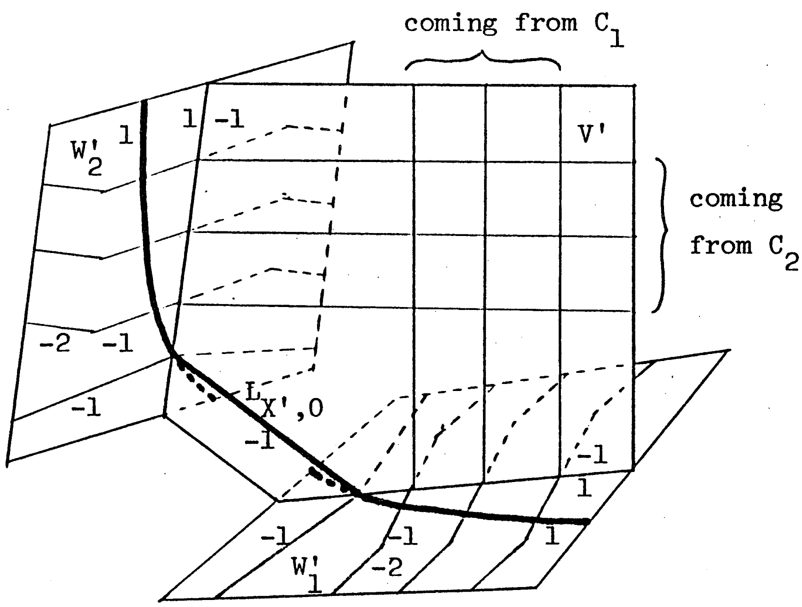
W' : *One description is as follows. We start with a configuration below of two cubics* D_V *and* D_L *on* $\mathbb{P}^2$ *such that* $D_V \simeq V' \cap W'$ *and* $D_L \simeq \mathcal{L}_{\mathcal{X}'} \cap W'$.



where A_0 *is a common inflex on* D_V *and* D_L *with intersection multiplicity* $I(A_0, D_V \cap D_L) = 3$; *there is unique abelian group structure on* D_V *and* D_L *such that* A_0 *is the zero element;* A_k $(0 \leq k \leq 3)$ *are the four 2-torsion points both on* D_V *and on* D_L *with respect to the above abelian group structures and* $I(A_k, D_V \cap D_L) = 2$ $(1 \leq k \leq 3)$. *Blow-up twice at each of the four common points* $D_V \cap D_L$, *we get* W'.

$V' \cap W' \simeq D_V$, $\mathcal{L}_{\mathcal{X}'} \cap V'$ *and* $\mathcal{L}_{\mathcal{X}'} \cap W' \simeq D_L$ *are elliptic curves.*

Figure 6



$f'^{-1}(0) = (X'_0, L_{X',0})$, $X'_0 = V' + W'_1 + W'_2$, $K_{\mathcal{X}'} = \mathcal{L}_{\mathcal{X}'} + \Sigma W'_j$

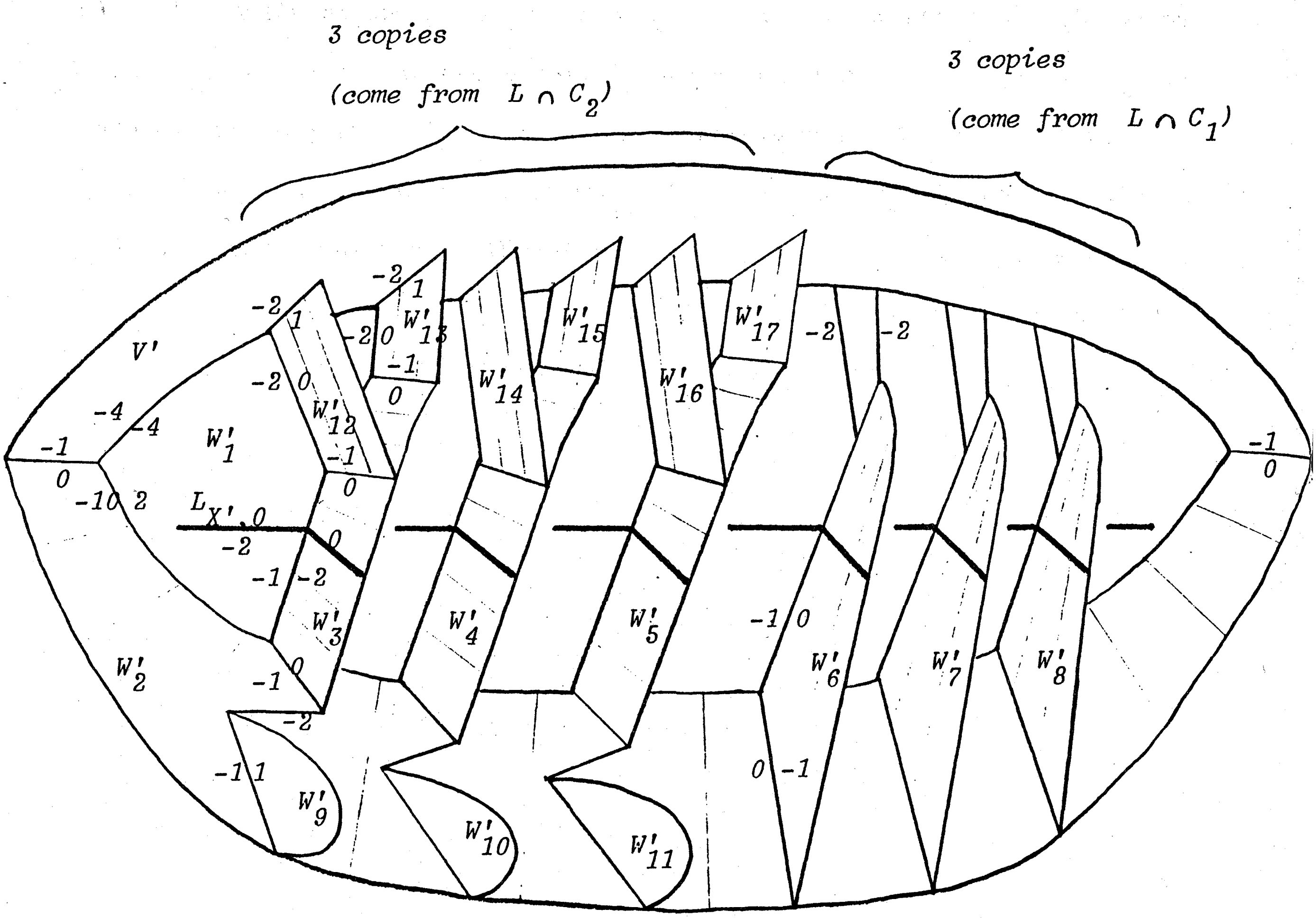
V' : *blowing-up one point on a decomposable abelian surface* $D_1 \times D_2$, *where* D_j *is an elliptic curve* $(j = 1, 2)$

W'_j $(j = 1, 2)$: *a rational surface like* W' *in Figure 5*

$V' \cap W'_j$ *and* $\mathcal{L}_{\mathcal{X}'} \cap W'_j$ *are elliptic curves* $(j = 1, 2)$

$\mathcal{L}_{\mathcal{X}'} \cap V'$: *a (-1)-curve*

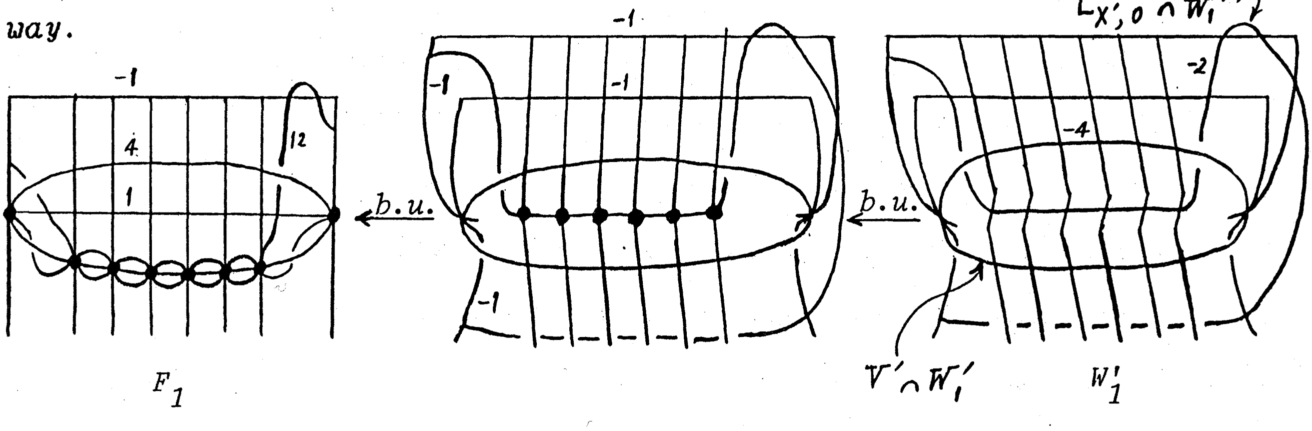
Figure 7



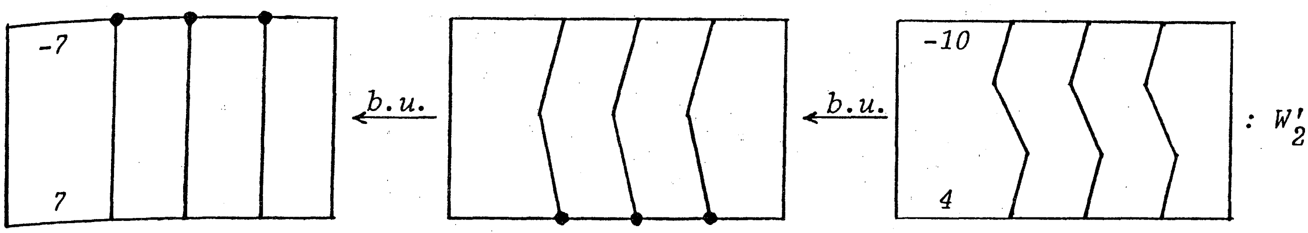
$X'_0 = V' + \Sigma_1^{17} W'_i$, $\quad K_{\mathfrak{X}'} = \mathcal{L}_{\mathfrak{X}'} + (\Sigma_1 + 2\Sigma_2^5 + 4\Sigma_6^{11} + \Sigma_{12}^{17})W'_i$

V' : *a K3 surface blown-up two points (come from* $Q \cap L$*)*

W'_1 : *One description is 14 times blown-ups of* F_1 *in the following way.*

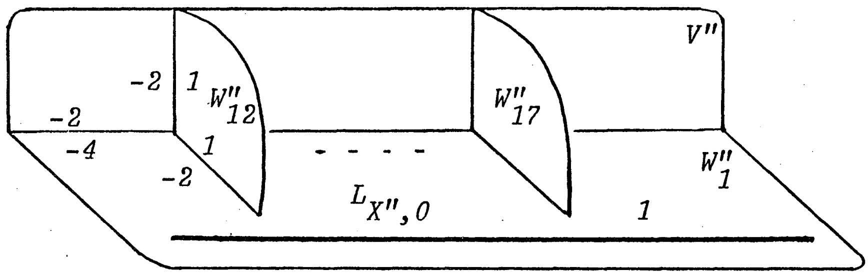


W_2' : *One description is six times blown-ups of* $\mathbb{P}^1$*-bundle of degree 7 over a curve of genus 3 in the following way.*



$W_i' \simeq F_2$ $(3 \le i \le 5)$, $W_i' \simeq F_1$ $(6 \le i \le 8,\ 12 \le i \le 17)$, $W_i' \simeq \mathbb{P}^2$ $(9 \le i \le 11)$

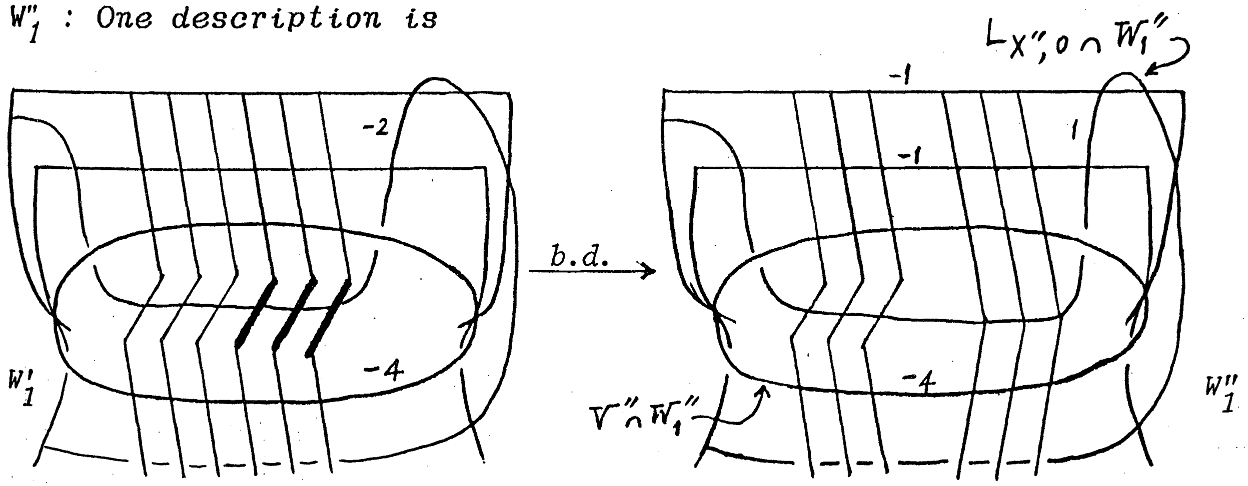
Figure 8



$X_0'' = V'' + (\Sigma_1 + \Sigma_{12}^{17})W_i''$, $K_{\mathfrak{X}''} = \mathcal{L}_{\mathfrak{X}''} + (\Sigma_1 + \Sigma_{12}^{17})W_i''$

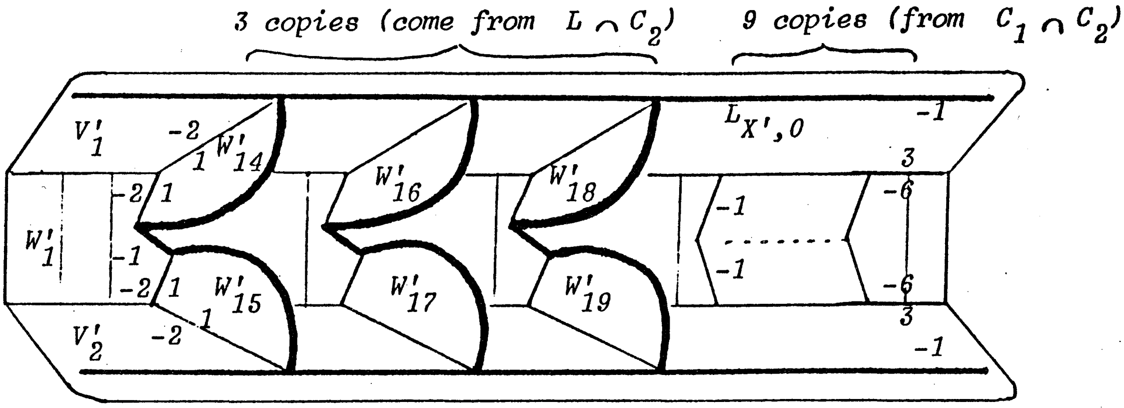
V'' : *a minimal K3 surface*

W_1'' : *One description is*



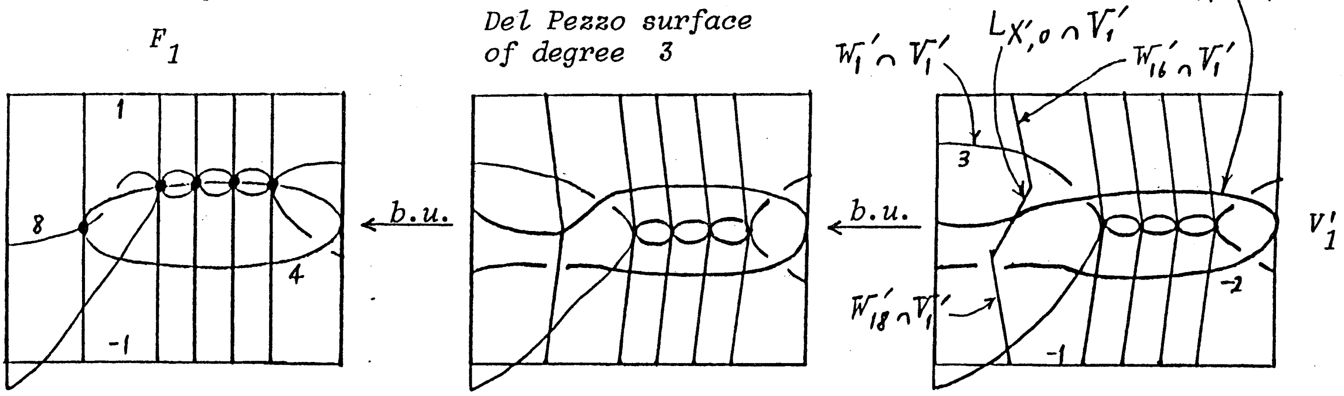
$W_i'' \simeq \mathbb{P}^2$ $(12 \le i \le 17)$, $L_{X'',0}$: *a curve of genus* 2

Figure 9

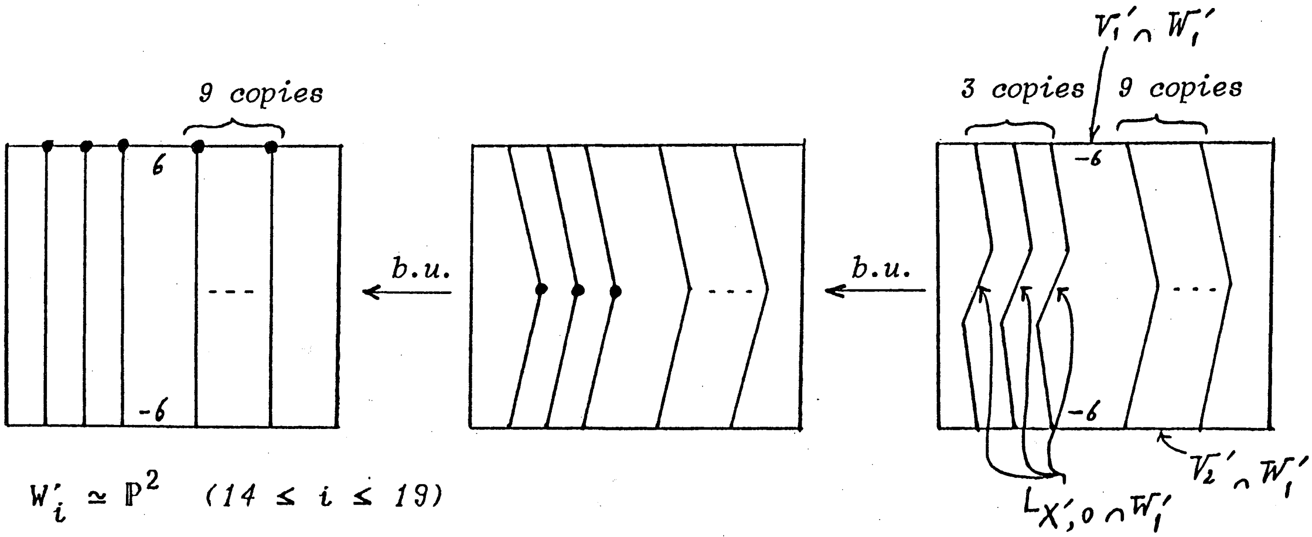


$X'_0 = \Sigma_1^2 V'_i + (\Sigma_1 + \Sigma_{14}^{19}) W'_{i'}, \quad K_{\mathfrak{X}'} = \mathcal{L}_{\mathfrak{X}'} + \Sigma_{14}^{19} W'_i$

V'_i : a rational surface. One description is six times blown-ups of F_1 in the following way (i = 1, 2).

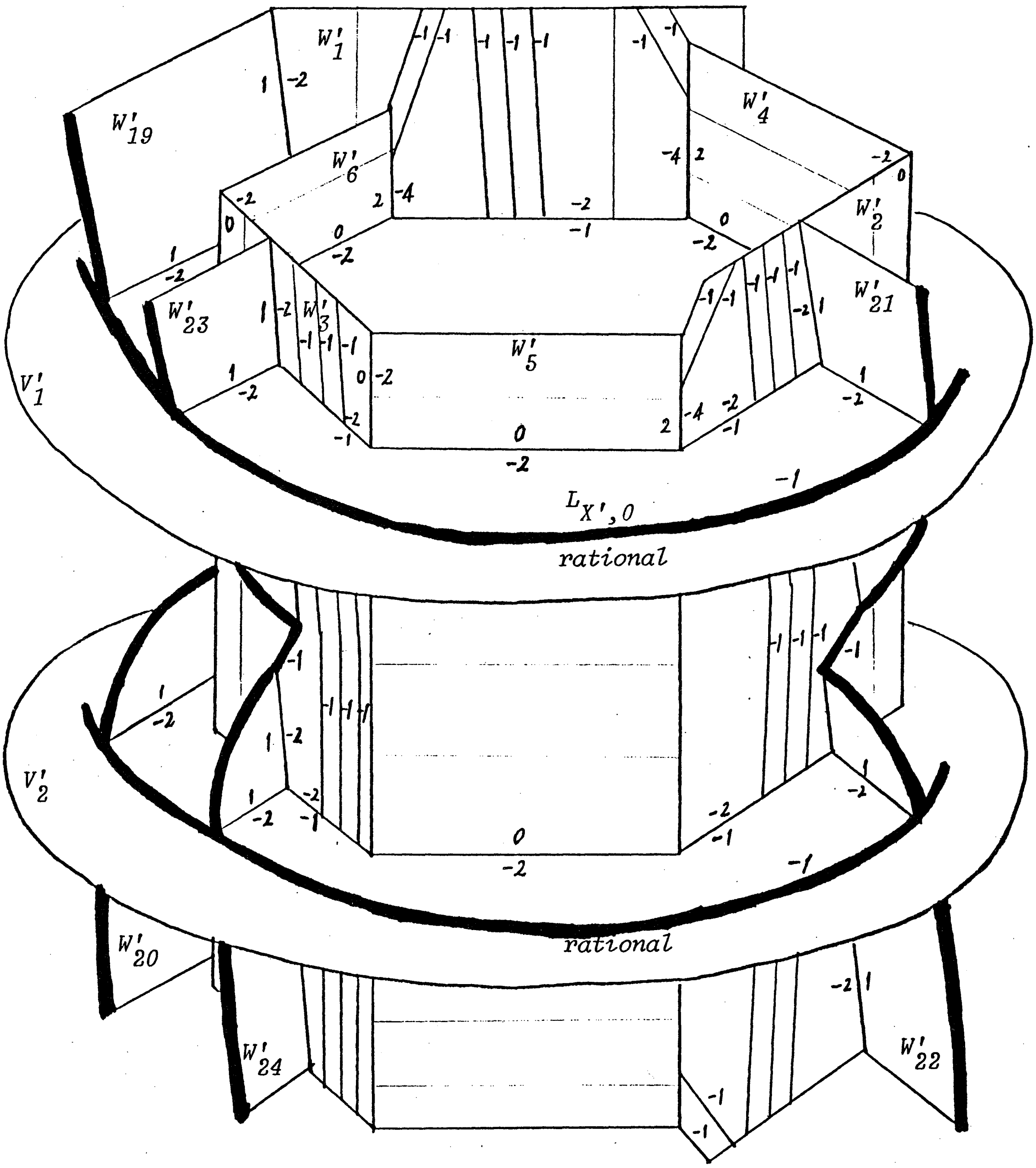


W'_1 : One description is 15 times bolwn-ups of an elliptic ruled surface of degree 6 in the following way.



$W'_i \simeq \mathbb{P}^2 \quad (14 \leq i \leq 19)$

Figure 10



$X'_0 = \Sigma_1^2 V'_i + (\Sigma_1^6 + \Sigma_{19}^{24}) W'_i$, $\quad K_{\mathfrak{X}'} = \mathcal{L}_{\mathfrak{X}'} + \Sigma_{19}^{24} W'_i$

V'_i : a rational surface which is the minimal resolution of the double cover of $\mathbb{P}^2$ branched along $\Sigma M_i + L$

W'_i : a rational surface (cf. the above drawing) $(1 \le i \le 3)$

$W'_i \simeq F_2$ $(4 \le i \le 6)$, $\quad W'_i \simeq \mathbb{P}^2$ $(19 \le i \le 24)$

REFERENCES

[Ca.1] Catanese, F., Surfaces with $K^2 = p_g = 1$ and their period mapping, Proc. Summer Meeting, Copenhagen 1978, Lect. Notes in Math. No 732, Springer-Verlag (1979) 1-29.

[Ca.2] Catanese, F., The moduli and the global period mapping of the surfaces with $K^2 = p_g = 1$: a counter-example to the global Torelli problem, Comp. Math. 41-3 (1980) 401-414.

[Cl] Clemens, C. H., Degenerations of Kähler manifolds, Duke Math. J. 44 (1977) 215-290.

[H] Horikawa, E., Surjectivity of period map of K3 surfaces of degree 2, Math. Ann. 228 (1977) 113-146.

[K] Kodaira, K., On homotopy K3 surface, Essays on topology and related topics, Mémoires dédiés à George de Rham, Springer, 1970, pp 58-69.

[L] Letizia, M., Intersections of a plane curve with a moving line and a generic global Torelli-type theorem for Kunev surfaces, Amer. J. Math. 106-5 (1984) 1135-1146.

[M] Morrison, D., On the moduli of Todorov surfaces, to appear.

[Sa] Saito, M.-H., On the infinitesimal Torelli problem of elliptic surfaces, J. Math. Kyoto Univ. 23-3 (1983) 441-460.

[Sh] Shah, J., A complete moduli space for K3 surfaces of degree 2, Ann. Math. 112 (1980) 485-510.

[SSU] Saito, M.-H., Shimizu, Y. & Usui, S., Variation of mixed Hodge structure and Torelli problem, to appear in Advanced Study of Pure Math., North-Holland Publ. C. & Kinokuniya C. LTD.

[T.1] Todorov, A. N., Surfaces of general type with $p_g = 1$ and (K,K)

= 1: I, Ann. scient. Éc. Norm. Sup. 4-13 (1980) 1-21.

[T.2] Todorov, A. N., A construction of surfaces with $p_g = 1$, $q = 0$ and $2 \leq (K^2) \leq 8$: Counterexamples of the global Torelli theorem, Invent. Math. 63 (1981) 287-304.

[U.1] Usui, S., Period map of surfaces with $p_g = c_1^2 = 1$ and K ample, Mem. Fac. Sci. Kochi Univ. (Math.) 3 (1981) 37-73.

[U.2] Usui, S., Effect of automorphisms on variation of Hodge structure, J. Math. Kyoto Univ. 21-4 (1981) 645-672.

[U.3] Usui, S., Variation of mixed Hodge structure arising from family of logarithmic deformations, Ann. scient. Éc. Norm. Sup. 4-16 (1983) 91-107; Id II: Classifying space, Duke Math. J. 51-4 (1984) 851-875.

[U.4] Usui, S., Degeneration of Kunev surfaces I; II, Proc. Japan Acad. 63-A-4 (1987) 110-113; 63-A-5 (1987) 167-169.

[U.5] Usui, S., Degeneration of Kunev surfaces, to appear in Astérisque.

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