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ON FINITE GALOIS COVERINGS OF COMPACT COMPLEX MANIFOLDS

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1. Introduction. A finite branched covering of an n-dimensional connected compact complex manifold M is, by definition, an irreducible normal complex space X together with a surjective proper finite holomorphic mapping $\pi : X \to M$. A point $p \in X$ is called an unramified point if π is locally biholomorphic around p. Otherwise, p is called a ramified point. The set of all ramified points forms a hypersurface R_π of X, called the ramification locus of π. The image $B_\pi = \pi(R_\pi)$ of R_π is called the branch locus of π, which is a hypersurface of M.

A morphism (resp. isomorphism) of $\pi : X \to M$ to $\pi' : X' \to M$ is a holomorphic (resp. biholomorphic) mapping $\phi : X \to X'$ such that $\pi'\phi = \pi$. An automorphism of π is called a _covering transformation of_ π. The set of all covering transformations of π forms a group G_π under composition, called the _covering transformation group of_ π. The covering $\pi : X \to M$ is called a _Galois covering_ if G_π acts transitively on every fiber.

Let $\pi : X \to M$ be a finite Galois covering and $B_\pi = D_1 \cup \dots \cup D_s$ be the irreducible decomposition of the branch locus B_π. For each non-singular point q of B_π, every point $p \in \pi^{-1}(q)$ is a non-singular point of both X and $\pi^{-1}(B_\pi)$. Moreover, there are coordinate systems $(w_1, \dots, w_n)$ and $(z_1, \dots, z_n)$ around q and p, respectively, such that $q = (0, \dots, 0)$, $p = (0, \dots, 0)$, $B_\pi = \{w_1 = 0\}$, $\pi^{-1}(B_\pi) = \{z_1 = 0\}$ and

$$\pi : (z_1, \dots, z_n) \to (w_1, \dots, w_n) = (z_1^e, z_2, \dots, z_n),$$

locally. The positive integer $e = e_j$ (≥ 2) is constant as a function of q on D_j - Sing B_π for every D_j and is called the _ramification index of_ π _around_ D_j. (Sing

B_π is the singular locus of B_π.) In this case, π is said to <u>branch along the positive divisor</u> $D = e_1D_1 + \ldots + e_sD_s$ <u>on</u> M.

The purpose of this note is to give a sufficient condition for the existence of a finite Galois covering $\pi : X \to M$ of M <u>with non-singular</u> X which branches along a <u>given</u> positive divisor $D = e_1D_1 + \ldots + e_sD_s$ $(e_j \geq 2)$ on M. The existence of such coverings for some special cases were proved in the interesting papers Hirzebruch [3], Höfer [4] and Kato [6]. In this note, we make use of Selberg's theorem on the monodromy representation of a Fuchsian differential equation.

<u>2. Kato's criterion.</u> Let B be a hypersurface of M and $B = D_1 \cup \ldots \cup D_s$ be its irreducible decomposition.

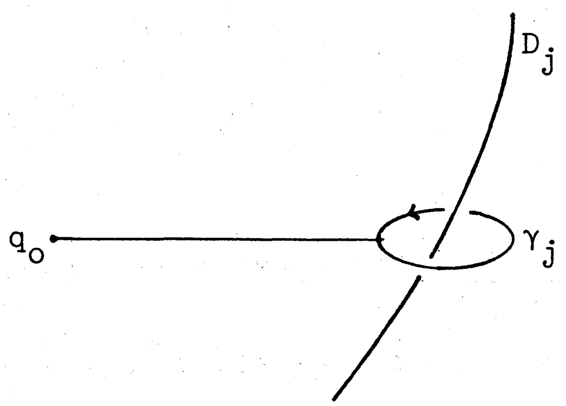

Figure 1

We fix a point $q_o \in M - B$ once for all. Let γ_j be a loop in $M - B$ starting and terminating at q_o encircling a point of D_j - Sing B in the positive sense as in Figure 1. We identify γ_j with its homotopy class in the fundamental group $\pi_1(M - B, q_o)$. Let $e_1, \ldots, e_s$ be integers greater than one. We denote by

$$J = \langle \gamma_1^{e_1}, \ldots, \gamma_s^{e_s} \rangle^{\pi_1}$$

the smallest normal subgroup of $\pi_1(M - B, q_o)$ which contains $\gamma_1^{e_1}, \ldots, \gamma_s^{e_s}$.

For a subgroup K of $\pi_1(M - B, q_o)$ with $J \subset K$, we consider the following condition:

Condition A. If γ_j^d belongs to K, then $d \equiv 0 \pmod{e_j}$ for every j $(1 \leq j \leq s)$.

Then we have

Theorem 1 (Namba [8]). There exists a finite Galois covering $\pi : X \to M$ which branches along $D = e_1D_1 + \ldots + e_sD_s$ if and only if there exists a normal subgroup K of

$\pi_1(M - B, q_o)$ of finite index which contains J and satisfies Condition A. The correspondence

$$\pi \rightarrow K = K(\pi) = \pi_*(\pi_1(X - \pi^{-1}(B)))$$

between (isomorphism classes of) such π's and such K's is one-to-one. G_π is isomorphic to $\pi_1(M - B, q_o)/K$.

<u>Henceforth, we suppose that</u> B <u>is simple normally crossing.</u> For any point $q \in B$, we take a local coordinate system $(w_1, \ldots, w_n)$ around q such that $q = (0, \ldots, 0)$ and $B = \{w_1 \ldots w_t = 0\}$ locally. We may suppose $D_j = \{w_j = 0\}$ locally for $1 \leq j \leq t$.

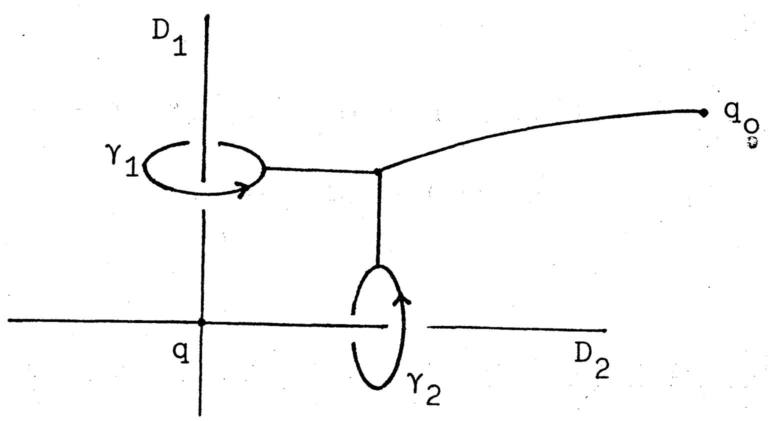

Figure 2

Let $\hat{\gamma}_j$ be a loop in $M - B$ starting and terminating at q_o encircling a point of D_j - Sing B near q in the positive sense as in Figure 2. $\hat{\gamma}_j$ is then conjugate to γ_j in $\pi_1(M - B, q_o)$. Note that $\hat{\gamma}_1, \ldots, \hat{\gamma}_t$ are mutually commutative.

For a subgroup K of $\pi_1(M - B, q_o)$ with $J \subset K$, we consider the following condition:

Condition A'. If $\hat{\gamma}_1^{d_1} \ldots \hat{\gamma}_t^{d_t}$ belongs to K, then $d_1 \equiv 0 \pmod{e_1}, \ldots, d_t \equiv 0 \pmod{e_t}$ for every point $q \in B$.

Then, as a special case of Kato [5], we have

Theorem 2 (Kato). In Theorem 1, if $K = K(\pi)$ satisfies Condition A', then X is non-singular.

3. Fuchsian differential equations. Let $B = D_1 \cup \ldots \cup D_s$ be as above simple normally crossing. Let Ω be an $(r \times r)$-matrix-valued meromorphic 1-form on M such that $d\Omega + \Omega \wedge \Omega = 0$ and such that Ω is holomorphic on $M - B$.

For an unknown r-vector-valued function Y, the differential equation

$$dY = Y\Omega \qquad (1)$$

(of order r) is called a <u>Fuchsian differential equation with regular singularity along</u> B, if, for every point $q \in B$, Ω can be locally written as

$$\Omega = A_1(w)\frac{dw_1}{w_1} + \dots + A_t(w)\frac{dw_t}{w_t} + A_{t+1}(w)dw_{t+1} + \dots + A_n(w)dw_n ,$$

around q, under the notations in §2, where $A_j(w)$ are $(r \times r)$-matrix-valued holomorphic functions around q. In this case,

$$\mathrm{Res}_{D_j}\Omega = A_j(q) \qquad (1 \leq j \leq t)$$

is a constant matrix on D_j, called the <u>residue of</u> Ω at D_j.

<u>Theorem 3 (Gérard [2], Yoshida-Takano [10])</u>. A fundamental matrix solution $F(w)$ around $q \in B$ of the Fuchsian differential equaiton (1) with regular singularity along B can be written as

$$F(w) = w_1^{C_1} \dots w_t^{C_t} w_1^{N_1} \dots w_t^{N_t} G(w),$$

where $C_1, \dots, C_t$ are mutually commutative constant matrices,

$N_1, \ldots, N_t$ are diagonal matrices whose components are non-negative integers, and $G(w)$ is an $(r \times r)$-matrix-valued holomorphic function around q with $\det G(w)$ nowhere vanishing. Moreover, if none of the differences of the eigenvalues of $\mathrm{Res}_{D_j}\Omega$ are non-zero integers for $1 \leq j \leq t$, then C_j and N_j can be so chosen that $N_j = 0$ $(1 \leq j \leq t)$ and C_j is conjugate to $\mathrm{Res}_{D_j}\Omega$ $(1 \leq j \leq t)$.

4. Existence of finite Galois coverings. Let $B = D_1 \cup \ldots \cup D_s$ be as above simple normally crossing. Suppose (i) $\pi_1(M - B, q_o)$ is finitely generated. (This condition is satisfied if M is projective as Prof. M. Oka informed us.) Suppose that there is a Fuchsian differential equation (1) with regular singularity along B such that (ii) the order r of (1) satisfies $r \geqslant n$, (iii) every $\mathrm{Res}_{D_j}\Omega$ $(1 \leq j \leq s)$ is diagonalizable and (iv) every eigenvalue of $\mathrm{Res}_{D_j}\Omega$ is a rational number.

We write the eigenvalue as

$$\frac{a_{j1}}{e_j}, \ldots, \frac{a_{jr}}{e_j}, \tag{2}$$

where e_j $(\geqslant 2)$ and $a_{j\nu}$ are integers such that

$$(a_{j1}, \ldots, a_{jr}, e_j) = 1,$$

where $(*, \ldots, *)$ denotes the GCD of the components.
Suppose moreover (v) if $a_{j\nu} \equiv a_{j\mu}$ (mod e_j), then $a_{j\nu} = a_{j\mu}$.

For a point $q \in$ Sing B, $\mathrm{Res}_{D_j}\Omega$ $(1 \leq j \leq t)$ under the notations in §2 are mutually commutative. Hence they can be simultaneously diagonalizable:

$$P(\mathrm{Res}_{D_j}\Omega)P^{-1} = \begin{pmatrix} a_{j1}/e_j & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & a_{jr}/e_j \end{pmatrix}$$

for $1 \leq j \leq t$, where P is a non-singular matrix.

Let

$$\Delta_1 , \ldots , \Delta_N \qquad (N = \binom{r}{t}).$$

be the $(t \times t)$-minors of the $(t \times r)$-matrix $(a_{j\nu})$. Put

$$f_j = e_j < e_1, \ldots , e_{j-1}, e_{j+1}, \ldots , e_t> / <e_1, \ldots , e_t>$$

for $1 \leq j \leq t$, where $\langle *, \ldots, * \rangle$ denotes the LCM of the components.

<u>Theorem 4</u>. Under the above notations and assumptions (i) ~ (v), suppose moreover (vi) $(\Delta_1, \ldots, \Delta_N, f_j) = 1$ $1 \leq j \leq t$ for every point $q \in \mathrm{Sing}\, B$. Then there exists a finite Galois covering $\pi : X \to M$ with non-singular X which branches along $D = e_1 D_1 + \ldots + e_s D_s$.

For the proof of Theorem 4, we make use of the following theorem of Selberg [9] (see also Borel [1]).

<u>Theorem 5 (Selberg)</u>. Let Γ be a finitely generated subgroup of $GL(r, \mathbb{C})$. Then there exists a normal subgroup Γ_o of Γ of finite index and of torsion free. If $\Gamma \neq \{1\}$, then Γ_o can be so chosen that $\Gamma_o \neq \Gamma$.

<u>Proof of Theorem 4</u>. Let $R : \pi_1(M - B, q_o) \to GL(r, \mathbb{C})$ be the monodromy representation of the equation (1) and put $\Gamma = R(\pi_1(M - B, q_o))$. Let Γ_o be a normal subgroup of Γ of finite index and of torsion free in Theorem 5. Put $K = R^{-1}(\Gamma_o)$.

Then K is a normal subgroup of $\pi_1(M - B, q_o)$ of finite index. We show that K satisfies the conditions of Theorem 2.

We first show that K contains J. By Theorem 3 and by the assumption (v),

$$R(\gamma_j^{e_j}) = R(\gamma_j)^{e_j} \backsim (\exp 2\pi i \operatorname{Res}_{D_j} \Omega)^{e_j}$$

$$= \exp 2\pi i\, e_j \operatorname{Res}_{D_j} \Omega$$

$$\backsim \exp 2\pi i \begin{pmatrix} a_{j1} & & 0 \\ & \ddots & \\ 0 & & a_{jr} \end{pmatrix} = 1,$$

where $\backsim$ means the conjugacy relation. Hence

$$\gamma_j^{e_j} \in \ker(R) \subset K,$$

so

$$J \subset K.$$

Next, suppose $\gamma_j^d \in K$. Then $R(\gamma_j)^d \in \Gamma_o$. Note that $(R(\gamma_j)^d)^{e_j} = (R(\gamma_j)^{e_j})^d = 1.$

Since Γ_o is torsion free, we have $R(\gamma_j)^d = 1$. This means that

//

$$da_{j1}/e_j, \; \ldots \; , da_{jr}/e_j$$

are integers. Hence $d \equiv 0 \pmod{e_j}$. This holds for $1 \leq j \leq s$.

Finally, for a point $q \in \operatorname{Sing} B$, suppose $\hat{\gamma}_1^{d_1} \ldots \hat{\gamma}_t^{d_t} \in K$, under the notations in §2. Then $R(\hat{\gamma}_1)^{d_1} \ldots R(\hat{\gamma}_t)^{d_t} \in \Gamma_o$. Note that

$$(R(\hat{\gamma}_1)^{d_1} \ldots R(\hat{\gamma}_t)^{d_t})^{e_1 \cdots e_t} = 1.$$

Since Γ_o is torsion free, we have $R(\hat{\gamma}_1)^{d_1} \ldots R(\hat{\gamma}_t)^{d_t} = 1$. This means that

$$d_1a_{11}/e_1 + \ldots + d_ta_{t1}/e_t$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$d_1a_{1r}/e_1 + \ldots + d_ta_{tr}/e_t$$

are integers. Now the assumption (vi) implies easily that

$$d_1 \equiv 0 \pmod{e_1}, \; \ldots \; , d_t \equiv 0 \pmod{e_t}.$$

Q.E.D.

5. Application to Appell's F_1. Let $(Z_0 : Z_1 : Z_2)$ be a homogeneous coordinate system on the complex projective plane $\mathbb{P}^2$ and let $(x, y) = (Z_1/Z_0, Z_2/Z_0)$ be the affine coordinate system. Appell's hypergeometric differential equation F_1 can be written as

$$d(f, x\frac{\partial f}{\partial x}, y\frac{\partial f}{\partial y}) = (f, x\frac{\partial f}{\partial x}, y\frac{\partial f}{\partial y})\, \Omega_o ,$$

where

$$\Omega_o = A\frac{dx}{x} + B\frac{dy}{y} + C\frac{dx}{x-1} + D\frac{dy}{y-1} + E\frac{d(x-y)}{x-y} ,$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1-\gamma+\beta' & -\beta' \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -\beta & 1-\gamma+\beta \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & -\alpha\beta & 0 \\ 0 & \gamma-\alpha-\beta-1 & 0 \\ 0 & -\beta & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & -\alpha\beta' \\ 0 & 0 & -\beta' \\ 0 & 0 & \gamma-\alpha-\beta'-1 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta' & \beta' \\ 0 & \beta & -\beta \end{pmatrix},$$

where α, β, β' and γ are constants, (see Kimura [7]). Ω_o is a (3×3)-matrix-valued meromorphic 1-form on $\mathbb{P}^2$ which is holomorphic on $\mathbb{P}^2 - B_o$, where

$$B_o = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6,$$

$$D_1 = \{x = 0\}, \quad D_2 = \{y = 0\},$$

$$D_3 = \{x = 1\}, \quad D_4 = \{y = 1\},$$

$D_5 = \{x = y\}$, D_6 = the line at infinity.

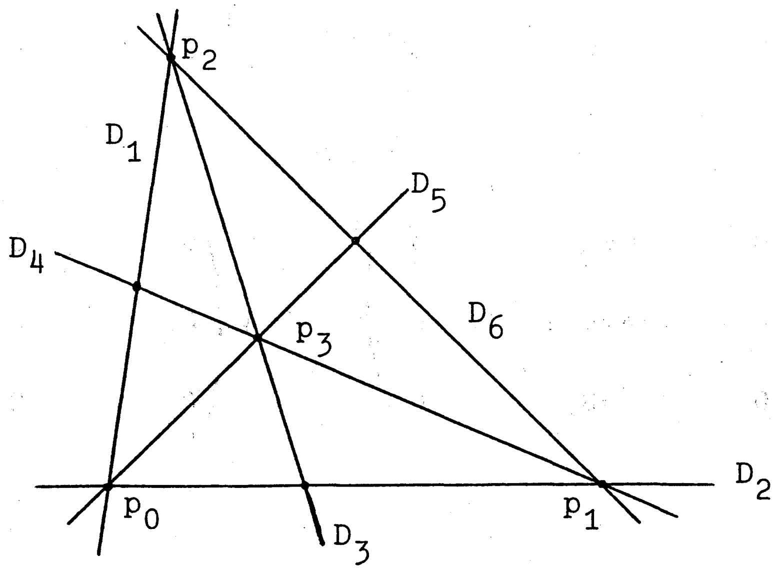

Figure 3

Consider the blowing up $\rho : M = \hat{\mathbb{P}}^2 \to \mathbb{P}^2$ at the four points

$p_0 = (1 : 0 : 0)$, $p_1 = (0 : 1 : 0)$, $p_2 = (0 : 0 : 1)$, $p_3 = (1 : 1 : 1)$

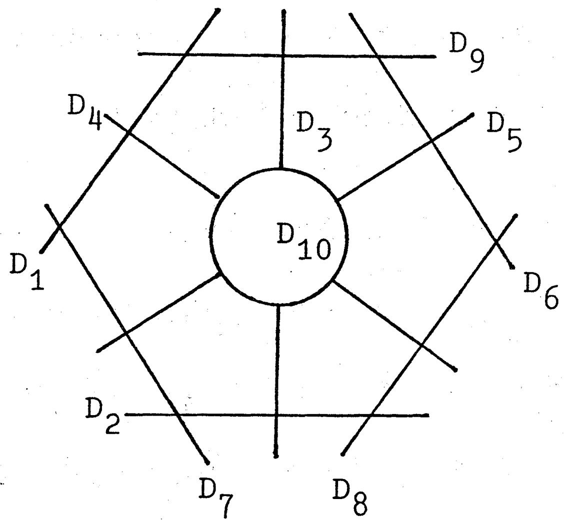

Figure 4

Then the differential equation

$$dY = Y\Omega \qquad (3)$$

on $\hat{\mathbb{P}}^2$, where $\Omega = \rho^*\Omega_0$, is Fuchsian with regular singularity along

$$B = D_1 \cup \ldots \cup D_{10} ,$$

where D_7, D_8, D_9, D_{10} are exceptional curves as in Figure 4.

(By abuse of notation, we write the strict transform of D_j as D_j again for $1 \leq j \leq 6$). Note that B is simple normally crossing.

We apply Theorem 4 to the equation (3). Suppose that α, β, β' and γ are rational numbers and suppose

$1 - \gamma + \beta' = a_1/e_1$, where $e_1 \geqslant 2$ and $(a_1, e_1) = 1$,

$1 - \gamma + \beta = a_2/e_2$, where $e_2 \geqslant 2$ and $(a_2, e_2) = 1$,

$\gamma - \alpha - \beta - 1 = a_3/e_3$, where $e_3 \geqslant 2$ and $(a_3, e_3) = 1$,

$\gamma - \alpha - \beta' - 1 = a_4/e_4$, where $e_4 \geqslant 2$ and $(a_4, e_4) = 1$,

$- \beta - \beta' = a_5/e_5$, where $e_5 \geqslant 2$ and $(a_5, e_5) = 1$,

$1 - \gamma = a_7/e_7$, where $e_7 \geqslant 2$ and $(a_7, e_7) = 1$,

$\gamma - 1 - \alpha - \beta - \beta' = a_{10}/e_{10}$, where $e_{10} \geqslant 2$ and $(a_{10}, e_{10}) = 1$,

$\beta + \beta' - \alpha = a_6/e_6$, $\alpha = b_6/e_6$, where $e_6 \geqslant 2$ and $(a_6, b_6, e_6) = 1$,

$\beta - \alpha = a_8/e_8$, $\alpha = b_8/e_8$, where $e_8 \geqslant 2$ and $(a_8, b_8, e_8) = 1$,

$\beta' - \alpha = a_9/e_9$, $\alpha = b_9/e_9$, where $e_9 \geqslant 2$ and $(a_9, b_9, e_9) = 1$.

Then, after some simple calculations, we have by Theorem 4:

<u>Theorem 6</u>. Under the above notations and assumptions, suppose moreover that $a_j \not\equiv 0 \pmod{e_j}$ for $j = 6, 8, 9$, and $(a_6, e_6, a_8, e_8) = 1$ and $(a_6, e_6, a_9, e_9) = 1$. Then there exists a finite Galois covering $\pi : X \to \hat{\mathbb{P}}^2$ with non-singular X which branches along $D = e_1 D_1 + \dots + e_{10} D_{10}$.

There are a lot of examples in which the conditions of Theorem 6 are satisfied. We give three such examples. (The numbers attached to the curves in the figures below mean the ramification indices along the curves.)

<u>Example 1</u>. $\alpha = 3 - 1/a$, $\beta = \beta' = 1 - 1/4a$ and $\gamma = 3 - 3/4a$, where $a \geqslant 2$ and $(a, 3) = 1$.

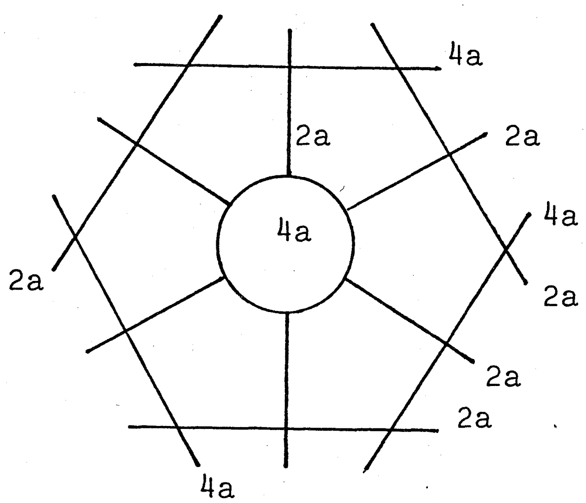

Figure 5

Example 2. $\alpha = 3 - 1/3a$, $\beta = \beta' = 1 - 1/12a$ and $\gamma = 3 - 1/4a$, where a is a positive integer.

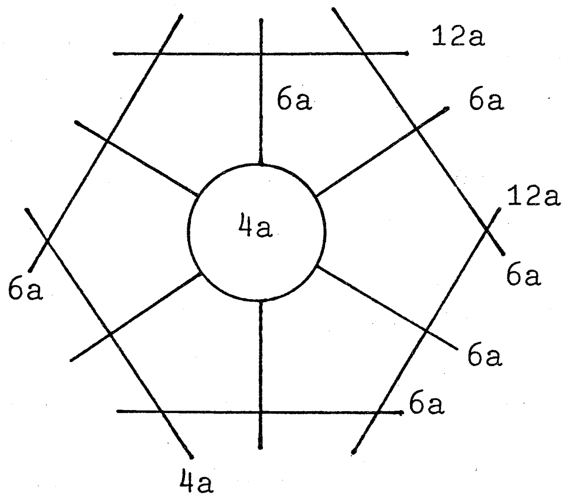

Figure 6

Example 3. $\alpha = 3 - 4/3a$, $\beta = \beta' = 1 - 1/3a$ and $\gamma = 3 - 1/a$, where $a \geqslant 3$ and $(a, 2) = 1$.

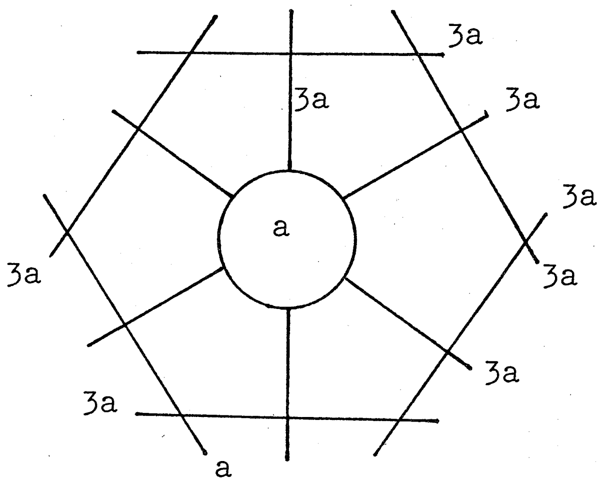

Figure 7

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