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Characteristic classes for singular varieties[(+)]

by

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INTRODUCTION. Some time back, at the Ninth Brazilian Mathematical Colloquium Robert MacPherson [Mac 2] gave a survey talk on characteristic classes for singular varieties. The introduction of his survey article goes as follows:"An important program in mathematics is to extend the large body of theory of smooth spaces or manifolds to singular spaces. Algebraic varieties form a fruitful context in which to work because the study of algebraic singularities is particularly well-developed. Our object here is to survey several results extending the theory of characteristic classes to singular algebraic varieties."

His survey was about "singular" Whitney classes (due to Sullivan), Chern classes (conjectured to exist by Deligne and Grothendieck, and constructed by MacPherson) and Todd classes (due to Baum, Fulton and MacPherson). The present paper is sort of an appendix to his article, and so the first part of this paper contains a repetition of some parts of MacPherson's survey article. Throughout this paper our varieties are projective (algebraic) varieties over **C**, the complex numbers.

(+) Some of the problems listed in this paper were suggested by Clint McCrory, while the author was visiting him at MSRI, Berkeley, in October, 1986.

1. Classical Theory of Characteristic Classes.

First of all, "usual" characteristic classes are defined for vector bundles as follows:

Definition 1.1. A characteristic class **cl** is a rule associating to any vector bundle **E** over any variety X a cohomology class $cl(E) \in H^*(X)$ such that

i) $cl(E \oplus F) = cl(E)\cdot cl(F)$,

ii) for any map $f: Y \longrightarrow X$ the pull-back property $cl(f^*E) = f^* cl(E)$ holds.

Or, it may be better to say the definition fashionably as follows: the classical definition of a characteristic class is a natural transformation from the Grothendieck K-functor to the ordinary cohomology functor with suitable coefficients:

$$K \xrightarrow{\;cl\;} H^*$$

Typical examples of such characteristic classes are Whitney classes, Chern classes, the Chern character, Todd classes and Segre classes (=inverse Chern classes).

Remark 1.2. The set of all possible characteristic classes forms a ring because cohomology has a ring structure by the cup-product.

In the above definition of characteristic classes the smoothness of the base space X is not required. When it comes to a smooth variety X, we can consider characteristic classes of the space X itself, by taking characteristic classes of the tangent bundle TX over X; i.e.,

$$cl(X) := cl(TX) \quad ,$$

where **cl** in the right hand side is a classical characteristic class.

Example 1.3. Let $cl_n(X) = c_n(X)$ be the top dimensional Chern class of X . Then $c_n(X)\cap[X] = \chi(X)$, the topological Euler-Poincaré characteristic of X .

Example 1.4. Let $cl_n(X) = td_n(X)$ be the top dimensional Todd class of X . Then $td_n(X)\cap[X] = g_a(X)$, the arithmetic genus of X .

Remark 1.5. Useful or meaningful characteristic classes are related to some other invariants of varieties.

2. Extension of characteristic classes from the case of smooth varieties to the case of singular varieties.

A principal difficulty in making the extension is that one cannot have a tangent bundle over a singular variety any longer and that there is no exact analogue of the tangent bundle, although there are some generalized "tangent bundles" (e.g. tangent cones).

Problem 1: What is a reasonable (or fruitful) theory of characteristic classes for singular varieties ?

MacPherson claims in [Mac 2] that (*) <u>the key to a useful extension of a given characteristic class lies in finding a theorem relating that class with other invariants which already</u>

extend to singular varieties. So far, a few characteristic classes of smooth varieties have been individually extended to singular varieties, in the sense of (*) above.

2.1. MacPherson's Chern classes (or Chern-MacPherson classes)

Motivated by Sullivan's definition of singular Whitney classes, Deligne and Grothendieck conjectured the existence of singular Chern classes as a natural transformation (described below), which was affirmatively proved by R. MacPherson in 1974 [Mac 1]:

Theorem 2.1.1. (Deligne-Grothendieck, MacPherson) Let **F** be the covariant functor from complex algebraic varieties to abelian groups whose value **F**(X) on a variety X is the group of constructible functions from X to Z, the ring of integers, and whose value f_* on a proper map $f: X \longrightarrow Y$ is determined by the condition that $f_*(1_V)(p) = \chi(f^{-1}(p) \cap V)$, for all $p \in Y$, where V is a closed subvariety of X . Then , for all X, there exists a unique natural transformation

$$c_{*X} : \mathbf{F}(X) \longrightarrow H_*(X;\ Z)$$

such that (c_{*X} is simply denoted by c_*)

Axiom(1): $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$,

Axiom(2): $c_*(f_*\alpha) = f_*c_*(\alpha)$,

Axiom(3): $c_*(1_X) = c(X) \cap [X]$, if X is smooth, where $c(X)$ is the usual total Chern (cohomology) class of X .

Axiom(1) is additivity, Axiom(2) is naturality and Axiom(3) is normalization. One might think this to be in analogy with the natural transformation from the K-functor to the H^*-functor in the classical set-up, by replacing K and H^* by F and H_* respectively. Here one might be tempted to pose the following

Problem 2: What are all possible nontrivial additive natural transformations from the constructible function functor F to the homology functor H_* (i.e., transformations from F to H_* satisfying Axioms (1) and (2), not necessarily satisfying Axiom (3)) ?

Remark 2.1.2. For any non-zero integer m , $m \cdot c_*$ is clearly a nontrivial additive natural transformation from **F** to H_* . If $m \neq 1$, then $m \cdot c_*$ does not satisfy Axiom (3) . It seems that the requirement of Axiom (3) in the theorem leads to the uniqueness of c_* .

Definition 2.1.3. For all X , $c_*(1_X)$ is called the <u>Chern-MacPherson class</u> and is simply denoted by C(X) , which is the total homology class .

Note that the 0-dimensional component of C(X) , $C_0(X) \in H_0(X;Z)$, is equal to the topological Euler-Poincare characteristic $\chi(X)$ of X .

MacPherson [Mac 1] proved the above theorem via his graph construction and using Chern-Mather classes. Later, A.Dubson [Du] gave an explicit formula for Chern-MacPherson class C(X):

Theorem 2.1.4. (Dubson's formula)

$$C(X) = C^M(X) + \sum_{\substack{S \in \mathcal{S}_X \\ S \subset X_{sing}}} \Theta(S,X) \cdot C^M(S)$$

where $C^M(\#)$ is the Chern-Mather class of # , $\mathcal{S}_X$ is a Whitney stratification of X with X_{smooth} being the top stratum, $\Theta(S,X)$ is some integer associated with each stratum $S \in \mathcal{S}_X$ (for more details see below).

Definition 2.1.5. (Mather) Let $\nu : \hat{X} \longrightarrow X$ be the Nash blow-up of X and $\widehat{TX}$ be the tautological Nash tangent bundle over $\hat{X}$. Then the <u>Chern-Mather class</u> $C^M(X)$ of X is defined by

$$C^M(X) := \nu_*(c(\widehat{TX}) \cap [\hat{X}]) ,$$

where $c(\widehat{TX})$ is the classical Chern class of the vector bundle $\widehat{TX}$.

Definition 2.1.6.(cf.Dubson [Du]) Let $\mathcal{S}_X$ be a Whitney stratification of X such that the top stratum of X is the smooth part of X. Then for each stratum $S \in \mathcal{S}_X$,

$$\Theta(S,X) := 1 - \chi(X \cap H_{s,\varepsilon}^{1+\dim S} \cap B_\delta(s)) ,$$

where $s \in S$, $0 < \varepsilon << \delta$, and $H_{s,\varepsilon}^{1+\dim S}$ is a plane of codimension $1+\dim S$, which does not go through s and is close to s within the distance ε. (see Figure 1)

Dubson [Du] showed that the definition of $\Theta(S,X)$ above is independent of the choice of $s \in S$.

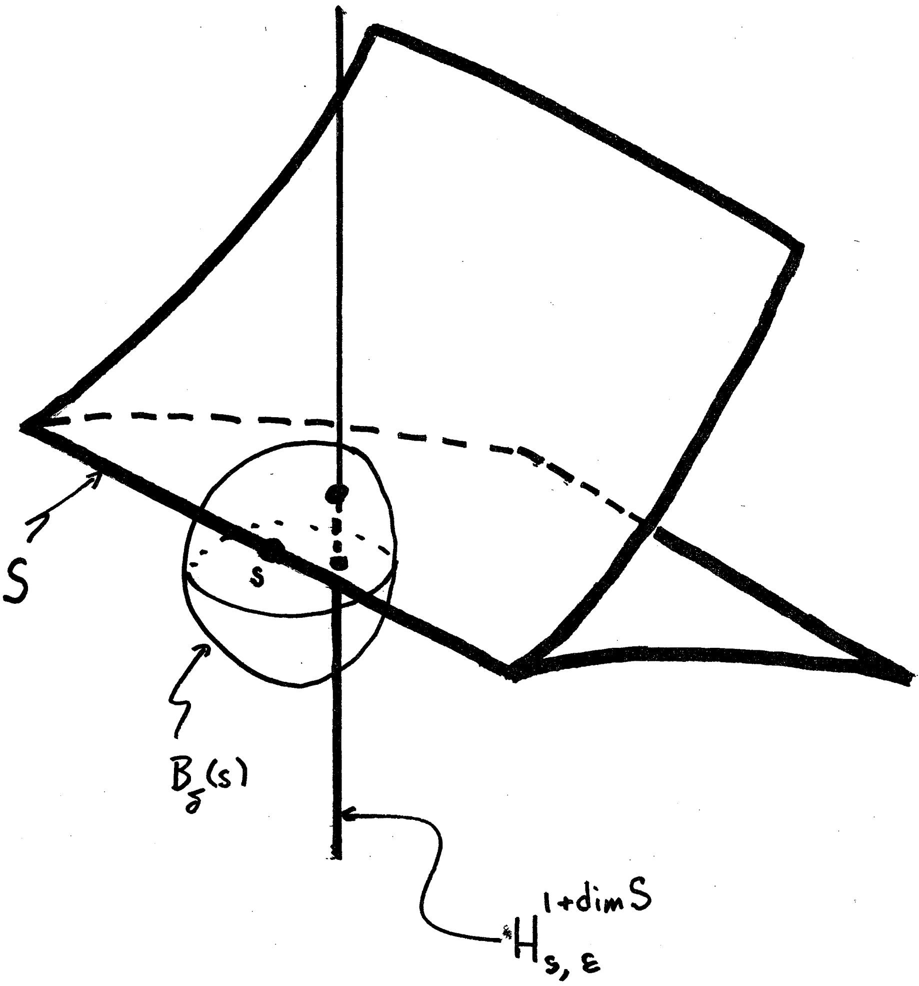

Figure 1.

Motivated by Mather's Chern classes, one might be tempted to define the following characteristic classes for singular varieties, which may be called "Nash" characteristic classes since they are defined via the Nash blow-up:

Definition 2.1.7.(Nash characteristic classes) Let **cl** denote any classical characteristic class for a vector bundle. Then the <u>Nash</u>

characteristic class $\widehat{cl}(X)$ of a variety X is defined by:

$$\widehat{cl}(X) := \nu_*(cl(\widehat{TX}) \cap [\widehat{X}]) \ ,$$

where $cl(\widehat{TX})$ is the classical characteristic class of the vector bundle $\widehat{TX}$ and ν is the Nash blow-up of X .

Remark 2.1.8. Nash characteristic classes are a naive theory of characteristic classes for singular varieties, apart from the problem of whether it fits the principle (*) suggested by MacPherson in his survey article. One can interpret Dubson's formula as a formula for the difference between Chern-MacPherson classes and Nash-Chern classes (i.e., Chern-Mather classes), which is measured systematically by singularities together with some reasonable invariants $\Theta(S,X)$. Chern-MacPherson classes and Chern-Mather classes have been studied by many mathematicians in connection with polar classes, Whitney stratifications, numerical invariants and equisingularity (e.g. see Dubson [Du], Lê-Teissier [LT], Piene [Pi 1,2] and Urabe [U 1,2])

Remark 2.1.9. MacPherson's theory of Chern classes is for singular algebraic varieties over the complex numbers. Recently in [Ke] Gary Kennedy introduced "stiff" Chern classes to extend MacPherson's theory to varieties over an arbitrary algebraically closed field of characteristic zero. In Kennedy's theory, MacPherson's constructible function functor **F** is replaced by his "stiff" functor **St**: for X , $\mathbf{St}(X) := \bigoplus_W \mathbb{Z}\hat{c}(W)$, where W is any closed subvariety of X and $\hat{c}(W)$ denotes the Chern-Mather class (i.e., Nash-Chern class) of W . For more details see [Ke].

2.2. Baum-Fulton-MacPherson's Todd classes

An extension of Todd classes of smooth varieties to singular varieties is based on Grothendieck's Riemann-Roch Theorem (GRR) about the classical Todd classes. Baum, Fulton and MacPherson [BFM] proved the following generalization of GRR:

Theorem 2.2.1. (BFM's Riemann-Roch) Let K_* denote the covariant functor from complex algebraic varieties to abelian groups whose value $K_*(X)$ on a variety X is the Grothendieck group of coherent algebraic sheaves on X and whose value $f_!$ on a map $f:X \longrightarrow Y$ is determined by the condition that

$$f_!\mathcal{F} = \sum(-1)^i R^i f_*\mathcal{F} .$$

Then, for all X, there exists a unique natural transformation

$$Td : K_*(X) \longrightarrow H_*(X;\mathbb{Q})$$

such that Axiom(1): $Td(\alpha + \beta) = Td(\alpha) + Td(\beta)$,

Axiom(2): $Td(f_!\alpha) = f_*Td(\alpha)$,

Axiom(3): $Td(E) = [ch(E)\cup td(X)]\cap[X]$, if X is smooth and E is a vector bundle (locally free sheaf), where ch(E) is the Chern character of E and td(X) is the classical Todd class of X .

Definition 2.2.2. For all X, the (singular) Todd class of X is defined to be Td(1) , where 1 is the trivial line bundle over X . This is denoted simply by Td(X) . (Hence, if X is smooth, then by Axiom(3) Td(X) is the Poincaré dual of the classical Todd class td(X) of X.)

Note that the 0-dimensional component of Td(X) is equal to the arithmetic genus of X.

In analogy with Dubson's formula for Chern-MacPherson classes, one may ask if one could express Baum-Fulton-MacPherson's Todd class Td(X) as the sum of the Nash-Todd class $\widehat{Td}(X)$ and some class supported on the singular locus of X. For example we may pose the following:

Problem 3: Let X be a singular variety with isolated singularities. Formulate the difference between BFM's Todd class Td(X) and the Nash-Todd class $\widehat{Td}(X) := \nu_*(td(T\widehat{X}) \frown [\widehat{X}])$. Perhaps,

$$Td(X) = \widehat{Td}(X) + \sum a_i [x_i] ,$$

where a_i is some integer attached to each singularity x_i, which may be describable in terms of known invariants of singularities.

Remark 2.2.3.([F, Example 18.3.3.]) If $\pi : \widetilde{X} \longrightarrow X$ is a proper birational morphism, isomorphic off $Z \subset X$, then

$$Td(X) = \pi_* Td(\widetilde{X}) + \alpha ,$$

where α is some class supported in Z . In particular,

$$Td_k(X) = \pi_* Td_k(\widetilde{X}) \quad \text{for } k > \dim Z .$$

If X is an n-dimensional singular variety with a finite number of singularities, and $\pi : \widetilde{X} \longrightarrow X$ is a resolution of singularities, then

$$Td_k(X) = \pi_*(td_{n-k}(T\widetilde{X}) \frown [\widetilde{X}]), \text{ for } k > 0,$$

and

$$Td_0(X) = \pi_*(td_n(T\widetilde{X}) \frown [\widetilde{X}]) + \sum_p n_p [p],$$

where p's are singularities of X , and

$$n_p = \sum_{i=1}^{n-1} (-1)^{i-1} \text{length}(R^i \pi_* \mathcal{O}_{\widetilde{X}})_p - \text{length}(\pi_* \mathcal{O}_{\widetilde{X}} / \mathcal{O}_X)_p .$$

Here $\text{length}(\mathcal{F})_p$ denotes the length of the stalk of the sheaf at p

2.3. Segre classes ("inverse" Chern classes)

Segre classes are inverse Chern classes when the given varieties are smooth. One might ask oneself what would be the "inverse" of Chern-MacPherson classes. Since we do not know, right now, the meaning of "inverse", let us consider (Fulton-) Johnson's Segre classes for singular varieties. First of all we recall the following definition from Fulton's book [F]:

Definition 2.3.1.(Relative Segre classes) Let X be a scheme and Y a subscheme of X. Then the relative Segre class S(Y,X) of Y to X is defined by: $S(Y,X) := \sum_{i \geq 0} p_*(c_1(\mathcal{O}(1))^i \cap [P(C)])$, where C is the normal cone of Y in X , P(C) is the projectivized normal cone, $p: P(C) \longrightarrow Y$ is the projection, and $\mathcal{O}(1)$ is the associated line bundle over P(C).

Then, Johnson's Segre class of a scheme (or a variety) X is defined to be the relative Segre class $S(\Delta, X \times X)$ of the diagonal Δ of X x X to X x X, i.e.,

Definition 2.3.2. (Johnson's Segre classes) Let X be a scheme. Blow up X x X along the diagonal Δ and let P(X) denote the exceptional divisor of the blow-up $\widetilde{X \times X}$, and $p: P(X) \longrightarrow X$ the projection, and let ξ be the first Chern class of the associated line bundle over P(X). Then the i-th Segre class $S_i(X)$ of X is defined by: $S_i(X) := p_*(\xi^{n-1+i} \cap [P(X)])$.

Here we note that if X is smooth, then P(X) is isomorphic to the projectivization of the tangent bundle TX and that it is known (e.g., see Kleiman's Oslo article [Kl]) that $S_i(X)$ is the Poincaré dual of the usual i-th Segre (cohomology) class of X, i.e., $S_i(X) = s_i(X)\cap[X]$. (Warning: $S_i(X)\in H_{2(n-i)}(X)$, not in $H_{2i}(X)$. Of course we can define $S_i(X)$ so that $S_i(X)\in H_{2i}(X)$ by changing ξ^{n-1+i} to ξ^{2n-1+i} in the definition, but we will stick to the above definition.)

K.Johnson [Jo], with this definition of Segre classes, showed the Todd formula and the double-point formula for projective varieties, which led him to discover the quite surprising fact that if a (possibly singular) variety X in $\mathbf{P}^N$ can be immersed into the lower dimensional projective variety $\mathbf{P}^m$ by a projection then it can be so embedded if N < 2 dim X. The present author has been interested in what properties Johnson's Segre classes have, such as its connection with Chern-MacPherson classes, its behaviours under push-forwards and so on (e.g., see [Y 1, Remark 4.9] and [Y 3]). Again, motivated by Dubson's formula for Chern-MacPherson classes, the present author asked what would be the relation between Johnson's Segre classes and Nash-Segre classes, which are defined below again.

Definition 2.3.3. (Nash-Segre classes) Let X be a singular variety. Then the i-th Nash-Segre class $\hat{S}_i(X)$ is defined by:

$$\hat{S}_i(X) := \nu_*(s_i(\widehat{TX})\cap[\hat{X}]),$$

where $\nu:\hat{X}\longrightarrow X$ is the Nash blow-up of X, $\widehat{TX}$ is the tautological Nash tangent bundle over $\hat{X}$ and $s_i(\widehat{TX})$ is the classical i-th Segre (cohomology) class of the bundle $\widehat{TX}$.

Here again, we note that if we let $\mathbf{P}(\widehat{TX})$ be the projectivization of the tautological Nash tangent bundle $\widehat{TX}$, $\mathcal{O}_{\mathbf{P}(\widehat{TX})}(1)$ the dual canonical line bundle (i.e., the dual of the tautological line bundle $\mathcal{O}_{\mathbf{P}(\widehat{TX})}(-1)$) over $\mathbf{P}(\widehat{TX})$, $t:\mathbf{P}(\widehat{TX})\longrightarrow X$

the projection map, $p_1 = \nu \cdot t$ and $\Theta = c_1(\mathcal{O}_{\mathbf{P}(\widehat{TX})}(1))$, then

$$S_i(X) = p_{1*}(\Theta^{n-1+i} \frown [\mathbf{P}(\widehat{TX})]).$$

Since $\mathbf{P}(\widehat{TX})$ and P(X) are isomorphic over the smooth part of X, one could expect that the difference between $S_i(X)$ and $\hat{S}_i(X)$ comes from the singular loci. A difficulty in comparing these classes lies in P(X): even if X is irreducible, P(X) is not necessarily irreducible and it may have many irreducible components over the the singular locus, whereas $\mathbf{P}(\widehat{TX})$ is always irreducible. It is known [Jo] that the fiber $P(X)_x$ of P(X) over x is set-theoretically the projectivization of the union of all the limiting secant lines $\lim_{\substack{x_i \to x \\ y_i \to x}} \widehat{x_i y_i}$, where $\widehat{x_i y_i}$ is the line going through the points x_i and y_i on X . This is nothing other than Whitney's tangent space C_5 (see [Wh, Chapter 7]). For example, if X is a plane curve with singularities x , then $P(X)_x$ is set-theoretically $\mathbf{P}^1$, the projective line, and if $X \subset \mathbf{P}^3$ (or C^3) is the union of three lines not lying in a plane which intersect in a point x, then $P(X)_x$ is, set-theoretically, the projectivization of the union of three planes spanned by the three pairs of lines of X , i.e., the union of three projective lines whose intersections are three different points which correspond to three lines of X itself. In passing, we note that $\mathbf{P}(\widehat{TX})$ is Whitney's generalized tangent space C_4 (see [Wh, Chapter 7]

Roughly speaking, the typical component of P(X), which is not supported on the singular locus, is more or less $\mathbf{P}(\widehat{TX})$. In fact, one can see (e.g., see [Y 1]) that there is a canonical map $q:\mathbf{P}(\widehat{TX}) \longrightarrow P(X)$ such that q restricted to $\mathbf{P}(\widehat{TX})|_{X_{smooth}}$ is an isomorphism and that $q(\mathbf{P}(\widehat{TX})) = \overline{P(X)|_{X_{smooth}}}$ is the typical component with multiplicity one. Hence $P(X)_{red}$ has the decomposition $q(\mathbf{P}(\widehat{TX}))\cup\{V_i\}$, where the V_i's are supported on the singular locus. It is not easy to identify these extra components V_i's and also it is not easy to compute the (geometric) multiplicity of V_i in $P(X)$, length$(\mathcal{O}_{P(X),V_i})$. These difficulties make it difficult to compare $S_*(X)$ and $\hat{S}_*(X)$. But, it is not hard to show the following naive formula relating $S_*(X)$ and $\hat{S}_*(X)$, by knowing the above decomposition $P(X)_{red} = q(\mathbf{P}(TX)\cup\{V_i\}$:

Proposition 2.3.4. ([Y 1]) Let X be a projective variety. Then

$$S_*(X) = \hat{S}_*(X) + \alpha \quad ,$$

where α is some class supported on the singular locus of X .

We can be more precise about the correction term α for certain cases:

Theorem 2.3.5. ([Y 1, 2]) Let $X^n \subset \mathbf{P}^N$ be a singular variety of equidimension n with Y^k denoting the singular locus, whose dimension is k . Then

(i) if $N - n > n - k$, then $\alpha = 0$, i.e., $S_*(X) = \hat{S}_*(X)$,

(ii) if $N - n = n - k$, then

$$S_i(X) = \hat{S}_i(X) + s_{i-n+k}(\mathbf{P}^N)\cap(\sum m_j[Y_j]) \quad ,$$

where Y^k_j is the component of Y^k, of dimension exactly k, m_j = length$(\mathcal{O}_{P(X),V_j})$, V_j the component of $P(X)$ supported on Y^k_j, $s_{i-n+k}(\mathbf{P}^N)$ is the pull-back of the classical (i-n+k)-th Segre (cohomology) class of $\mathbf{P}^N$ by the inclusion of X into $\mathbf{P}^N$.

We have been unable to solve the following problem, which seems to be quite subtle (cf.[Y 2]):

Problem 4: Formulate the correction term in Proposition 2.3.4 above in the case when $N - n < n - k$.

If $N = 2n$ and $k = 0$ in (ii) of Theorem 2.3.5, then we can give a more "dynamic" description of the multiplicity m_j.

Definition 2.3.6. (shift multiplicity) Let $X^n \subset \mathbf{P}^{2n}$ with isolated singularities. Let x_0 be an isolated singularity. Let H be a generic hyperplane off x_0, and let us set $X^a = X \cap (\mathbf{P}^{2n} - H)$, the associated affine variety in $\mathbf{C}^{2n} = \mathbf{P}^{2n} - H$. Let Q ($\neq x_0$) be a point such that any limiting tangent line at the singularity x_0 does not go through the point Q. Let ε be a small positive number and δ be a sufficiently small number such that $0 < \delta << \varepsilon$. Shift X^a slightly in the direction of the shift-vector $\overrightarrow{x_0 Q}$ such that its shift-distance is δ . Then the number of the intersections of X^a and the shifted X^a inside the ε-ball $B_\varepsilon(x_0)$ is called the <u>shift multiplicity</u> of x_0 (see Figure 2.)

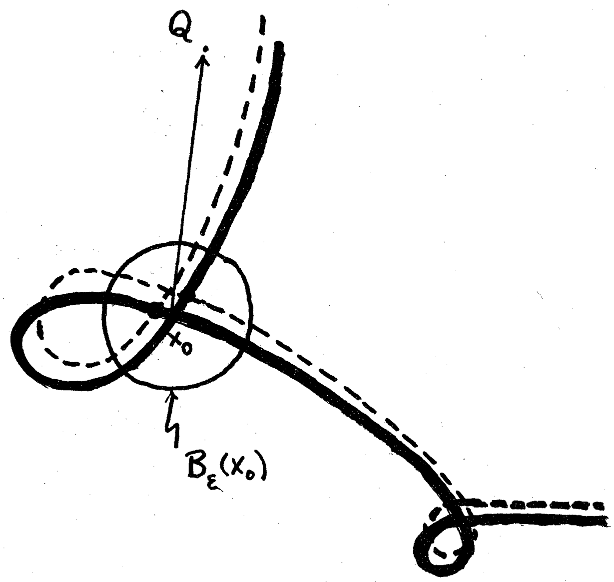

Figure 2.

We can show that by a generic shift, X^a and the shifted X^a intersect each other transversally within a small ball $B_\varepsilon(x_0)$.

Theorem 2.3.7.([Y 1]) Let $X^n \subset \mathbf{P}^{2n}$ be a singular variety with isolated singularities $x_1, x_2, \ldots, x_r$. Then

$$S_*(X) = \hat{S}_*(X) + \sum m_j[x_j] ,$$

where m_j is the shift multiplicity of the singularity x_j .

Remark 2.3.8. Under the same hypothesis above, if we use Fulton's notation (see 6.1 of [F] and a remark right after Lemma7.1, p.120), we can see the following:

$$m_i[x_i] = (\mathbf{P}^{2n} \bullet (X \times X))^{\{(x_i,x_i)\}} ,$$

which is called <u>the part of $\mathbf{P}^{2n} \bullet (X \times X)$ supported on (x_i, x_i)</u>. We may call this <u>the localized self-intersection class of X at x_i</u> and denote it simply by $(X \bullet X)^{\{x_i\}}$.

Problem 5: Give a more "dynamic" (or "down-to-earth") description for m_j appearing in Theorem 2.3.5.. Can we express m_j in terms of some (possibly known) invariants associated to the irreducible components Y_j^k of the singular loci X_{sing} ?

For the case when N = n+1 and k=n-1, i.e., if X is a hypersurface with the singular locus X_{sing} being of codimension exactly 1, then, using a result to R.Piene [Pi 1,2], we can show the following:

Theorem 2.3.9.([Y 1]) Let $X^n \subset \mathbf{P}^{n+1}$ be a hypersurface having a codimension 1 singular loci. Then we have

$$S_i(X) = \hat{S}_i(X) + s_{i-1}(\mathbf{P}^{n+1}) \cap (\sum e_j [Y_j^{n-1}]),$$

where Y_j^{n-1} is an irreducible component of X_{sing}, of dimension = n-1, and e_j is the Jacobian multiplicity of Y_j^{n-1} (which is defined below) and $s_{i-1}(\mathbf{P}^{n+1})$ is the pull-back of the classical (i-1)-th Segre class of $\mathbf{P}^{n+1}$ by the inclusion of X into $\mathbf{P}^{n+1}$.

Remark 2.3.10. As to Problem 5 in the hypersurface case, via Theorem 2.3.9 and a little work we can show that $m_j = e_j$ (see [VY]). Here it should be remarked that Theorem 2.3.5 and Theorem 2.3.9 do not immediately imply $m_j = e_j$, because they are at the level of homology classes. As for the plane curve case, we can conclude that m_j in Theorem 2.3.5 is equal to the shift multiplicity and also to the Jacobian multiplicity, hence that the Jacobian multiplicity of a plane curve singularity is equal to the shift multiplicity. (Professor Hironaka gave me a reasonable idea for a direct proof of "shift multiplicity = Jacobian multiplicity", but I have not been successful yet.)

The Jacobian multiplicity is defined as follows (cf.[F]): Let X be an irreducible hypersurface in $\mathbf{P}^{n+1}$ defined by an equation $F(x_0, \ldots, x_{n+1})$ of degree d. The singular, or Jacobian subscheme of X is the scheme J of zeros of the partial derivatives $F_{x_0}, \ldots, F_{x_{n+1}}$. Let V be any irreducible component of J. Then the Jacobian multiplicity of V is defined to be the coefficient of [V] in the relative Segre class S(J,X), which turns out to be the same as Samuel's multiplicity e(J) of the ideal determined by J in the local ring $A = \mathcal{O}_{X,V}$, i.e., if n=dim(A)=codim(V,X), then

$$\text{length}(A/J^t) = e(J)t^n/n! + \text{lower terms, for } t \gg 0.$$

In a more general set-up where X is a closed subscheme of a pure-dimensional scheme Y and V is an irreducible component of X, the multiplicity of Y along X at V, denoted $(e_X Y)_V$, is defined to be the coefficient of [V] in the relative Segre class S(X,Y). If V = X, then we write simply $e_X Y$. This multiplicity is the same as Samuel's multiplicity e(q) of the ideal q determined by X in the local ring $A = \mathcal{O}_{X,V}$.

As for the Jacobian multipicity of singularities for the case of hypersurfaces with isolated singularities, B.Teissier[Te] showed the following nice formula: $e = \mu + m - 1$,
where μ is the Milnor number, m is the ordinary multiplicity of the singularity. If $X^n \subset C^{n+1}$ is an irreducible hypersurface with the origin being an isolated singularity, defined by an equation $f(x_1, \ldots, x_{n+1}) = 0$, then

$$\mu = \dim \left[\mathcal{O}_{C^{n+1},0} \Big/ (f_{x_1}, f_{x_2}, \ldots , f_{x_{n+1}}) \right].$$

Examples: If $X \subset C^2$ is defined by $y^2 - x^3 = 0$ ($y^2 - x^2 - x^3 = 0$, resp.), then the Milnor number μ of the singularity =(0,0) is equal to 2 (1, resp.). So the Jacobian multiplicity of the singularity is equal to 3 (2, resp.) If $X \subset C^2$ is defined by $y^a - x^b = 0$, where $a<b$ and $(a,b)=1$, then $\mu=(a-1)(b-1)$ and $e= (a-1)(b-1) + a - 1 = b(a-1)$.

Here we give a "shift" argument for $e=b(a-1)$ in the above example: Let $X = \{(x,y) \in C^2 \mid y^a - x^b = 0\}$, where $a<b$ and $(a,b)=1$. Any point Q on the y-axis except the origin is not in the limiting tangent line (= the x-axis) of X at the origin. So we consider the following shift:

$$(x, y) \longmapsto (x, y + t)$$

for a sufficiently small $t \neq 0$. (see Figure 3)

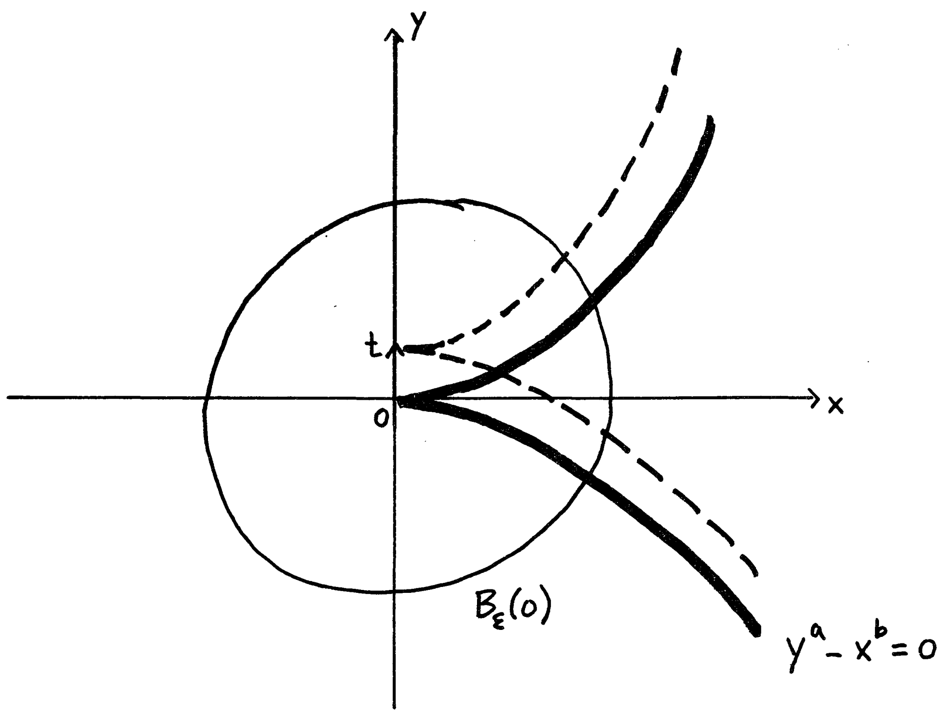

Figure 3.

Then by a simple computation we get that

$$\#(X \cap (\text{shifted } X) \cap B_\varepsilon(0)) = b(a-1) \ , \text{ where } 0 < t \ll \varepsilon \ .$$

Indeed, shifted $X = \{(x,y) \in \mathbf{C}^2 \mid (y-t)^a - x^b = 0\}$. Then

$$(\text{shifted } X) \cap X = \{(x,y) \in \mathbf{C}^2 \mid (y-t)^a - x^b = 0,\ y^a - x^b = 0\}.$$

From $(y-t)^a = y^a$, $(1 - t/y)^a = 1$ (since $y \neq 0$), hence $1 - t/y$ is an a-th root of the unity, except 1. Letting ζ_i $(1 \leqq i \leqq a-1)$ be a-th roots of the unity other than 1, the solutions of $(y-t)^a = y^a$ are $t/(1-\zeta_i)$ $(1 \leqq i \leqq a-1)$. For each $y_i = t/(1-\zeta_i)$, let us consider $x^b = y_i^a$, which can be rewritten as

$$\left(\frac{x}{y_i^{a/b}}\right)^b = 1$$

So, letting η_j $(1 \leq j \leq b)$ be b-th roots of the unity, we get that the solutions of $x^b = y_i^a$ are $y_i^{a/b} \cdot \eta_j$ $(1 \leq j \leq b)$.

Hence

$$(\text{shifted } X) \cap X = \left\{ \left(\left(\frac{t}{1 - \zeta_i} \right)^{a/b} \cdot \eta_j , \frac{t}{1 - \zeta_i} \right) \right\}_{\substack{1 \leq i \leq a-1 \\ 1 \leq j \leq b}} .$$

Therefore, as long as t is sufficiently small,

$$\#((\text{shifted } X) \cap X \cap B_\varepsilon(0)) = b(a-1) .$$

3. A few more problems

Problem 6: Can we describe Johnson's Segre class as a natural transformation from some kind of functor to the homology functor just like Chern-MacPherson class ?

Problem 7: Can we develop a unified theory of characteristic classes of singular varieties as Nash characteristic classes plus some invariants of singularities ?

Since Goresky-MacPherson-Deligne's intersection homology theory IH_*^p is well-developed for singular varieties, it is natural to consider the following:

Problem 8: Can we develop a theory of characteristic classes of singular varieties in intersection homology theory ?

Finally, I want to cite one more thing from MacPherson's article [Mac 2]: "It remains to be seen whether there is a unified theory of characteristic classes for singular varieties like the classical one."

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