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ON THE VECTOR BUNDLES WHOSE ENDOMORPHISMS YIELD QUATERNION ALGEBRAS OF CYCLIC TYPE OVER A PRODUCT OF TWO ELLIPTIC CURVES

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§0. Introduction

Let X be a non-singular projective variety over an algebraically closed field k with arbitrary characteristic p, let n be a positive integer prime to p, and let us consider the following diagram of étale cohomology sets:

$$\text{(FD)}\qquad \begin{array}{ccccc} & & & & H^1(X, \mathbb{G}_m) \\ & & & & \downarrow c_X \\ & & H^1(X, \mu_n)\times H^1(X, \mu_n) & \xrightarrow{\cup} & H^2(X, \mu_n) \\ & & \downarrow A & & \downarrow \\ H^1(X, GL_n) & \xrightarrow{\mathcal{E}nd} & H^1(X, PGL_n) & \xrightarrow{d_n} & H^2(X, \mathbb{G}_m). \end{array}$$

The definition of each map above is this: The lower horizontal sequence is induced from a well-known, fundamental sequence of étale sheaves of group schemes over X

$$\text{(FS)}\qquad 1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1,$$

so this sequence is exact; The right vertical sequence is induced from the Kummer sequence for the étale topology over X

$$\text{(KS)}\qquad 1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1,$$

so this sequence is also exact; The upper horizontal map $\cup$ is (non-canonically) defined by the cup-product on X with a fixed primitive n-th root ζ of unity in k; The left vertical map A is defined by a generalization of the construction of a cyclic algebra over a field to the case over a scheme, which makes the diagram above commutative. We shall give a construction of an Azumaya algebra, denoted by A(L, M), of rank n^2 over X from a pair of n-torsion line bundles L and M over X such that the diagram above commutes (see Sections 1 and 2).

We here note that (see, e.g., [1, §4] and [3, III and IV]): $H^1(X, GL_n)$ is equal to the set of isomorphism classes of vector bundles of rank n over X (for the Zariski topology), in particular, $H^1(X, \mathbb{G}_m)$ coincides with the Picard group Pic(X) of X; $H^1(X, \mu_n)$ coincides with its n-torsion part $Pic(X)_n$; $H^1(X, PGL_n)$ is equal to the set of isomorphism classes of Azumaya algebras of rank n^2 over X, whose elements correspond bijectively with the isomorphism classes of fibre bundles over X for the étale topology with a geometric fibre $\mathbb{P}^{n-1}$, namely, *projective space bundles of rank* n over X.

Now, for a pair of n-torsion line bundles L and M over X, the commutativity and the exactness of the diagram above imply the equivalence of the following conditions:

(1) The Azumaya algebra A(L, M) is isomorphic to $\mathcal{E}nd$(V) for some vector bundle V over X;

(2) The cup-product $L \cup M$ is equal to $c_X(Z)$ for some line bundle Z over X.

So, one may expect that there would exist some relation between V and Z above. We here propose the following problem (see [7]):

How can one construct the vector bundle V *from the line bundle* Z?

where one should note that V is uniquely determined by the algebra A(L, M) up to tensoring line bundles over X.

In case n = 2, it will turn out that our problem is reduced to finding rational solutions of certain quadratic forms over the function field of X. The purpose of this article is to give an answer to this problem in case X is a product of two elliptic curves and $n = 2$. Namely, in this case, we shall construct *all* such vector bundles V from the line bundles Z.

Throughout this article, we always assume that *the base* X *is a non-singular, quasi-projective variety over a field* k, *the integer* n *is positive, prime to the characteristic* p *of* k, *and* k *contains a primitive* n*-th root* ζ *of unity.* For full details on the contents of this article, we refer to [7].

§1. Construction of Azumaya Algebras

In this section, we give a construction of an Azumaya algebra of rank n^2 over X from a pair of n-torsion line bundles, strictly speaking, we define a map

$$H^1(X, \mu_n) \times H^1(X, \mu_n) \quad \to \quad H^1(X, PGL_n),$$

which, in the special case $n = 2$, has been given by D. Mumford [10, §3].

Remark 1.1. Taking cohomology of the sequence (KS), one can interpret $H^1(X, \mu_n)$ as the set of isomorphism classes of couples (L, Φ) such that L is an n-torsion line bundle over X and Φ is an isomorphism $O_X \to L^{\otimes n}$. In case X is a complete variety defined over an algebraically closed field, it follows that

$$H^1(X, \mu_n) \simeq \mathrm{Pic}(X)_n .$$

In case X is a spectrum of a field K, it follows that

$$H^1(K, \mu_n) \simeq K^*/K^{*n}$$

$$H^2(K, \mu_n) \simeq H^2(K, \mathbb{G}_m)_n .$$

Now, let L and M be n-torsion line bundles over X with isomorphisms $\Phi: O_X \to L^{\otimes n}$ and $\Psi: O_X \to M^{\otimes n}$. For a pair of such couples

(L, Φ) and (M, Ψ), we consider a vector bundle
$A := \bigoplus_{0\le i, j\le n-1} L^{\otimes i}\otimes M^{\otimes j}$ over X. Using the isomorphisms Φ and Ψ, defining the following maps:

$$\left(L^{\otimes i}\otimes M^{\otimes j}\right)\otimes\left(L^{\otimes k}\otimes M^{\otimes \ell}\right) \overset{\zeta^{jk}}{\to} L^{\otimes i}\otimes L^{\otimes k}\otimes M^{\otimes j}\otimes M^{\otimes \ell} \to L^{\otimes r}\otimes M^{\otimes s},$$

with $i+k \equiv r$, $j+\ell \equiv s$ modulo n, and $0 \le r, s \le n-1$, we obtain an O_X-algebra A over X.

Now, we locally investigate this algebra A. Take an affine open neighborhood U of an arbitrary point in X over which

$$L|_U = O_U\cdot \ell \simeq O_U ,$$
$$M|_U = O_U\cdot m \simeq O_U ,$$

where ℓ, m are generators of L, M over U, respectively. Since both $\Phi(1)|_U$ and $\ell^{\otimes n}$ generate $L^{\otimes n}|_U$, there exists a unit a in $\Gamma(U, O_U)$ such that

$$a\cdot\Phi(1)|_U = \ell^{\otimes n}.$$

Similarly, as for M, there exists a unit b in $\Gamma(U, O_U)$ such that

$$b\cdot\Psi(1)|_U = m^{\otimes n}.$$

Then, we see that the restriction $A|_U$ is isomorphic to an O_U-algebra generated by elements ℓ, m with relations

$$\ell^n = a,\ m^n = b, \text{ and } \ell m = \zeta m\ell.$$

Particularly, in case n = 2, $A|_U$ is isomorphic to an O_U-algebra generated by ℓ, m with relations

$$\ell^2 = a,\ m^2 = b, \text{ and } \ell m = -m\ell,$$

namely, a *quaternion algebra* over U. Hence, A is an Azumaya algebra of rank n^2 over X, in particular, a quaternion algebra over X when n = 2 (see, e.g., (8, IV, (2.1))). The algebra A obtained from a pair of these couples (L, Φ), (M, Ψ) is denoted by $A((L, \Phi), (M, \Psi))$, and the projective space bundle naturally corresponding to the algebra A is denoted by $P((L, \Phi), (M, \Psi))$. Thus, our construction gives the required map.

Remark 1.2. In case X is a spectrum of a field K, by the isomorphism $H^1(K, \mu_n) \to K^*/K^{*n}$ in Remark 1.1, the couples (L, Φ), (M, Ψ) are assigned to the elements a, b modulo K^{*n}, respectively. So, in this case, the map above gives the construction of ordinary cyclic algebras over the field K.

Next, we study the case n = 2 in detail. In this case, we have another method for constructing projective space bundles of rank 2, namely, *projective line bundles*, from a pair of 2-torsion line bundles as follows: For any 2-torsion line bundles L and M with isomorphisms $\Phi: O_X \to L^{\otimes 2}$ and $\Psi: O_X \to M^{\otimes 2}$, let E be a direct summand $O_X \oplus L \oplus M$ of the algebra $A := A((L, \Phi), (M, \Psi))$. Let q be the restriction to E, of the reduced norm of the quaternion algebra A, which is a quadratic form on E. In other words, the quadratic form q on E is this: We have three global sections

$$1/\iota(1) \in \Gamma(X, O_X^{\vee\otimes 2}) \subset \Gamma(X, S^2(E^\vee))$$
$$1/\Phi(1) \in \Gamma(X, L^{\vee\otimes 2}) \subset \Gamma(X, S^2(E^\vee))$$
$$1/\Psi(1) \in \Gamma(X, M^{\vee\otimes 2}) \subset \Gamma(X, S^2(E^\vee)),$$

where ι is a natural isomorphism $O_X \to O_X^{\otimes 2}$. Put

$$q := 1/\iota(1) - 1/\Phi(1) - 1/\Psi(1).$$

Then, we obtain a quadratic divisor C of $\mathbb{P}(E^\vee)$ defined by q, which is a conic bundle over X.

Now, we locally investigate the bundle C. For the 2-torsion line bundles L and M, we take an affine open neighborhood U of an arbitrary point in X over which both L and M are trivial, as before. Then, we get an isomorphism

$$E^\vee|_U = O_U \cdot 1/1 \oplus O_U \cdot 1/\ell \oplus O_U \cdot 1/m \simeq O_U \oplus O_U \oplus O_U ,$$

Furthermore, we have an expression of the restriction of q:

$$q|_U = 1/\iota(1) - 1/\Phi(1) - 1/\Psi(1)|_U \simeq \begin{pmatrix} 1 & & \\ & -a & \\ & & -b \end{pmatrix}$$

via the isomorphism $E^{\vee}|_U \simeq O_U \oplus O_U \oplus O_U$ above, which is nothing but the restriction of the reduced norm of the quaternion algebra $A|_U$. Since both a and b are units in $\Gamma(U, O_U)$, the quadratic form q has maximal rank at every point in U: In fact, q is the restriction of a reduced norm of a quaternion algebra, so it is non-degenerate everywhere. Hence, the conic bundle C over X has no singular fibres. Using an étale cover of X associated to a 2-torsion line bundle, for example, L, we see that C is locally trivial over X for the étale topology, namely, a projective line bundle over X.

Thus, we obtain a projective line bundle C from a pair of these couples (L, Φ) and (M, Ψ), which is denoted by C((L, Φ), (M, Ψ)). The vector bundle E used above and the quadratic form q on E defining C are denoted by E((L, Φ), (M, Ψ)) and q((L, Φ), (M, Ψ)), respectively.

Since C((L, Φ), (M, Ψ)) is naturally corresponding to the quaternion algebra A((L, Φ), (M, Ψ)) (see, e.g., [1, §4], or [14, XIV, §2, Remark 3, p207]), the bundles C((L, Φ), (M, Ψ)) and P((L, Φ), (M, Ψ)) are isomorphic over X, and our projective line bundles are explicitly given in terms of conic bundles.

Therefore, we get the diagram (FD) in the introduction, which is called the *fundamental diagram* for X.

To conclude this section, we define Hilbert symbols over the base X

Definition 1.3. For any elements a and b of $H^1(X, \mu_n)$, the value $d_n(A(a, b))=d_n(P(a, b))$ is called the *Hilbert symbol* of a and b over X, and denoted by $\{a, b\}_n$.

Remark 1.4. Our Hilbert symbols over X coincides with the classical one when X is a spectrum of a field.

The next proposition follows directly from the exactness of the diagram (FD) and Definition 1.3 above.

Proposition 1.5. For any elements a, b of $H^1(X, \mu_n)$, the following conditions are equivalent:

(1) $P(a,b) \simeq \mathbb{P}(V^\vee)$, or equivalently, $A(a, b) \simeq \mathcal{E}nd(V)$ for some vector bundle V over X;

(2) $\{a, b\}_n = 0$.

Under the equivalent conditions above, we say that P(L, M), or A(L, M) *comes from the vector bundle* V.

§2. Commutativity of the Fundamental Diagram

In this section, we shall show the commutativity of the fundamental diagram (FD).

Lemma 2.1. If X is a spectrum of a field, then the fundamental diagram (FD) for X is commutative.

Proof. See, e.g., [14, XIV, §2, Proposition 5].

Lemma 2.2. Let K be the function field of a general X. Then, the natural map

$$H^2(X, \mathbb{G}_m) \to H^2(K, \mathbb{G}_m)$$

is injective.

Proof. See [8, III, (2.22)].

Corollary 2.3. For any elements a, a', b and b' of $H^1(X, \mu_n)$, the following conditions are equivalent:

(1) P(a, b) and P(a', b') are birational over X, or equivalently, A(a, b) and A(a', b') are isomorphic at the generic point of X;

(2) $\{a, b\}_n = \{a', b'\}_n$.

Now, we have

Theorem 2.4. The fundamental diagram (FD) for a general X is commutative.

Proof. Combine Lemmas 2.1 and 2.2 (see [7, Theorem 2.6]).

From the above theorem, we get the following corollaries.

Corollary 2.5. For any elements a, b and c of $H^1(X, \mu_n)$, we have:

(a) $\{a \otimes b, c\}_n = \{a, c\}_n + \{b, c\}_n$;

(b) $\{a, b \otimes c\}_n = \{a, b\}_n + \{a, c\}_n$;

(c) $\{a, b\}_n + \{b, a\}_n = 0$.

Proof. The required results follow directly from the fact that the cup-product $\cup$ is bilinear and alternating.

Corollary 2.6. For any elements a and b of $H^1(X, \mu_n)$, the following conditions are equivalent:

(1) The projective space bundle P(a, b), or equivalently, the Azumaya algebra A(a, b) comes from some vector bundle over X;

(2) The cup-product $a \cup b$ in $H^2(X, \mu_n)$ is equal to $c_X(Z)$ for some line bundle Z over X;

(3) $\{a, b\}_n = 0$.

Proof. Combine Proposition 1.5 and Theorem 2.4.

Under the equivalent conditions above, we say that the cup-product $a \cup b$ *comes from the line bundle* Z over X. Therefore, in terms of the cup-product of torsion line bundles, we can compute the obstruction for our projective space bundle, or our Azumaya algebra to come from some vector bundle.

Corollary 2.7. For any elements a, b and c of $H^1(X, \mu_n)$, we have:

(a) If two of three P(a, c), P(b, c) and P($a \otimes b$, c) come from vector bundles, then so does the other;

(b) If two of three P(a, b), P(a, c) and P(a, $b \otimes c$) come from vector bundles, then so does the other;

(c) The bundle P(a, b) comes from vector bundle if and only if so does P(b, a).

Proof. Combine Corollaries 2.5 and 2.6.

Definition 2.8. For elements a, b and c of $H^1(X, \mu_n)$, we define the *composition of the pairs* (a, c) and (b, c) to be ($a \otimes b$, c) in

$H^1(X, \mu_n)\times H^1(X, \mu_n)$. In this case, we define the *composition of the projective space bundles* P(a, c) and P(b, c) to be P(a⊗b, c). Furthermore, if P(a, c) and P(b, c) come from vector bundles V_a and V_b, respectively, then we define the *composition of the vector bundles* V_a and V_b to be a vector bundle V_{ab} modulo line bundles over X such that P(a⊗b, c) comes from V_{ab}. By virtue of Corollary 2.7, the existence of the composition V_{ab} is guaranteed. But, one should note that, for isomorphism classes of projective space bundles, or vector bundles, the composition of them are *not* well-defined since it depends upon the choice of the pairs (a, c) and (b, c). So, we shall specify the pairs of the elements of $H^1(X, \mu_n)$ whenever we use this terminology. Similarly, we define the *composition of the pairs* (a, b) and (a, c) to be (a, b⊗c) in $H^1(X, \mu_n)\times H^1(X, \mu_n)$, and so on.

Finally, we give a sufficient condition for Br(X) = Br'(X) (see [8, IV, (2.9)]), where Br'(X) is the cohomological Brauer group $H^2(X, \mathbb{G}_m)_{tor}$ of X.

Corollary 2.9. If the map

$$H^1(X, \mu_n)\otimes H^1(X, \mu_n) \quad \to \quad H^2(X, \mu_n)$$

defined by the cup-product ∪ is surjective, then the set

$$\{ \{a, b\}_n \mid a, b \in H^1(X, \mu_n) \}$$

generates the n-torsion part $Br'(X)_n$. In particular, we have

$$Br(X)_n = Br'(X)_n.$$

Proof. See Theorem 2.4 (see [7, Corollary 2.12).

Remark 2.10. For example, abelian varieties defined over an algebraically closed field satisfy the assumption of Corollary

2.9 above.

Remark 2.11. The rank of our projective space bundles P(a, b) whose obstructions generate $Br(X)_n$ is equal to the order n of the group $Br(X)_n$, so that it does not depend on the dimension of X (compare [3], [4, Theorem 1] and [6, pp235-236, A2]).

§3. Rational Sections and Vecctor Bundles

In this section, we study the relation between rational sections of conic bundles C and vector bundles V such that C comes from V. So, we always consider the case n = 2.

Lemma 3.1. Let P be a projective line bundle over X with projection π, and let ω_π be the relative canonical bundle of π. Then we have:

(a) P is isomorphic to a quadratic divisor C of $\mathbb{P}(E^\vee)$ for some vector bundle E of rank 3 over X, which is a conic bundle over X;

(b) Any such E as above is uniquely determined by P, and isomorphic to the vector bundle $\left(\pi_*(\omega_\pi{}^\vee)\right)^\vee$ modulo line bundles over X.

Proof. See [7, Lemma 3.3].

Lemma 3.2. Let K be the function field of X, let C be a projective line bundle over X, and let q be a quadratic form over X which defines C as a conic bundle C as in Lemma 3.1. Then, the following conditions are equivalent:

(1) C comes from a vector bundle over X;

(2) C has a rational section over X;

(3) q has a K-rational solution at the generic point Spec K of X.

Proof. See (13) and (14, X, §6, Exercise 1).

Now, we have

Theorem 3.3. Let C be a projective line bundle over X, and let E be a vector bundle of rank 3 over X such that C is isomorphic to a quadratic divisor of $\mathbb{P}(E^{\vee})$ as in Lemma 3.1. Assume that C has a rational section over X, and identify C with the divisor of $\mathbb{P}(E^{\vee})$ above.

(a) For a rational section of C over X, there exist a unique line bundle S over X and a unique homomorhism $s:S \to E$ satisfying the following conditions:

(1) s is injective as a homomorphism of sheaves;

(2) The zero locus $(s)_0$ of s as a homomorphism of vector bundles has codimension at least 2 in X;

(3) The cokernel V_0 of s is a torsion-free sheaf of rank 2;

(4) The rational map from $X \simeq \mathbb{P}(S^{\vee})$ to $\mathbb{P}(E^{\vee})$ defined by s gives a section of C defined over the complement $X-(s)_0$, which coincides with the given rational section of C over X.

Thus, we have an exact sequence over X

$$0 \to S \xrightarrow{s} E \to V_0 \to 0.$$

(b) Let V be the double dual of the torsion-free sheaf V_0. Then, V is a vector bundle of rank 2 over X and the bundle C comes from V.

Proof. See [7, Theorem 3.6].

Example 3.4. Let L, L' and M be 2-torsion line bundles over X with isomorphisms $\Phi:O_X\to L^{\otimes 2}$, $\Phi':O_X\to L'^{\otimes 2}$ and $\Psi:O_X\to M^{\otimes 2}$, and let ι be a natural isomorphism $O_X\to O_X^{\otimes 2}$. Then, the vector bundle $O_X\oplus L\otimes L'$ is the composition of the vector bundles $O_X\oplus L$ and $O_X\oplus L'$ defined by the pairs $((O_X, \iota), (L, \Phi))$ and $((O_X, \iota), (L', \Phi'))$ (see [7, Example 3.7 and Proposition 3.8]).

§4. Composition of Vector Bundles

We first study geometric meaning of the composition of our projective line bundles (see Definition 2.8). In this section, we always consider the case n = 2.

Proposition 4.1. Let L, L' and M be 2-torsion line bundles over X with isomorphisms $\Phi:O_X\to L^{\otimes 2}$, $\Phi':O_X\to L'^{\otimes 2}$ and $\Psi:O_X\to M^{\otimes 2}$. Let C and C' be the projective line bundles $C((L, \Phi), (M, \Psi))$ and $C((L', \Phi'), (M, \Psi))$, let C" be the composition of C and C', and let E, E', and E" be the vector budles $E((L, \Phi), (M, \Psi))$, $E((L', \Phi'), (M, \Psi))$, and $E((L", \Phi"), (M, \Psi))$, respectively, where we put $(L", \Phi"):= (L, \Phi)+(L', \Phi')$ in $H^1(X, \mu_2)$. Let (X:Y:Z), (X':Y':Z'), and (X":Y":Z") be the global coordinates of $E = O_X\oplus L\oplus M$, $E'= O_X\oplus L'\oplus M$, and $E"= O_X\oplus L"\oplus M$, respectively. Let φ be the morphism from $\mathbb{P}(E^\vee)\times_X\mathbb{P}(E'^\vee)$ to $\mathbb{P}(E"^\vee)$ defined by $\varphi((X:Y:Z)\times(X':Y':Z'))=(X":Y":Z")$ with

$$X":= X\otimes X'+\Psi^{-1}\circ Z\otimes Z'$$

$$Y":= Y\otimes Y'$$

$$Z":= X\otimes Z'+Z\otimes X'.$$

Then we have:

(a) The image of the restriction $\varphi|_{C \times_X C'}$ is dense in C";

(b) The base locus of $\varphi|_{C \times_X C'}$ is contained in the fibre product $H \times_X H'$, where H, H' are tautological divisors of $\mathbb{P}(E^{\vee})$, $\mathbb{P}(E'^{\vee})$ defined by the natural inclusions

$$O_X \oplus M \to E, \qquad O_X \oplus M \to E',$$

respectively.

Using Proposition 4.1, one can define *the composition of the maps* s in Theorem 3.3 (a) in the obvious way, by which we define the composition of rational sections of our projective line bundles.

Theorem 4.2. With the same notations as above, assume that the bundles C and C' have rational sections over X and the element (M, Ψ) in $H^1(X, \mu_2)$ is not zero (see Example 3.4). Let $s=(X:Y:Z)$, $s'=(X':Y':Z')$ be the maps $S \to E$, $S' \to E'$ corresponding to the rational sections as in Theorem 3.3 (a), respectively, let s" be the composition of s and s', and let V, V' and V" be the double dual of the cokernels of s, s' and s", respectively. Then:

(a) $(s")_0 \subset (s)_0 \cup (s')_0 \cup \Big((Y)_0 \cap (Y')_0 \Big)$;

(b) $(s")_0$ is a proper subset of X;

(c) s" defines a rational map from $X \simeq \mathbb{P}(S^{\vee} \otimes S'^{\vee})$ to $\mathbb{P}(E"^{\vee})$, which gives a rational section of C" over X;

(d) If $(s")_0$ has codimension at least 2 in X, then V" is a vector bundle of rank 2 over X, and the bundle C" comes from V". In other words, the vector bundle V" is the composition of V and V' defined by the pairs $((L, \Phi),$

(M, Ψ)) and ((L', Φ'), (M, Ψ)).

Proof. See [7, Theorem 4.4].

Definition 4.3. By virtue of Theorem 4.2 (c) above, from rational sections of the bundles C and C' over X, we obtain a rational section of C" over X, which is called the *composition of the rational sections* of C and C' over X (defined by the pairs ((L, Φ), (M, Ψ)) and ((L', Φ'), (M, Ψ))).

Therefore, using rational sections, we can construct the composition of vector bundles under the conditions above, which will play a key role in Section 7.

§5. Cycle Map on a Product of Two Elliptic Curves

In this section, we investigate the cycle map c_X from Pic(X) to $H^2(X, \mu_n)$ when the base X is a product of two elliptic curves defined over an algebraically closed field. From now on, we shall assume that *the ground field* k *is algebraically closed.*

For any elliptic curve E, we always fix the unity of group structure of E. By the isomorphism from E to its dual $\hat{E}$ defined by the point of unity, we sometimes identify them.

Lemma 5.1. Let X be a product of elliptic curves E_1 and E_2, and let R be the group $\mathrm{Hom}(E_1, \hat{E}_2) = \mathrm{Hom}(E_2, \hat{E}_1)$ of correspondences between E_1 and E_2. Then, we have the following commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & c_{E_1} \oplus c_{E_2} & \downarrow \\
\mathrm{Pic}(E_1) \oplus \mathrm{Pic}(E_2) & \rightarrow & H^2(E_1, \mu_n) \oplus H^2(E_2, \mu_n) \\
\downarrow & c_X & \downarrow \\
\mathrm{Pic}(X) & \rightarrow & H^2(X, \mu_n) \\
\downarrow & \gamma & \downarrow \\
R & \rightarrow & H^1(E_1, \mu_n) \otimes H^1(E_2, \mu_n) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

with exact rows, where we denote by γ the map induced by the cycle map c_X. Moreover, the top horizontal map is surjective.

Proof. See, e.g., [7, Lemma 5.5].

Now, looking at the meaning of the induced map γ above, we find that γ is composed of the following:

$$\begin{aligned}
R = \mathrm{Hom}(E_2, \hat{E}_1) &\rightarrow \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(H^1(\hat{E}_1, \mu_n), H^1(E_2, \mu_n)) \\
&\rightarrow \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(H^1(\hat{E}_1, \mu_n), \mu_n) \otimes_{\mathbb{Z}/n\mathbb{Z}} H^1(E_2, \mu_n) \\
&\rightarrow H^1(E_1, \mu_n) \otimes_{\mathbb{Z}/n\mathbb{Z}} H^1(E_2, \mu_n),
\end{aligned}$$

where one should note that, by the e_n-pairing over E_1 (see, e.g., [8, V, (2.4) (f)]), there is a canonical isomorphism of $\mathbb{Z}/n\mathbb{Z}$-modules

$$\mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(H^1(\hat{E}_1, \mu_n), \mu_n) \simeq H^1(E_1, \mu_n).$$

Proposition 5.2. Let X be a product of elliptic curves E_1 and E_2 with i-th projection p_i, let L and M be n-torsion line bundles over X, written

$$L = p_1{}^*L_1 \otimes p_2{}^*L_2, \qquad M = p_1{}^*M_1 \otimes p_2{}^*M_2,$$

with n-torsion line bundles L_i, M_i over E_i, i=1, 2, and let γ be as in Lemma 5.1. Then:

(a) $\qquad L \cup M = (L_1 \cup M_1) \oplus (L_2 \cup M_2) \oplus (L_1 \otimes M_2 - M_1 \otimes L_2)$

via the decomposition

$$H^2(X, \mu_n) = H^2(E_1, \mu_n) \oplus H^2(E_2, \mu_n) \oplus H^1(E_1, \mu_n) \otimes H^1(E_2, \mu_n);$$

(b) Assume that L_1, M_1 are a basis for $H^1(\hat{E}_1, \mu_n)$, and let P_1, Q_1 be the points of E_1 corresponding to L_1, M_1, respectively. For a homomorphism $\varphi : E_2 \to \hat{E}_1$ such that

$$\hat{\varphi}(P_1) = L_2, \quad \hat{\varphi}(Q_1) = M_2,$$

we have

$$\gamma(\varphi) = L_1 \otimes M_2 - M_1 \otimes L_2$$

in $H^1(E_1, \mu_n) \otimes H^1(E_2, \mu_n)$.

Proof. (a) This is obvious.

(b) From the meaning of the map γ, one can compute the value $\gamma(\varphi)$ in $H^1(E_1, \mu_n) \otimes H^1(E_2, \mu_n)$.

Remark 5.3. Using Proposition 5.2, one can easily compute the relations on the set of generators of the group $Br(X)_n$ of a product X of two elliptic curves (see Example 8.4).

Theorem 5.4. With the same notations above, the following conditions are equivalent:

(1) The projective space bundle P(L, M), or equivalently, the Azumaya algebra A(L, M) comes from some vector bundle over X;

(2) The elements $L_1 \otimes M_2 - M_1 \otimes L_2$ in $H^1(E_1, \mu_n) \otimes H^1(E_2, \mu_n)$ is equal to $\gamma(\varphi)$ for some correspondence φ between E_1 and E_2;

(3) $\{L, M\}_n = 0$.

Proof. Combine Corollary 2.6, Lemma 5.1 and Proposition 5.2 (a).

Under the equivalent conditions above, we say that the element $L_1 \otimes M_2 - M_1 \otimes L_2$ *comes from the correspondence* φ between E_1 and E_2. In Section 7, we shall explain the relation between the correspondences and the vector bundles above.

As an application of Theorem 5.4, we obtain an elementary, concrete example of projective space bundles which do not come from any vector bundles (see also Example 8.4)). Such an example in the case over the complex number field $\mathbb{C}$ has been given by J.-P. Serre [13, 6.4].

Example 5.5. With the same notations as above, assume that E_1 and E_2 are not isogenous and $L \cup M$ is not zero. Then, it follows from Theorem 5.4 that the projective space bundle P(L, M) does not come from any vector bundle. Note that, in case n = 2, P(L, M) is concretely given in terms of a conic bundle.

§6. Some Properties

In this section, we state some properties of our projective space bundles (and our Azumaya algebras) over an abelian variety. For proofs, see [7, §6].

Proposition 6.1. Let X be an abelian variety, and let L and M be n-torsion line bundles over X. Then the projective space bundle P(L, M) is homogeneous. In particular, if the bundle P(L, M) comes from a vector bundle V over X, then V is homogeneous up to tensoring line bundles over X, namely, *semi-homogeneous* (see [9, (5.2)]).

Proposition 6.2. With the same notations as above, assume that the projective space bundle P(L, M) comes from a vector bundle V over X. Then the following conditions are equivalent:

(1) P(L, M), or equivalently, V is simple (see [12, I, (4.1.1)] and [15, §1]);

(2) $L \cup M$ has order n in $H^2(X, \mu_n)$.

Proposition 6.3. With the same notations as above, for an integer d, the following conditions are equivalent:

(1) P(L, M), or equivalently, A(L, M) is a pull-back from an abelian variety of dimension d;

(2) Both L and M are pull-backs from an abelian variety of dimension d.

§7. Vector Bundles over a Product of Two Elliptic Curves

This section contains the main results of this article. From now on, we consider only the case n = 2, so the characteristic p is not 2.

Example 7.1. Let X be an elliptic curve, and let P_∞ be the point of X corresponding to the unity of the group X. For any 2-torsion line bundles L and M over X such that the cup-product $L \cup M$ is not zero, let P_0 and P_1 be the points of X corresponding to them, respectively, where we note that the conditions $L \cup M \neq 0$ and $P_0 \neq P_1$ are equivalent.

It follows from Tsen's theorem that the Hilbert symbol $\{L, M\}_2$ is zero. In other word, the projective line bundle C(L, M) has a rational section over X, and comes from some vector bundle over X.

We here construct a global section of C(L, M) and the vector bundle over X. One may assume that X is given by the equation

$$y^2 = x(x-1)(x-\lambda) \qquad \text{with } \lambda \neq 0,\ 1$$

in $\mathbb{P}^2$ such that P_∞ is the point at infinity and P_0, P_1 have coordinate (0, 0), (1, 0), respectively. Let P_λ be the point of X with coordinate $(\lambda, 0)$. In terms of the group structure of X, we have that $P_0+P_1 = P_\lambda$.

Now, for such a pair (L, M), according to the local investigation of conic bundles over X at Section 1, the quadratic form q:= q(L, M) on the vector bundle E:= E(L, M) is represented by a matrix

$$\begin{pmatrix} 1 & & \\ & -x & \\ & & -(x-1) \end{pmatrix}$$

at the generic point Spec K of X. Clearly, it has a K-rational solution (1:1:i), with $i^2 = -1$. By the ratio (1:1:i), we embed a line bundle $S := O_X(-P_\infty)$ into E as a subbundle; we define a map s from S to $E = O_X \oplus L \oplus M$ by $s(\alpha) := (\alpha:\alpha:i\alpha)$. Then, we find that s gives a global section of the bundle C(L, M) over X. According to Theorem 3.3 (b), with the same notations as there, C(L, M) comes from the cokernel $V_0 = V$ of s, where one should note that $(s)_0$ is empty, in other words, s is an injection of vector bundles, so its cokernel V_0 is already locally free. From the exact sequence of vector bundles over X

$$0 \rightarrow S \xrightarrow{s} E \rightarrow V \rightarrow 0,$$

we find that the vector bundle V is indecomposable, of rank 2, with the first Chern class P_λ, where one should note that P_λ is the point of X corresponding to the 2-torsion line bundle $L \otimes M$. According to M. F. Atiyah (2, II, Theorem 7), such a vector bundle V is characterized by the properties above. In this article, a vector bundle V over an elliptic curve X is called *of* *type Atiyah (determined by a point* P of X) if the bundle V is indecomposable, of rank 2 and degree 1 (whose first Chern class is represented by the point P). Using the characterization above, we see that a vector bundle of type Atiyah is semi-homogeneous (see [2, II, Corollary, p434]), which follows also from Proposition 6.1. On the other hand, it is well-known that a vector bundle of type Atiyah is simple (see [2, III, §2, Lemma 22]), which follows also from Proposition 6.2.

Now, we study what vector bundle comes to our quaternion algebra (or, projective line bundle) obtained from a pair of 2-torsion line bundles over a product of two elliptic curves.

Let X be a product of elliptic curves E_1 and E_2. For any 2-torsion line bundles L and M over X, there exist 2-torsion line bundles L_i and M_i over E_i, with i=1, 2, such that

$$L = p_1{}^*L_1+p_2{}^*L_2, \qquad M = p_1{}^*M_1+p_2{}^*M_2,$$

where p_i is the i-th projection from X to E_i, and the tensor products of line bundles are written additively.

Now, assume that the Hilbert symbol $\{L, M\}_2$ is zero. It follows from Theorem 5.4 that the element $L_1 \otimes M_2 - M_1 \otimes L_2$ in $H^1(E_1, \mu_2) \otimes H^1(E_2, \mu_2)$ comes from a correspondence φ between E_1 and E_2:

$$\gamma(\varphi) = L_1 \otimes M_2 - M_1 \otimes L_2.$$

Using φ, we shall construct the vector bundle which comes to the quaternion algebra A(L, M) (or, projective line bundle C(L, M)).

In order to do this, we use Examples 3.4, 7.1, Theorem 4.2 and Proposition 7.2 below. Essentially, we first find a rational solution of the quadratic form q(L, M) defining C(L, M) over X, secondly we construct the rational section of C(L, M), and, thirdly we construct the required vector bundle (see [7, §7]).

Proposition 7.2. With the same notations as above, let N_i be a non-trivial 2-torsion line bundle over E_i , i=1, 2. Assume that the Hilbert symbol $\{p_1^*N_1, p_2^*N_2\}_2$ is equal to zero. Then we have:

(a) The element $N_1 \otimes N_2$ in $H^1(E_1, \mu_2) \otimes H^1(E_2, \mu_2)$ comes from a correspondence φ between E_1 and E_2;

(b) For each index i, if N_i' is a 2-torsion line bundle over E_i such that N_i and N_i' are a basis for $H^1(E_i, \mu_2)$, and V_i is the vector bundle of type Atiyah over E_i determined by the point corresponding to the 2-torsion line bundle N_i+N_i', then the bundle $C(p_1^*N_1, p_2^*N_2)$ comes from the composition of $\Phi_i^*V_i$ and $p_i^*V_i$, where Φ_i is the homomorphism from X to E_i defined by φ, and the compositions above are defined by the pairs

$$(p_1^*N_1,\ p_1^*N_1'+p_2^*N_2) \text{ and } (p_1^*N_1,\ p_1^*N_1'),$$
$$(p_1^*N_1+p_2^*N_2',\ p_2^*N_2) \text{ and } (p_2^*N_2',\ p_2^*N_2),$$

respectively;

(c) The composition above is constructed as described in Theorem 4.2 (d).

Proof. See [7, Proposition 7.2].

Thus, we have

Theorem 7.3. Let X be a product of elliptic curves E_1 and E_2 with i-th projection p_i , let L and M be 2-torsion line bundles over X, written

$$L = p_1^*L_1+p_2^*L_2, \quad M = p_1^*M_1+p_2^*M_2,$$

with n-torsion line bundles L_i, M_i over E_i , i=1, 2, and let P_i

be the 2-torsion point of E_i corresponding to the line bundle L_i+M_i, i=1, 2. Assume that the Hilbert symbol $\{L, M\}_2$ is equal to zero, and let Φ_i be the homomorphism from X to E_i defined by the correspondence between E_1 and E_2 which comes to the element $L_1 \otimes M_2 - M_1 \otimes L_2$ in $H^1(E_1, \mu_2) \otimes H^1(E_2, \mu_2)$. Then, we have:

(a) In case the cup-product $L \cup M$ is zero, the quaternion algebra A(L, M) comes from either $O_X \oplus L$ or $O_X \oplus M$, corresponding to whether L is non-trivial or not , or whether M is trivial or not;

(b) In case the cup-product $L_i \cup M_i$ is not zero for some index i, let V_i be the vector bundle of type Atiyah over E_i determined by P_i. Then, the algebra A(L, M) comes from the pull-back $\Phi_i{}^* V_i$;

(c) In case both cup-products $L_1 \cup M_1$ and $L_2 \cup M_2$ are zero but the element $L_1 \otimes M_2 - M_1 \otimes L_2$ is not zero, let V_i' be the vector bundle of type Atiyah over E_i determined by a non-zero 2-torsion point other than P_i. Then the algebra A(L, M) comes from the composition of $\Phi_i{}^* V_i'$ and $p_i{}^* V_i'$, which is constructed as in Proposition 7.2. In this case, A(L, M) is uniquely determined by the value $L_1 \otimes M_2 - M_1 \otimes L_2$.

Corollary 7.4. With the same notations as above, if the Hilbert symbol $\{L, M\}_2$ is equal to zero, then the quaternion algebra A(L, M) comes from one of the vector bundles of the following three types:

(1) A direct sum $O_X \oplus L$ or $O_X \oplus M$;

(2) A pull-back of a vector bundle of type Atiyah over either E_1 or E_2 by a morphism defined by $L \cup M$, which is semi-homogeneous and simple;

(3) A composition of vector bundles of type (2) above, which is

semi-homogeneous and simple.

Proof. See Propositions 6.1, 6.2 and Theorem 7.3.

Remark 7.5. For a vector bundle of type (3) in Corollary 7.4 above, we have both examples, such that the bundle is a pull-back of a vector bundle over some elliptic curve, and such that the bundle is not any pull-back of any vector bundle over any elliptic curve (see Example 8.5).

§8. Examples

Throughout this section, we consider the case n = 2, so that the characteristic p is not 2. We shall discuss about some examples over a product of two elliptic curves E given by the equation

$$y^2 = x^3 - x$$

in $\mathbb{P}^2$.

First, we fix some notations and state some elementrary facts on the elliptic curve E. Let P_∞ be the point of E at infinity. Considering P_∞ as a unity, define a group structure on E. Via the isomorphism from E to its dual defined by P_∞, we sometimes identify them. Let P_{-1}, P_0 and P_1 be the points of E with coordinates (-1, 0), (0, 0) and (1, 0), respectively. Let L and M be the 2-torsion line bundles over E corresponding to the points P_0 and P_1, respectively, which form a basis for the group $H^1(E, \mu_2)$.

Computing the Hasse invariant of E, we have

Lemma 8.1. E is supersingular if and only if $p \equiv 3$ modulo 4.

Let R be the ring of endomorphisms of E, and let ι be the endomorphism of E defined by $\iota(x, y) := (-x, iy)$, with $i^2 = -1$. It clearly follows that

$$\iota^2+1 = 0,$$
$$\hat{\iota}(P_0) = L, \qquad \hat{\iota}(P_1) = L+M. \tag{1}$$

Moreover, we have

Lemma 8.2. If E is not supersingular, then R is freely generated by 1 and ι as a $\mathbb{Z}$-module.

Proof. See, e.g., [5, IV, (4.19)] and [11, IV, §22, Second example].

If, on the contrary, E is supersingular, then R is a maximal order of the quaternion division algebra $R\otimes_{\mathbb{Z}}\mathbb{Q}$ over $\mathbb{Q}$ (see, e.g., [loc. cit.]). To get a typical example of funny phenomenon in this case, we assume p = 3 (see Lemma 8.1). Let η be the linear transformation η of $\mathbb{P}^2$ defined by $\eta(x, y) := (x+1, y)$. Then, η is an endomorphism of E, and satisfies

$$\eta^2+\eta+1 = 0.$$

It clearly follows that

$$\hat{\eta}(P_0)=L+M, \qquad \hat{\eta}(P_1)=L. \tag{2}$$

Remark 8.3. Furthermore, one can easily show that R is freely generated by 1, ι, η and $\iota\eta$ as a $\mathbb{Z}$-module.

Now, let X be a product of elliptic curves $E_1 := E$ and $E_2 := E$ with i-th projection p_i.

Example 8.4 (for Remark 5.9). It can be shown that: If E is supersingular, then the 2-torsion part $Br(X)_2$ is zero; Otherwise, it is a $\mathbb{Z}/2\mathbb{Z}$-module of rank 2 (see, e.g., [7, Corollary 5.7]). We assume that E is not supersingular.

Here, we shall find a free generator of $Br(X)_2$ over $\mathbb{Z}/2\mathbb{Z}$. By virtue of Lemmas 5.1 and 8.2, we see that $Br(X)_2$ is isomorphic to a $\mathbb{Z}/2\mathbb{Z}$-module generated by $L\otimes L$, $L\otimes M$, $M\otimes L$ and $M\otimes M$ with relations $\gamma(1) = \gamma(\iota) = 0$. Using Proposition 5.2 (b) and the equalities (1), we have

$$\{p_1{}^*L,\ p_2{}^*M\}_2 = \{p_1{}^*M,\ p_2{}^*L\}_2 \tag{3}$$

$$\{p_1{}^*L,\ p_2{}^*L\}_2 = 0. \tag{4}$$

Thus, $Br(X)_2$ is freely generated by $\{p_1{}^*L,\ p_2{}^*M\}_2 = \{p_1{}^*M,\ p_2{}^*L\}_2$ and $\{p_1{}^*M,\ p_2{}^*M\}_2$ over $\mathbb{Z}/2\mathbb{Z}$. According to Corollary 2.3, the equality (3) means that the quaternion algebra $A(p_1{}^*L,\ p_2{}^*M)$ and $A(p_1{}^*M,\ p_2{}^*L)$ are isonmorphic at the generic point of X. According to Proposition 1.5, the equality (4) means that the algebra $A(p_1{}^*L,\ p_2{}^*L)$ comes from some vector bundle over X. We note that the algebras $A(p_1{}^*L,\ p_2{}^*M)$, $A(p_1{}^*M,\ p_2{}^*L)$ and $A(p_1{}^*M,\ p_2{}^*M)$ do not come from any vector bundles over X.

Example 8.5 (for Remark 7.5). According to Theorem 7.3, the quaternion algebra $A(p_1{}^*L,\ p_2{}^*L)$ comes from a vector bundle of type (3) in Corollary 7.4. We shall show that: In case p = 3, $A(p_1{}^*L,\ p_2{}^*L)$ comes from a pull-back of a vector bundle over an elliptic curve; In case p = 0, $A(p_1{}^*L,\ p_2{}^*L)$ does not come from any pull-back of any vector bundle over any elliptic curve.

First, assume p = 3. Let ψ be the endomorphism $\iota+\eta+\iota\eta$ of E, and define a homomorphism $\Psi: X \to E$ to be the composition of $\psi \times \iota\psi$ with the group law of E. Using the equalities (1) and (2), we find

$$\Psi^*A(L+M,\ M) = A(p_1{}^*L,\ p_2{}^*L).$$

According to Example 7.1, $A(L+M, M)$ comes from a vector bundle V_3 over E which is of type Atiyah determined by the point P_0. Thus, our bundle $A(p_1^*L, p_2^*L)$ comes from the pull-back Ψ^*V_3.

Next, assume $p = 0$. In order to prove our claim, by Proposition 6.3, we have only to show that both line bundles p_1^*L and p_2^*L are not pull-backs of any line bundles over any elliptic curve. In this case, we may assume that k is the complex number field $\mathbb{C}$, and E is given by

$$E = \mathbb{C}^1/\Gamma, \qquad \Gamma = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot i,$$

with $i^2 = -1$ (see, e.g., (5, IV, (4.20.1))), so that

$$X = \mathbb{C}^2/\Gamma \times \Gamma.$$

Identifying X with its dual, the line bundles p_1^*L and p_2^*L correspond to the vectors $\left(\frac{1+i}{2}, 0\right)$ and $\left(0, \frac{1+i}{2}\right)$ of $\mathbb{C}^2$ modulo $\Gamma \times \Gamma$, respectively. Now, assume that both line bundles are pull-backs of some line bundles over an elliptic curve. Then, taking dual, there should exist a 1-dimensional vector subspace of $\mathbb{C}^2$ which contains both $\left(\frac{1+i}{2}, 0\right)$ and $\left(0, \frac{1+i}{2}\right)$ modulo $\Gamma \times \Gamma$. Therefore, we have

$$\det \begin{pmatrix} \frac{1+i}{2} + a & b \\ c & \frac{1+i}{2} + d \end{pmatrix} = 0$$

for some elements a, b, c and d of Γ. It follows that the complex number $\frac{1+i}{2}$ is integral over $\Gamma = \mathbb{Z}(i)$. This contradicts the fact that the ring $\mathbb{Z}(i)$ of Gaussian integers is integrally closed in its quotient field $\mathbb{Q}(i)$. Hence, our algebra $A(p_1^*L, p_2^*L)$ does not come from any pull-back.

Finally, we refer to a rational solution of a quadratic form over the function field K of X. Chasing the construction of the vector bundles at Section 7, one can find a K-rational solution of *all* the quadratic form $q(L, M)$ with 2-torsion line bundles L and M over X.

Example 8.6. Let q be the quadratic form $q(p_1^*L, p_2^*L)$. According to the local investigation of conic bundles at Section 1, q is represented by the matrix

$$\begin{pmatrix} 1 & & \\ & -x_1 & \\ & & -x_2 \end{pmatrix}$$

at the generic point Spec K of X, where (x_i, y_i) is the affine coordinate of E_i in $\mathbb{P}^2$, with i = 1, 2. The quadratic form q defines the projective line bundle $C(p_1^*L, p_2^*L)$, which is of type (3) in Corollary 7.4. So, chasing the construction of vector bundles of this type, we find a solution (X:Y:Z) of q, where

$$X := \frac{1-i}{2}\left(x_1{}^2x_2 + \frac{i}{2}y_1{}^2 \right) + x_1\left(\frac{1+i}{4}(x_1{}^2+1)x_2 - \frac{i}{2}y_1y_2 \right)$$

$$Y := \frac{1+i}{4}y_1(x_1+i)(x_2-1) - \frac{i}{2}y_2x_1(x_1-i)$$

$$Z := \frac{1-i}{2}\left(x_1{}^2x_2 + \frac{i}{2}y_1{}^2 \right) + \frac{x_1}{x_2}\left(\frac{1+i}{4}(x_1{}^2+1)x_2 - \frac{i}{2}y_1y_2 \right),$$

with $i^2 = -1$.

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