RECONSTRUCTION OF A 3-MANIFOLD
BY A NON-SINGULAR FLOW

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1. SPINES INDUCED BY A NON-SINGULAR FLOW

Let $M$ be a smooth and closed 3-manifold, and $\psi_t$ be a non-singular flow on $M$. Take a compact local section $\Sigma$ of $\psi_t$ so that it is homeomorphic to a compact 2-disk and intersects with every orbit of $\psi_t$. For such a pair $(\psi_t, \Sigma)$ we define functions $T_+(x)$ and $T_-(x)$ by

\[ T_+(x) = \inf \{ t > 0 \mid \psi_t(x) \in \Sigma \} \]
\[ T_-(x) = \sup \{ t < 0 \mid \psi_t(x) \in \Sigma \} \]

Moreover define $\hat{T}_\pm(x)$ to be $\hat{T}_\pm(x) = \psi_\sigma(x)$ ($\sigma = T_\pm(x)$).

We can take $\Sigma$ so that it satisfies that

(i) $\exists \Sigma$ is $\psi_t$-transversal at $(x, T_+(x))$ for any $x \in M$ (see [2] for the definition of $\psi_t$-transversality), and

(ii) if $x \in \partial \Sigma$ and $x_1 = \hat{T}_+(x) \in \partial \Sigma$, then $\hat{T}_+(x_1)$ is included in int($\Sigma$).

We call a pair $(\psi_t, \Sigma)$ with the above conditions a normal pair.

For a normal pair $(\psi_t, \Sigma)$, the flow-spines $P_- = P_-(\psi_t, \Sigma)$ and $P_+ = P_+(\psi_t, \Sigma)$ are defined by
\[ P_- = \Sigma \cup \{ \psi_t(x) \mid x \in \partial \Sigma, \ T_-(x) \leq t \leq 0 \} \]
\[ P_+ = \Sigma \cup \{ \psi_t(x) \mid x \in \partial \Sigma, \ 0 \leq t \leq T_+(x) \}. \]

Each of \( P_- \) and \( P_+ \) is a closed fake surface and forms a standard spine of \( M \) (cf. [1]). The set \( S_j(P_-) \) \((j = 2, 3)\) of the \( j \)-th singularities of \( P_- \) are given by

\[ S_3(P_-) = \{ x \in \text{int}({\Sigma}) \mid \hat{T}_+(x) \text{ and } \hat{T}_+^2(x) \text{ are both on } \partial \Sigma \} \]
\[ S_2(P_-) = \hat{T}_-(\partial \Sigma) \cup \{ \psi_t(x) \mid x \in S_3(P_-), \ 0 \leq t \leq T_+(x) \} \]

(see [1] and [2] for the precise).

2. RECONSTRUCTION OF \( M \).

Let \( B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1 \} \) be the unit 3-ball in \( \mathbb{R}^3 \), \( \rho : \partial B \to \partial B \) be a map defined by \( \rho(x, y, z) = (x, y, -z) \), and \( \imath \) be an embedding of \( \Sigma \) into \( S^2 = \partial B \) such that \( \imath(\partial \Sigma) = \partial B \cap \{z = 0\} \). We define an equivalence relation "\( \sim \)" on \( \partial B \) as follows:

(i) for \( x \in S_3(P_-) \subset \text{int}(\Sigma) \),
\[ \imath(x) \sim \imath(\hat{T}_+(x)) \sim \imath(\hat{T}_+^2(x)) \sim \imath(\rho(\hat{T}_+^3(x))) \]

(ii) for \( x \in \text{int}(\Sigma) \cap (\hat{T}_-(\partial \Sigma) - S_3(P_-)) \),
\[ \imath(x) \sim \imath(\hat{T}_+(x)) \sim \imath(\rho(\hat{T}_+^2(x))) \]

(iii) for \( x \in \text{int}(\Sigma) - \hat{T}_+^2(\partial \Sigma) \),
\[ \imath(x) \sim \imath(\rho(\hat{T}_+(x))) \]

Then we get the following theorem which gives a polyhedral representation of \( M \).
THEOREM 1 ([2]).

\[ M \text{ is homeomorphic to } B/\sim, \text{ and each of } P_- \text{ and } P_+ \text{ is homeomorphic to } \partial B/\sim. \]

3. AN APPLICATION

We consider a normal pair for which the following condition (B) is satisfied.

(B) \( \hat{T}_+(S_3(P_-)) \) and \( \hat{T}_+^2(S_3(P_-)) \) are separated by two points, namely, there are two points \( z_1 \) and \( z_2 \) on \( \partial \Sigma \) such that \( \hat{T}_+(S_3(P_-)) \) is included in one of the components of \( \partial \Sigma - \{z_1, z_2\} \) and \( \hat{T}_+^2(S_3(P_-)) \) is in the other.

This condition is a generalization of the condition (A) in [4]. And, using the reducing method shown in [3] and [4], we can deform a normal pair with (B) into one giving a polyhedral representation of some normal form which include those given by Fig.7 in [4]. As examples, we exhibit in Fig.1 and 2 the normal form obtained by the above way which represent the lens space \( L(5, 1) \) and \( L(5, 2) \) respectively. In general, we can prove that the polyhedral representation of these normal form yield \( S^3 \) or \( S^2 \times S^1 \) or lens space \( L(p, q) \). Conversely, using the method in [4], we can construct a normal pair with the condition (B) on these manifolds. Thus we have
THEOREM 2.

M admits a normal pair satisfying (B) if and only if

\[ M = S^3 \text{ or } S^2 \times S^1 \text{ or } L(p, q) . \]

REFERENCES
