数理解析研究所講究録
第 635 巻 1987 年 165-177

On A Bifurcation of Heteroclinic Orbits

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Abstract

A bifurcation of heteroclinic orbits in a two or more parameter family of autonomous ODEs is studied, where the unperturbed system has two heteroclinic orbits joined at a common saddle point. Some cases of bifurcation of homoclinic orbits are also treated including the one producing a twice-rounding homoclinic orbit along the original one.

1. Introduction

In the qualitative theory of autonomous ODEs, a trajectory connecting two equilibria O and O' is called a *heteroclinic orbit* or an (O,O')-*connection*. In case $O = O'$, it is called a *homoclinic orbit* based at O. These heteroclinic and homoclinic orbits play an important role in the theory of dynamical systems, though they are structurally unstable. For some information concerning such orbits, we refer to the textbook by Guckenheimer and Holmes[7)], and by Chow and Hale[5)].

From the bifurcation theoretical point of view, it is more difficult to study the bifurcation of homoclinic or heteroclinic orbits than that of equilibria, since the analysis of the former needs some global information. Recent development of the theory of the Melnikov function[8)] and the exponential dichotomy[6)] invoked many works on such bifurcations. See [2,Chap.4], [7,Chap.4 & 7], [5,Chap.11], and references therein.

These authors primarily pursue persistency conditions of homoclinic and heteroclinic orbits in perturbed systems. In other words, they deal with the case where, in some sense, the same type of heteroclinic (homoclinic) orbit persists as the one in the unperturbed system. It is, however, also the case where a new type of heteroclinic (homoclinic) orbit is born from several heteroclinic (homoclinic) orbits coexisting in the unperturbed system. Figures 1 - 4 illustrate examples of such bifurcations. The purpose of the present paper is to analyse these bifurcations of heteroclinic and homoclinic orbits.

Fig.1

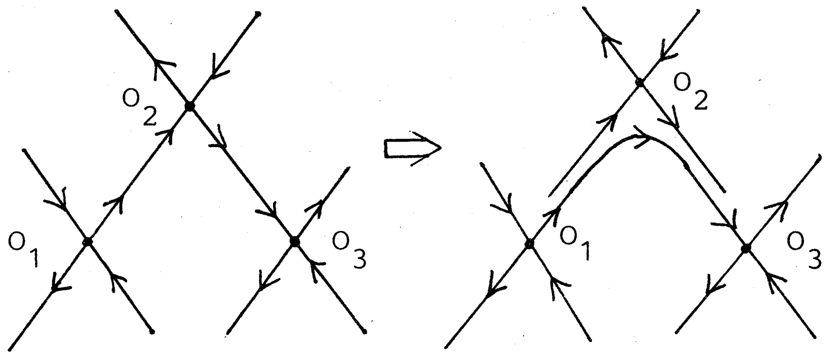


Fig.2

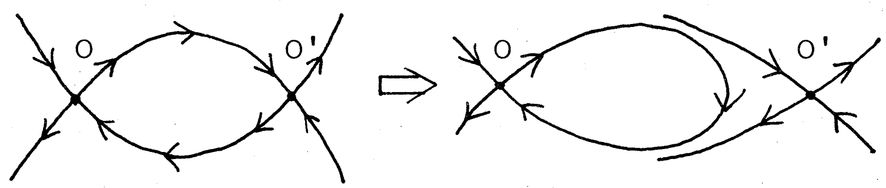


Fig.3

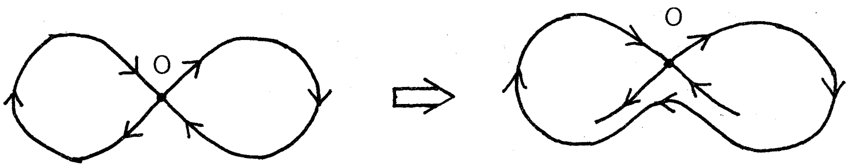
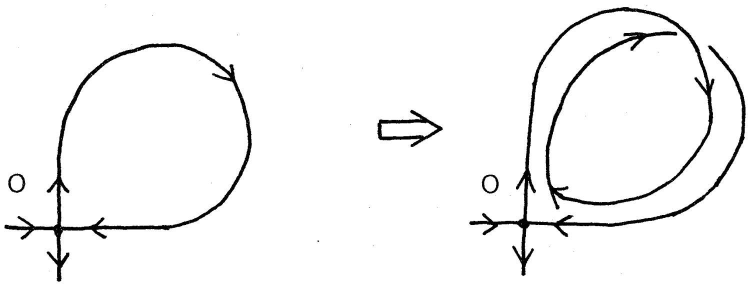


Fig.4



These kinds of bifurcation have not been studied until very recent years, but such a study is important not only for the bifurcation theory but also for various other fields such as the study of the nerve impulse.

In our analysis, we basically treat the bifurcation shown in Fig.1, and divide the analysis into two cases: 'the case of non-degenerate eigenvalues' and 'the case of critical eigenvalues' (see below for the precise statement). The former case can be easily studied, while the latter requires more delicate analysis. It is, however, inevitable to deal with the case of critical eigenvalues for the study of the bifurcation producing a twice-rounding homoclinic orbit given in Fig.4. The bifurcations indicated in Fig.2 and 3 are also studied along the same line. Throughout our paper, the notion of the exponential dichotomy and a representation given by Shil'nikov[10,11]) of trajectories near a saddle equilibrium, play a very important role.

2. Statement of the results

Let us take a k-parameter family of (m+n)-dimensional ordinary differential equations

$$(1) \qquad \dot{x} = f(x) + g(x,\mu), \qquad x \in \mathbb{R}^{m+n}, \quad \mu \in \mathbb{R}^k \ (k\geq 2),$$

where f and g are smooth, and $g(x,0) = 0$. We consider the following situation:

(S1) The system (1) has three saddle equilibria $O_1(\mu)$, $O_2(\mu)$ and $O_3(\mu)$, and the eigenvalues

$$-\eta^i_{m-1}(\mu),\ldots,-\eta^i_1(\mu),-\rho^i(\mu),\nu^i(\mu),\kappa^i_1(\mu),\ldots,\kappa^i_{n-1}(\mu)$$

of the linearized system at $O_i(\mu)$ $(i=1,2,3)$ satisfy

$$-\Re e\,\eta^i_{m-1}(0) \leq \ldots \leq -\Re e\,\eta^i_1(0) < -\rho^i(0) < 0$$

$$< \nu^i(0) < \Re e\,\kappa^i_1(0) \leq \ldots \leq \Re e\,\kappa^i_{n-1}(0)$$

(S2) The unperturbed system

$$(2) \qquad \dot{x} = f(x)$$

for $\mu = 0$ has a heteroclinic orbit $h_1(t)$ connecting $O_1(0)$ and $O_2(0)$ (i.e. an (O_1,O_2)-connection) and a heteroclinic orbit $h_2(t)$ connecting $O_2(0)$ and $O_3(0)$

(i.e. an (O_2,O_3)-connection) simultaneously. See Fig.1.

(S3) For $\mu = \mathbf{0}$, $W^u(O_i)$ $(i=1,2)$ and $W^s(O_i)$ has one-dimensional intersection, i.e. for any point P on the (O_i,O_{i+1})-connection, it holds that

$$\dim\left(T_PW^u(O_i) \cap T_PW^s(O_{i+1})\right) = 1.$$

Under this situation, the problem can be stated as:

(P1) Find a condition for the parameter μ so that there exists an (O_i,O_{i+1})-connection in (1) for $i = 1,2$, respectively.

(P2) Find a condition for the parameter μ so that there exists an (O_1,O_3)-connection in (1), passing through a neighborhood of the equilibrium O_2.

In order to give an answer to these problems, we impose the following non-degeneracy hypotheses:

(H1) The heteroclinic orbits in (2) are generic in the sense that, as $t \to -\infty$, $h_i(t)$ $(i=1,2)$ approaches the equilibrium $O_i(\mathbf{0})$ along the eigenspace associated with $\nu^i(\mathbf{0})$, and, as $t \to +\infty$, it approaches to $O_{i+1}(\mathbf{0})$ along the eigenspace associated with $-\rho^{i+1}(\mathbf{0})$.

(H2) For $\mu = \mathbf{0}$, the manifold $W^u(O_2(\mathbf{0})) \cap \overline{W^u(O_1(\mathbf{0}))}$ is transversal to the eigenspace associated with $\nu^2(\mathbf{0})$ at $O_2(\mathbf{0})$ in $W^u(O_2(\mathbf{0}))$. Also, $W^s(O_2(\mathbf{0})) \cap \overline{W^s(O_3(\mathbf{0}))}$

is transversal to the eigenspace associated with $-\rho^2(\mathbf{0})$ at $O_2(\mathbf{0})$ in $W^s(O_2(\mathbf{0}))$.

(H3) For a bounded solution $\hat{q}^i(t)$ $(i=1,2)$ of the linear ordinary differential equation

$$\dot{\hat{z}} = -{}^t Df(h_i(t))\cdot\hat{z}, \quad (i=1,2),$$

the vectors given by the integrals

$$(3) \qquad \int_{-\infty}^{+\infty} \hat{q}^i(s)\cdot\frac{\partial}{\partial\mu}g(h_i(s),\mathbf{0})ds \quad (i=1,2)$$

are linearly independent, and hence, non-zero.

<u>**Remark**</u> It can be proved that the bounded solution $\hat{q}^i(t)$ is unique up to the multiplication by constants.

Now the next theorem gives an answer for (P1).

<u>**Theorem A**</u>

Under the hypotheses (H1)-(H3), *there exist two hypersurfaces* M_i $(i=1,2)$ *of codimension 1 in a sufficiently small neighborhood of* $\mu = \mathbf{0}$ *in* $\mathbb{R}^k$, *so that* M_i *consists of parameter values* μ *for which the system has an* (O_i, O_{i+1})*-connection. Moreover* M_1 *and* M_2 *intersects transversally at* $\mu = \mathbf{0}$.

<u>**Remark**</u> This theorem is not new, since essentially the same result was obtained by Palmer[9], et al. However, the result as well as its proof is necessary for the following

analysis.

In order to investigate the second problem (P2), we divide our analysis into the following two cases:

(i) $\nu^2(0) \neq \rho^2(0)$ [the case of non-degenerate eigenvalues]

(ii) $\nu^2(0) = \rho^2(0)$ [the case of critical eigenvalues]

The first case is comparatively easy and we can show the following theorem.

Theorem B (The case of non-degenerate eigenvalues)

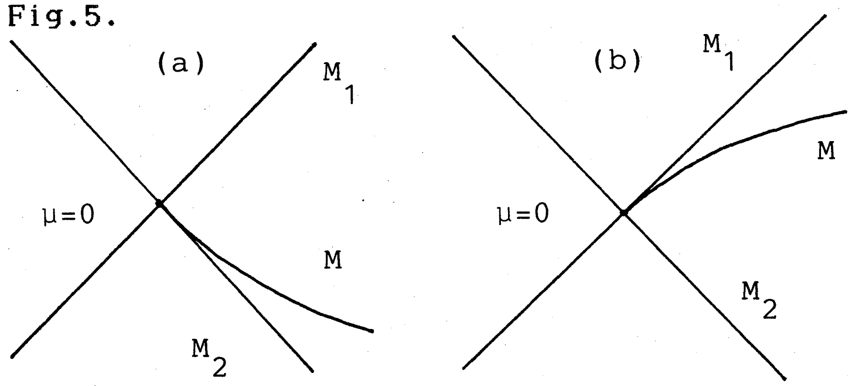
Under the hypotheses (H1)-(H3) *and for the case of non-degenerate eigenvalues, there exists a hypersurface* M *of codimension 1 with the boundary* $\partial M = M_1 \cap M_2$ *in a sufficiently small neighborhood of* $\mu = 0$ *in* $\mathbb{R}^k$, *so that* M *consists of parameter values* μ *for which the system* (1) *has an* (O_1, O_3)*-connection. Moreover,*

(a) *if* $\nu^2(0) < \rho^2(0)$, *then* M *is tangent to* M_2 *at* $\mu=0$.

(b) *if* $\nu^2(0) > \rho^2(0)$, *then* M *is tangent to* M_1 *at* $\mu=0$.

From Theorems A and B, we obtain the bifurcation diagram as shown in Fig.5.

Fig.5



For the case of critical eigenvalues, we restrict ourselves to the case $k = 3$, i.e. a three-parameter family, and impose further hypothesis as follows.

(H4) The set $\{\mu | \nu^2(\mu) = \rho^2(\mu)\}$ forms a surface π in the parameter space $\mathbb{R}^3$ and is transversal to both of M_1 and M_2 at $\mu = \mathbf{0}$. In other words, the vector

$$\frac{d}{dt}\Big|_{\mu=\mathbf{0}} \{\nu^2(\mu) - \rho^2(\mu)\}$$

is linearly independent of both vectors given by (3).

Theorem C (The case of critical eigenvalues)

Under the hypotheses (H1)-(H4) *and for the case of critical eigenvalues, there exists a surface* M *with the boundary* $\partial M = M_1 \cap M_2$ *in a sufficiently small neighborhood of* $\mu = \mathbf{0}$ *in* $\mathbb{R}^3$, *so that it consists of parameter values* μ *for which the system (1.1) with* $k = 3$ *has an* (O_1, O_3)*-connection and the curve given by the intersection of* M *and* π *is tangent to neither of* M_1 *nor* M_2 *at* $\mu = \mathbf{0}$.

Fig.6 shows the bifurcation diagram for the case of critical eigenvalues.

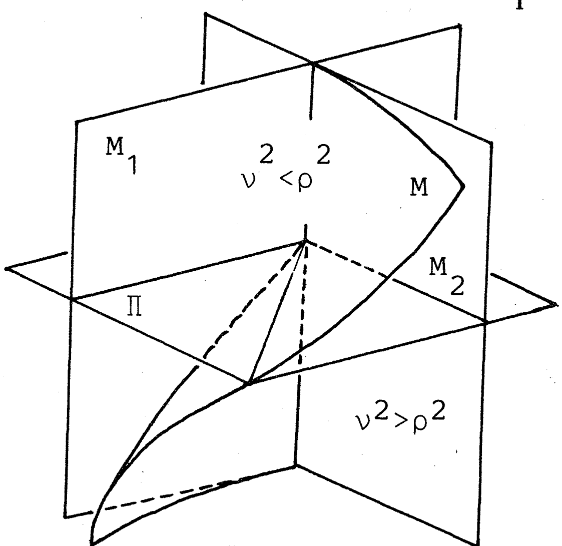


Fig.6

The method to prove these theorems also applicable to some other types of homoclinic and heteroclinic orbits, such as the ones indicated in Fig.2 and 3.

In order to study the bifurcation shown in Fig.4, we distinguish these two kinds of homoclinic orbits; we call the homoclinic orbit rounding twice around the original one produced by the bifurcation a *twice-rounding homoclinic orbit*. For definiteness, we also call the original type of homoclinic orbit a *once-rounding homoclinic orbit*.

Now let us consider a system (1) with $m = 2$, $n = 1$ and $k = 2$, and assume that the unperturbed system (2) has a (once-rounding) homoclinic orbit $h(t)$ based at a saddle equilibrium, say the origin O, with eigenvalues $-\eta < -\rho < 0 < \nu$. Furthermore we impose the next five hypotheses:

(H1)' The homoclinic orbit is generic, i.e. it approaches O along the eigenspace associated with $-\rho$ as $t \to +\infty$.

(H2)' $W^s_{loc}(O) \cap \overline{W^s(O)-W^s_{loc}(O)}$ is transversal to the eigenspace associated with $-\rho$ in $W^s(O)$.

(H3)' For a bounded solution $\hat{q}(t)$ of the linear ordinary differential equation

$$\dot{\hat{z}} = -{}^tDf(h(t))\hat{z},$$

the vector given by the integral

(4) $$\int_{-\infty}^{+\infty} \hat{q}(s)\cdot\frac{\partial}{\partial\mu}g(h(s),0)ds$$

is non-zero.

(H4)′ The vector

$$\frac{d}{d\mu}\Big|_{\mu=0}\{\nu(\mu)-\rho(\mu)\}$$

is linearly independent of the vector given by (4).

(H5)′ An exponent λ, which is defined by the information only of the unperturbed system, is non-zero.

Theorem D (The doubling of the homoclinic orbit)

Under the hypotheses (H1)′-(H3)′, *there exists a curve* M_0 *passing through the origin* $\mu = 0$ *in a sufficiently small neighborhood of* $\mu = 0$ *in* $\mathbb{R}^2$, *so that it consists of parameter values* μ *for which the system has a once-rounding homoclinic orbit.*

If $\nu \neq \rho$ *at* $\mu = 0$, *and even when* $\nu = \rho$ *at* $\mu = 0$, *if a tubelar neighborhood* $\mathcal{T}$ *of the unperturbed homoclinic orbit in* $W^s(O)$ *is homeomorphic to a cylinder, then there is no parameter value corresponding to a twice-rounding homoclinic orbit in a neiborhood of* $\mu = 0$.

If $\nu = \rho$ *at* $\mu = 0$ *and* $\mathcal{T}$ *is homeomorphic to a Möbius strip, then, under additional hypotheses* (H4)′ *and* (H5)′, *there exists a curve* M *containing* $\mu = 0$ *at an end, so that it consists of parameter values* μ *for which the system has a twice-rounding homoclinic orbit.*

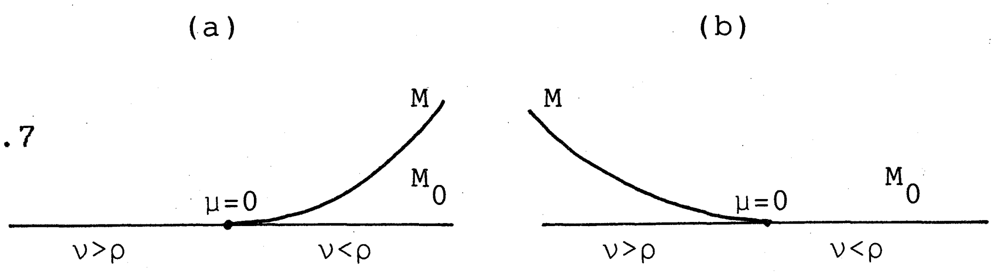
Moreover,

(a) *if the exponent* r *is positive, then* M *is tangent to* M_0 *at the side of* $\nu(\mu) < \rho(\mu)$.

(b) *if the exponent* r *is negative, then* M *is tangent to* M_0 *at the side of* $\nu(\mu) > \rho(\mu)$.

The bifurcation diagram is given in Fig.7.

Fig.7



3. Concluding remarks

(1) The bifurcation of homoclinic and heteroclinic orbits treated in this paper was first investigated by Yanagida[12] intending to the study of the pulse travelling waves in a nerve equation, where he obtained almost the same result as Theorem D of this paper. His argument is, however, insufficient at the point that he linearizes the system around the equilibrium using the C^0-linearization theorem due to Hartman and Grobman, which loses the smoothness of the system. In order to overcome the difficulty, our earlier version adopted the Belitskii's result of C^1-linearization[1] under an additional condition on eigenvalues.

(2) Chow, Deng and Terman[3] studied the same problem (P1),

(P2) of this paper from the topological point of view. Their results are, hence, weaker than ours under weaker assumptions. Very recently, I have received their new preprint[4], in which they have obtained analytical results for this problem. They are dealing with the bifurcation problem of heteroclinic orbits of the type in Fig.2 for the case of non-degenerate eigenvalues. They also obtained a result of the bifurcation of periodic orbits from the pair of heteroclinic orbits.

By using the Shil'nikov's representation[10,11] of trajectories near a saddle equilibrium, which plays an imprtant role in [4], it is not necessary to C^1-linearize the system, hence we can exclude the Belitskii's condition. The major differences between [4] and ours are as follows:

(i) In [4], only the case of non-degenerate eigenvalues is treated while ours contains the case of critical eigenvalues.

(ii) In [4], they assume that, in our notation, M_1 and M_2 intersect transversally. In our paper, we give an explicit criterion (H3) of transversal intersection in terms of the Melnikov-like integrals.

(iii) The bifurcation analysis of [4] covers the periodic orbits produced from heteroclinic orbits. Since the unperturbed system for Theorem A-C (Fig.1) cannot give rise to any periodic orbits by slightly perturbing it, we don't study the bifurcation of periodic orbits.

(3) The details as well as the proofs of the theorems in this article will appear in elsewhere. Especially, the explicit definition of the exponent $\imath$ in (H5)' is given there.

4. References

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