数理解析研究所講究録
第 638 巻 1988 年 13-32

Higher dimensional analogue of the m-truncated KP hierarchy

Masatoshi NOUMI （野海 正俊）
Sophia University（上智大・理工）

§0. Introduction

The theory of KP hierarchy of Prof. Mikio Sato can be thought of as a universal framework for the integrable systems of non-linear differential equations involving a single space variable. It seems, however, that no theory of such a nature has been brought out yet so far as "higher dimensional" integrable systems are concerned. As is already introduced in Dr. Takasaki's article [5], Prof. Sato presented, in his recent lectures at Kyoto University, a point of view that integrable systems, hopefully including higher dimensional ones, appear via correspondence of Riemann-Hilbert type, namely, interrelation between linear differential equations and their solution spaces. Following his idea, some young people from Kyoto and Tokyo are now working in search for a natural framework for higher dimensional systems. The purpose of this note is to report some topics related to this subject, mainly on an analogue of the "m-truncated" KP hierarchy. The m-truncated KP hierarchy is an object, holonomic in some sense, from which the KP hierarchy can be obtained by a limiting process as m tends to infinity. Though the present stage of this note is, in nature, far from the vision of Prof. Sato, it might be regarded as a miniature of the theory of higher dimensional systems in its proper form if it will be brought to light in the near future.

Before starting my report, I should make it clear that the basic ideas behind are due to Prof. Sato and most part of its theme has already been prospected by himself. The contents of this note are formed out of the discussion or taken from the joint work with the following five persons : Prof. K. Ueno, Dr. K. Takasaki, Mr. H. Harada, Mr. N. Suzuki and Mr. Y. Ohyama.

§ 1. Wronskian determinants and the Grassmann formalism

Our first object of interest is the Wronskian determinant for functions in several variables. To fix the ideas, we take a differential field K with a family $\partial = (\partial_0, \ldots, \partial_{r-1})$ of mutually commutative derivations $\partial_k : K \to K$ $(0 \leq k < r)$. We denote by $C = \{ c \in K;\ \partial_0(c) = \ldots = \partial_{r-1}(c) = 0 \}$ the field of constants of K, and by $D = K[\partial]$ the ring of differential operators in ∂ with K-coefficients. By definition, D is a (free) left K-module with basis $\{ \partial^\alpha ;\ \alpha \in \mathbb{N}^r \}$, and its multiplicative structure is characterized by the commutation rules

$$(1.1) \quad [\partial_i, \partial_j] = 0 \quad \text{and} \quad [\partial_k, f] = \partial_k(f) \quad \text{for any} \quad f \in K.$$

Setting $L = N^r$, we wil use freely the notation of multi-indices : $\partial^\alpha = \partial_0^{\alpha(0)} \ldots \partial_r^{\alpha(r-1)}$ for each $\alpha = (\alpha(0), \ldots, \alpha(r-1)) \in L = \mathbb{N}^r$.

Let $\underline{f} = (f_0, \ldots, f_{m-1}) \in K^m$ be an m-tuple of "functions" (i.e. elements of K). Then, for each m-tuple $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{m-1}) \in L^m$ of multi-indices, we define the Wronskian determinant $Wr_{\underline{\alpha}}(\underline{f}) = Wr_{\alpha_0, \ldots, \alpha_{m-1}}(f_0, \ldots f_{m-1})$ of $\underline{f}$ with indices $\underline{\alpha}$ by

$$(1.2) \quad Wr_{\alpha_0, \ldots, \alpha_{m-1}}(f_0, \ldots, f_{m-1}) = \det(\partial^{\alpha_i}(f_j)\ ;\ 0 \leq i, j < m) \in K.$$

It is well known that, in the case where $r = 1$, one has only to check one Wronskian determinant $Wr_{0,1,\dots,m-1}(f_0,\dots,f_{m-1})$ to see if $f_0,\dots,f_{m-1}$ are linearly independent or not. In the case where $r>1$, however, the situation becomes a little complicated.

Let us call a subset S of the lattice L an <u>order ideal</u> if it satisfies the condition

$$(1.3) \qquad \alpha \in L, \quad \beta \in S \quad \text{and} \quad \alpha \leqq \beta \implies \alpha \in S,$$

where $\leqq$ is the natural partial order of $L = \mathbb{N}^r$. (The above condition is equivalent to saying that the complement $E = L \setminus S$ is a <u>monoideal</u> of the monoid L.) By abuse of terminology, we say that an m-tuple $\underline{\alpha} = (\alpha_0,\dots,\alpha_{m-1}) \in L^m$ is an <u>order ideal</u> of degree m if the α_i's are mutually distinct and the set $\{\alpha_0,\dots,\alpha_{m-1}\}$ is an order ideal of L.

<u>Theorem 1.1</u>. Let $\underline{f} = (f_0,\dots,f_{m-1})$ be an m-tuple of elements of K. Then, $f_0,\dots,f_{m-1}$ are linearly independent over the field of constants C if and only if there exists an order ideal $\underline{\alpha} \in L^m$ of degree m such that $Wr_{\underline{\alpha}}(\underline{f})$ does not vanish.

(This theorem was conjectured by Prof. K. Ueno, and proved by Mr. N. Suzuki and the author independently.)

Let V be a C-subspace of K of dimension m. We define an left ideal J_V of D by

$$(1.4) \qquad J_V = \{ P \in D \; ; \; Pf = 0 \text{ for all } f \in V \}.$$

Now, choose a C-basis $\underline{f} = (f_0,\dots,f_{m-1})$ of V. For each $\underline{\beta} = (\beta_0,\dots,\beta_m) \in L^{m+1}$, define a differential operator $P_{\underline{\beta}}$ by

$$(1.5) \qquad P_{\underline{\beta}} = \Sigma_{i=0}^{m} (-)^i \, Wr_{\beta_0,\dots,\hat{\beta}_i,\dots,\beta_m}(\underline{f})\, \partial^{\beta_i},$$

which readily belongs to the ideal J_V.

Corollary to Theorem 1.1. With the notations as above, the ideal J_V of D is generated by the set { P_β ; β is an order ideal of degree m+1 } and the quotient D-module $M_V = D / J_V$ is an m-dimensional vector space over K. Moreover one has

$$(1.6) \qquad V = \{ f \in K \; ; \; Pf = 0 \text{ for all } P \in J_V \}.$$

The latter half of Corollary suggests that there is a one to one correspondence between the finite dimensional C-vector spaces and the holonomic systems of linear differential equations "solvable in K".

We say that a (left) D-module M is holonomic if it is finite dimensional as a vector space over K. (Here, we adopt it as the definition of holonomicity.) For each D-module M, we denote by $Sol(M) = Hom_D(M,K)$ the C-vector space of solutions to M in K. Using Theorem 1.1, one can show that, for any holonomic D-module M, the dimension of Sol(M) over C does not exceed the dimension of M over K. We say that a holonomic D-module M is K-solvable if $\dim_C Sol(M) = \dim_K M$.

Theorem 1.2. The functor Sol gives an anti-equivalence between the category of K-solvable holonomic D-modules and the category of finite dimensional vector spaces over C.

(This theorem is more or less tautological indeed, unless one should give a criterion for a holonomic D-module to be K-solvable.)

Theorem 1.2 give rise to various types of isomorphism between two "Grassmann manifolds", one consisting of D-modules and the other consisting of C-vector spaces.

For a D-module M, we denote by $\mathrm{Quot}_D(m;M)$ $(\mathrm{Quot}_D(m;M)^{K\text{-solv}})$ the set of (K-solvable) quotient D-modules M' of M with $\dim_K M' = m$. For a C-vector space V, we denote by $\mathrm{Sub}_C(m;V)$ the set of C-subspaces V' of V with $\dim_C V' = m$. With these notations,

<u>Corollary to Theorem 1.2</u>. (1) Sol gives an isomorphism

(1.7) $\quad \mathrm{Sol} : \mathrm{Quot}_D(m;D)^{K\text{-solv}} \xrightarrow{\sim} \mathrm{Sub}_C(m;K)$ for each $m \in \mathbb{N}$.

(2) Let M be a K-solvable holonomic D-module and set $V = \mathrm{Sol}(M)$. Then one has an isomorphism

(1.8) $\quad \mathrm{Sol} : \mathrm{Quot}_D(m;M) \xrightarrow{\sim} \mathrm{Sub}_C(m;V)$ for each $m \in \mathbb{N}$.

As Prof. Sato pointed out, the left hand side of the isomorphism (1.7) or (1.8), which is a kind of Grassmann manifold itself, can be regarded as represents the space of solutions to a certain system of non-linear differential equations. Theorem then says that the totality of its solutions can be parametrized by the Grassmann manifold on the right hand side. In order to give an explicit form to this type of "Grassmann formalism", we will discuss the "canonical forms" of linear differential equations. In that context, it is natural that Wronskian determinants play an essential role in describing the inverse of the isomorphisms of Theorem 1.2.

<u>§2. Gröbner representation of a linear differential equation</u>

In this section, we discuss the canonical forms of <u>cyclic</u> D-modules, with respect to a total order of L. We fix a <u>total order</u> $\preccurlyeq$ of $L = \mathbb{N}^r$, <u>additive</u> in the following sense :

(2.1) i) $(L, \preceq)$ is a well-ordered set.

ii) $0 \preceq \alpha$ ($\alpha \in L$) ; $\alpha \preceq \beta$, $\gamma \in L \implies \alpha + \gamma \preceq \beta + \gamma$.

Note that the condition ii) implies : $\alpha \leqq \beta \implies \alpha \preceq \beta$ ($\alpha, \beta \in L$). A typical example of such a total order is the lexicographic order of $L = \mathbb{N}^r$, and another is the one composed of the length of multi-indices and the lexicographic order.

Using the additive total order $\preceq$, we define bilateral K-submodule $D_{\preceq \alpha}$ and $D_{\prec \alpha}$ as follows :

(2.2) $D_{\preceq\alpha} = \bigoplus_{\beta \preceq \alpha} K\partial^{\beta}$ and $D_{\prec\alpha} = \bigoplus_{\beta \prec \alpha} K\partial^{\beta}$ for each $\alpha \in L$.

For a non-zero differential operator P in D, we define the $\preceq$-order of P, denoted by $\mathrm{ord}_{\preceq}(P)$ ($\in L$), as the mininum among all $\alpha \in L$ such that $P \in D_{\preceq\alpha}$. An operator P of $\preceq$-order α is said to be monic if $P \equiv \partial^{\alpha}$ mod $D_{\prec\alpha}$. For each subset S of L, we define a left K-submodule $K\langle S\rangle$ of D by

(2.3) $$K\langle S\rangle = \bigoplus_{\alpha \in S} K\partial^{\alpha} \subset D,$$

which is a right K-submodule as well if S is an order ideal of L.

Lemma 2.1. (Division Lemma.) Let A be a finite subset of L and define an order ideal S of L by $S = L \setminus \bigcup_{\alpha \in A}(\alpha + L)$. Let $(P_\alpha)_{\alpha \in A}$ be a family of differential operators in D such that P_α is monic of $\preceq$-order α for each $\alpha \in A$. If P is an operator in $D_{\preceq\beta}$ ($\beta \in L$), there exist operators Q_α ($\alpha \in A$) and R in D such that

(2.4) $$P = \Sigma_{\alpha \in A} Q_\alpha P_\alpha + R,$$

satisfying the condition

(2.5) $R \in K\langle S\rangle$ and ($Q_\alpha \neq 0 \Longrightarrow \mathrm{ord}_{\preceq}(Q_\alpha) + \alpha \preceq \beta$).

As a matter of fact, we need this type of Division Lemma to prove Theorem 1.1. Note that one cannot always expect the unicity of the residue R. (We will return to this point later.)

Let $M = D / J$ be a <u>cyclic</u> D-module with generator $u = 1 \bmod J$. (We do <u>not</u> assume M to be holonomic.) Then we associate a subset S(M) of L defined by

(2.6) $$S(M) = \{ \alpha \in L ; \ M_{\preceq\alpha} / M_{\prec\alpha} \neq 0 \},$$

where $M_{\preceq\alpha} = D_{\preceq\alpha}u$ and $M_{\prec\alpha} = D_{\prec\alpha}u$. For each subset S of L, we denote by $\varepsilon_S : K\langle S\rangle \to M$ the K-homomorphism $P \mapsto Pu$.

<u>Proposition 2.2</u>. The set S(M) is an order ideal of L. Moreover $\varepsilon_{S(M)}$ gives a K-isomorphism $K\langle S(M)\rangle \xrightarrow{\sim} M$.

One can also show that S(M) is the maximum among all subset $S \subset L$ such that $\varepsilon_S : K\langle S\rangle \to M$ is injective, under a natural total order of the power set $\mathcal{P}(L)$ induced by $\preceq$. In this sense, we call S(M) the <u>canonical order ideal</u> of the cyclic D-module (M,u) with respect to the total order $\preceq$.

Consider the case where $M = D / J_V$ is the D-module coming from an m-dimensional C-subspace V of K with basis $\underline{f} = (f_0,\dots,f_{m-1})$. If one expresses the canonical order ideal S(M) as $\{\alpha_0,\dots,\alpha_{m-1}\}$, where $\alpha_0 \prec \alpha_1 \prec \dots \prec \alpha_{m-1}$, then the m-tuple $\underline{\alpha} = (\alpha_0,\dots,\alpha_{m-1})$ is characterized as the minimum among all $\beta \in L^m$ such that $Wr_{\underline{\beta}}(\underline{f}) \neq 0$, under the lexicographic order of L^m induced by the total order $\preceq$ of L.

When S is an order ideal of L, we denote by A(S) the set of all minimal elements, under the natural partial order of $L = \mathbb{N}^r$, of the complement $L \setminus S$; the set $A(S)$ is necessarily finite, and one has $S = L \setminus \bigcup_{\alpha \in A(S)} (\alpha + L)$. If (M,u) is a cyclic D-module with canonical order ideal S, then by Proposition 2.2 one can find a unique family $(W_\alpha)_{\alpha \in A}$, $A = A(S)$, such that

$$(2.7) \qquad W_\alpha \in \partial^\alpha + K\langle S\rangle \text{ and } W_\alpha u = 0 \text{ for each } \alpha \in A.$$

<u>Proposition 2.3</u>. With the notations as above, one has $\mathrm{ord}_{\preccurlyeq}(W_\alpha) = \alpha$ for each $\alpha \in A$. Moreover, $(W_\alpha)_{\alpha \in A}$ gives a generator system of the ideal J of M.

We call the above $(W_\alpha)_{\alpha \in A}$ the <u>canonical generator system</u> of J or the <u>Gröbner representation</u> of the system M, with respect to the total order $\preccurlyeq$.

In the case where $M = D / J_V$ comes from an m-dimensional C-subspace V of K, the generator system $(W_\alpha)_{\alpha \in A}$ can be explicitly expressed by Wronskian determinants. Let $\underline{f} = (f_0, \ldots, f_{m-1})$ be a C-basis of V. If one writes the canonical order ideal $S=S(M)$ as $\{\alpha_0, \ldots, \alpha_{m-1}\}$, with $\alpha_0 \prec \alpha_1 \prec \ldots \prec \alpha_{m-1}$, then one has for each $\alpha \in A=A(S)$

$$(2.8) \qquad W_\alpha = \partial^\alpha + \Sigma_{i=0}^{m-1} w_{\alpha,i} \partial^{\alpha_i}, \text{ where}$$

$$w_{\alpha,i} = (-)^{m-i} \frac{Wr_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_{m-1},\alpha}(f_0,\ldots,f_{m-1})}{Wr_{\alpha_0,\ldots,\alpha_{m-1}}(f_0\ldots,f_{m-1})}.$$

Note that the property $\mathrm{ord}_{\preccurlyeq}(W_\alpha) = \alpha$ is equivalent to saying that $w_{\alpha,i} = 0$ for $\alpha_i \succ \alpha$.

Suppose that a generator system of an ideal J is given. Then how can one know what subset of L is the canonical order ideal of $M = D / J$ or what is the canonical generator system of J ? The answer can be given in an algorithm by using the characterization theorem of the canonical order ideal.

<u>Theorem 2.4</u>. Let A be a finite subset of L and define an order ideal S by $S = L \setminus \bigcup_{\alpha \in A}(\alpha + L)$. Let $(P_\alpha)_{\alpha \in A}$ be a family of monic operators in D with $\mathrm{ord}_{\preceq}(P_\alpha) = \alpha$. Denote by J the left ideal of D generated by $(P_\alpha)_{\alpha \in A}$ and set $M = D / J$. Then one has $S \supset S(M)$. Moreover, the following three conditions are equivalent :

a) S is the canonical order ideal $S(M)$ of M.

b) (Unicity of the residue in Division Lemma.) $K\langle S\rangle \cap J = 0$.

c) (Compatibility condition.) For any couple $(\alpha,\beta) \in A^2$, there exist operators $Q_\gamma^{(\alpha,\beta)}$ $(\gamma \in A)$ such that

$$\partial^{\alpha'} P_\beta - \partial^{\beta'} P_\alpha = \Sigma_{\gamma \in A}\, Q_\gamma^{(\alpha,\beta)}\, P_\gamma, \tag{2.9}$$

where $\alpha' = \alpha \vee \beta - \beta$ and $\beta' = \alpha \vee \beta - \alpha$, $\alpha \vee \beta = \sup\{\alpha,\beta\}$ under the natural partial order of L, and that, for each $\gamma \in A$,

$$Q_\gamma^{(\alpha,\beta)} \neq 0 \implies \mathrm{ord}_{\preceq}(Q_\gamma^{(\alpha,\beta)}) + \gamma \prec \alpha \vee \beta. \tag{2.10}$$

(This theroem was proved by Mr. Y. Ohyama and the author.)

By using Theorem 2.4, one can express the "Grassmann manifold" $\mathrm{Quot}_D(m;D)$ as a disjoint union of solution spaces of certain non-linear equations. Note first that $\mathrm{Quot}_D(m;D)$ is decomposed into a disjoint union

$$\mathrm{Quot}_D(m;D) = \coprod_S \mathrm{Quot}_D(S;D), \tag{2.11}$$

where S ranges over the set of all order ideals with $\#S = m$, and $\mathrm{Quot}_D(S;D)$ stands for the set of all quotient D-modules of D whose canonical order ideal equals S. Now, take an order ideal S of L with $\#S = m$ and set $A = A(S)$. Consider a family

$(W_\alpha)_{\alpha \in A}$ of differential operators in the form

$$(2.12) \qquad W_\alpha = \partial^\alpha + \Sigma_{\beta \in S, \beta \prec \alpha} w_{\alpha,\beta} \partial^\beta ,$$

for each $\alpha \in A$. Regarding $w_{\alpha,\beta}$ as "symbols", one can find, for each couple $(\alpha,\beta) \in A^2$, a differential operator $R^{(\alpha,\beta)}$ such that

$$(2.13) \qquad \partial^{\alpha'} W_\beta - \partial^{\beta'} W_\alpha = \Sigma_{\gamma \in A} B_\gamma^{(\alpha,\beta)} W_\gamma + R^{(\alpha,\beta)},$$

for some differential operators $B_\gamma^{(\alpha,\beta)}$ satisfying (2.10). (Use Division Lemma 2.1 for a suitable differential field.) Here, the operator $R^{(\alpha,\beta)}$ should have the form

$$(2.14) \qquad R^{(\alpha,\beta)} = \Sigma_{\gamma \in S} r_\gamma^{(\alpha,\beta)} \partial^\gamma .$$

Note that the coefficients $r_\gamma^{(\alpha,\beta)}$ $(\alpha,\beta \in A, \gamma \in S)$ are determined as differential polynomials in $w_{\alpha,\beta}$ $(\alpha \in A, \beta \in S, \beta \prec \alpha)$. Then the component $Quot_D(S;D)$ of (2.11) can be identified with the totality of families $(W_\alpha)_{\alpha \in A}$ of operators in $D = K[\partial]$ in the form (2.12), satisfying the system of non-linear equations $R^{(\alpha,\beta)} = 0$ $(\alpha,\beta \in A)$.

§ 3. The m-truncated KP hierarchy in higher dimensions

Now, we apply the arguments of the preceding sections to formulating the m-truncated KP hierarchy in higher dimensions.

Let $R = \mathbb{C}[[x]]$ be the ring of formal power series in r variables $x = (x_0,\dots,x_{r-1})$ and K the field of total fractions of R. Endowed with the family $\partial_x = (\partial_{x_0},\dots,\partial_{x_{r-1}})$ of mutually commutative derivations, K is regarded as a differential field. As before, $D = K[\partial_x]$ denotes the ring of differential operators with coefficients in K.

We say that a cyclic holonomic D-module $M = D / J$ is <u>R-solvable</u> if it is K-solvable and $Sol(M) \subset R$, where $Sol(M)$ is identified with a $\mathbb{C}$-subspace of K. The totality of R-solvable quotients M of D with $\dim_K M = m$ will be denoted by $Quot_D(m;D)^{R\text{-}solv}$. In the sequel, we denote by $\mathbb{V}$ the underlying $\mathbb{C}$-vector space of $R = \mathbb{C}[[x]]$ and by $GM(m;\mathbb{V}) = Sub_{\mathbb{C}}(m;R)$ the infinite dimensional Grassmann manifold consisting of all m-dimensional $\mathbb{C}$-subspaces of $\mathbb{V}$. Then by Theorem 1.2 we have

<u>Proposition 3.1</u>. There is a natural isomorphism

$$(3.1) \qquad Sol : Quot_D(m;D)^{R\text{-}solv} \xrightarrow{\sim} GM(m;\mathbb{V}).$$

To make things clear, we confine our discussion to the "generic part" of the isomorphism (3.1). Fix a total order $\preceq$ of L, additive in the sense of § 2. Let $S = \{\mu_0, \ldots, \mu_{m-1}\}$, $\mu_0 \prec \mu_1 \prec \cdots \prec \mu_{m-1}$, be the set of m smallest elements of L under the total order $\preceq$, which we consider as a generic order ideal. Then we define the <u>generic cell</u> $GM(m;\mathbb{V})^{\emptyset}$ of $GM(m;\mathbb{V})$ to be the set of all $\mathbb{C}$-subspaces V of $\mathbb{V}$ having a $\mathbb{C}$-basis $\underline{f} = (f_0, \ldots, f_{m-1})$ such that $Wr_{\mu_0, \ldots, \mu_{m-1}}(f_0, \ldots, f_{m-1})(0) \neq 0$. (This condition is equivalent to saying that the corresponding Plücker coordinate $\xi_{\mu_0, \ldots, \mu_{m-1}}$ does not vanish.) Then one can show that the inverse image of the generic cell $GM(m;V)^{\emptyset}$ by the isomorphism (3.1), say $Quot_D(m;D)^{R\text{-}solv,\emptyset}$, can be identified with the totality of families $(W_\mu)_{\mu \in A}$, $A = A(S)$, of differential operators

$$(3.2) \qquad W_\mu = \partial_x^{\mu} + \Sigma_{i=0}^{m-1} w_{\mu,i}\, \partial_x^{\mu_i}, \text{ where } w_{\mu,i} \in R = \mathbb{C}[[x]],$$

satisfying the following compatibility condition : For any $\mu, \nu \in A$,

$$(3.3)\quad \exists\, Q^{(\mu,\nu)}_{\kappa} \in D\ (\kappa \in A),\quad \partial^{\mu'} W_{\nu} - \partial^{\nu'} W_{\mu} = \Sigma_{\kappa \in A}\, Q^{(\mu,\nu)}_{\kappa}\, W_{\kappa}\,.$$

(This compatibility condition is equivalent to the Frobenius condition of integrability when the system $W_{\mu} u = 0$ $(\mu \in A)$ is rewritten into a system of order 1 for the vector $(\partial_x^{\mu_i} u)_{0 \leqq i < m}$ of unknown functions.)

We denote by $\Lambda = (\Lambda_0, \ldots, \Lambda_{r-1})$ the family of $\mathbb{C}$-endomorphisms Λ_k of V corresponding to the derivation ∂_{x_k}. Then, the powers Λ^{α} ($\alpha \in L^*$, $L^* = L \setminus \{0\}$) induce commuting global vector fields on the Grassmann manifold $GM(m;V)$. By exponentiating these, we obtain a family of commuting dynamical flows

$$(3.4)\qquad e^{\eta(t,\Lambda)},\ \text{where}\ \ \eta(t,\Lambda) = \Sigma_{\alpha \in L^*}\ t_{\alpha} \Lambda^{\alpha}\,,$$

parametrized by a countable set $t = (t_{\alpha})_{\alpha \in L^*}$ of time variables.

Via the isomorphism (3.1), the above dynamical system on $GM(m;V)$ induces another on $\mathrm{Quot}_D(m;D)^{R\text{-solv}}$. On the generic part $\mathrm{Quot}_D(m;D)^{R\text{-solv},\phi}$, this gives rise to a time evolution of a family $(W_{\mu})_{\mu \in A}$ of differential operators (3.2) satisfying (3.3). The equation for this time evolution, called the <u>Sato equation</u>, then can be written down as follows : For each $\alpha \in L^*$ and $\mu \in A$, there exist some differential operators $B_{\alpha;\mu,\nu}$ $(\nu \in A)$ satisfying

$$(3.5)\qquad \frac{\partial W_{\mu}}{\partial t_{\alpha}} + W_{\mu} \partial_x^{\alpha} = \Sigma_{\nu \in A}\ B_{\alpha;\mu,\nu}\, W_{\nu}\,.$$

This condition can be read as a system of non-linear equations, just as before, by using the properties of the Gröbner representation. Considering this equation (3.5) together with (3.3) as a system of non-linear equations for a family $(W_{\mu})_{\mu \in A}$ of differential operators with coefficients in the ring $R[[t]]$ of formal power series, we have

<u>Theorem 3.2</u>. The totality of the formal solutions $(W_\mu)_{\mu\in A}$ to the Sato equation (3.5), together with the compatibility comdition (3.3), is parametrized by the generic cell $GM(m;\mathbb{V})^{\phi}$ of the infinite dimensional Grassmann manifold $GM(m;\mathbb{V})$.

Note that this parametrization can be explicitly given as in (2.8) by using Wronskian determinants.

§ 4. Some results on the τ functions

In the final section, we report some results on the τ functions of the higher dimensional m-truncated KP hierarchy, obtained by the collaboration with Prof. K. Ueno and Mr. H. Harada. In contrast to the one dimensional case, a single τ function is not sufficient to describe the time evolution of the Plücker coordinates. We introduce here a "family of τ functions" which satisfies a system of bilinear equations of Hirota.

Here we fix a $\mathbb{C}$-basis $(e_\alpha)_{\alpha\in L}$ of $\mathbb{V}$ as follows :

(4.1) $$\mathbb{V} = \Pi_{\alpha\in L}\,\mathbb{C}e_\alpha \ , \text{ where } e_\alpha = \frac{x^\alpha}{\alpha!} \text{ for each } \alpha\in L.$$

Let V be a m-dimensional $\mathbb{C}$-subspace of $\mathbb{V} = R$, and $\underline{f} = (f_0,\ldots,f_{m-1})$ a $\mathbb{C}$-basis of V. To the $\mathbb{C}$-basis $\underline{f}$, we associate a frame $\Xi = (f_{\alpha,j})_{\alpha\in L, 0\leqq j<m} \in \mathrm{Mat}(L,m;\mathbb{C})$, expanding f_j in the form

(4.2) $$f_j(x) = \Sigma_{\alpha\in L}\frac{x^\alpha}{\alpha!}f_{\alpha,j} \quad \text{for each } 0\leqq j<m.$$

Then the Plücker coordinates $(\xi_{\underline{\alpha}})_{\underline{\alpha}\in L^m}$ of Ξ is defined by

(4.3) $\xi_{\underline{\alpha}} = \xi_{\underline{\alpha}}(\Xi) = \det(f_{\alpha_i, j}\ ;\ 0 \leqq i,j < m)$ for each $\underline{\alpha} \in L^m$,

which equals the value of the Wronskian determinant $Wr_{\underline{\alpha}}(\underline{f})(0)$. ($\xi_{\underline{\alpha}}$ are anti-symmetric under the permutation of indices.) Recall that, for each point V of GM(m;V), the Plücker coordinates $(\xi_{\underline{\alpha}})_{\underline{\alpha} \in L^m}$ are determined up to a constant multiple, and satisfy the following Plücker relations : For any $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{m-2}) \in L^{m-1}$ and $\underline{\beta} = (\beta_0, \ldots, \beta_m) \in L^{m+1}$, one has

(4.4) $$\Sigma_{i=0}^{m} (-)^i \xi_{\alpha_0, \ldots, \alpha_{m-2}, \beta_i} \cdot \xi_{\beta_0, \ldots, \hat{\beta}_i, \ldots, \beta_m} = 0.$$

Now, consider the time evolution (3.4) of our m-truncated KP hierarchy. First note that the time evolution of a frame Ξ can be given by $\Xi(t) = e^{\eta(t, \Lambda)} \Xi$. Taking the m-minors of the $L \times L$ matrix $e^{\eta(t, \Lambda)}$, we define the relative Schur polynomials $\chi_{\underline{\alpha};\underline{\beta}}(t)$ as follows : For each couple $(\underline{\alpha}, \underline{\beta})$ of elements of L^m,

(4.5) $$\chi_{\underline{\alpha};\underline{\beta}}(t) = \det(p_{\beta_j - \alpha_i}(t)\ ;\ 0 \leqq i,j < m),$$

where $p_\alpha(t)$ $(\alpha \in L)$ are the polynomials determined by the generating function

(4.6) $$\Sigma_{\alpha \in L}\, p_\alpha(t)\, u^\alpha = e^{\eta(t,u)}, \quad u = (u(0), \ldots, u(r-1)).$$

In terms of these relative Schur plynomials, we can express the time evolution $(\xi_{\underline{\alpha}}(t))_{\underline{\alpha} \in L^m}$ of the Plücker coordinates $(\xi_{\underline{\alpha}})_{\underline{\alpha} \in L^m}$. Noting that $\xi_{\underline{\alpha}}(t) = \xi_{\underline{\alpha}}(\Xi(t))$, we have

(4.7) $$\xi_{\underline{\alpha}}(t) = \Sigma'_{\underline{\beta} \in L^m}\, \chi_{\underline{\alpha};\underline{\beta}}(t)\, \xi_{\underline{\beta}},$$

where Σ' represents the sum restricted so that, for any subset S of L, only one m-tuple $\underline{\beta} = (\beta_0, \ldots, \beta_{m-1})$ with $S = \{\beta_0, \ldots,$

$\beta_{m-1}\}$ should appear in the summation.

Proposition 4.1. With the notations in § 3, the solutions $(W_\mu)_{\mu\in A}$ to the Sato equation (3.5) can be represented by the Plücker coordinates $(\xi_{\underline{\alpha}}(t))_{\underline{\alpha}\in L^m}$ with $\xi_{\mu_0,\dots,\mu_{m-1}} \neq 0$. The coefficients $w_{\mu,i}(0 \leqq i < m)$ of W_μ are given by

$$(4,8) \qquad w_{\mu,i}(x,t) = (-)^{m-i} \frac{\xi_{\mu_0,\dots,\hat{\mu}_i,\dots,\mu_{m-1},\mu}(x+t)}{\xi_{\mu_0,\dots,\mu_{m-1}}(x+t)},$$

where $(x+t)_\alpha = x_k + t_{\varepsilon_k}$ if α is the unit vector $\varepsilon_k = (\delta_{j,k})_j$ for $0 \leqq k < r$ and $(x+t)_\alpha = t_\alpha$ otherwise.

Note that, in the case where $r=1$, any Plücker coordinate $\xi_{\underline{\alpha}}(t)$ is expressed by a linear combination of the derivatives of the τ function $\tau(t) = \xi_{0,1,\dots,m-1}(t)$. In the case where $r \geqq 2$, however, we need to take a subset $\mathcal{S}$ of L^m and to consider a family of Plücker coordinates $(\xi_{\underline{a}}(t))_{\underline{a}\in\mathcal{S}}$ in order to obtain an object with similar properties. Here we take the subset $\mathcal{S}$ of L defined by

$$(4.9) \qquad \mathcal{S} = \{\, \underline{a} = (a_0,\dots,a_{m-1}) \in L^m \,;\; |a_i| < m \text{ for all } i \,\}.$$

(One might guess that the set of all order ideals of degree m would suffice as in § 1. But it is known that this set is too small to our purpose. Theoretically, one can minimize the set indeed, but it is not clear if it carries nice properties.) With the set $\mathcal{S}$ of (4.9), we define, tentatively, the family $(\tau_{\underline{a}}(t))_{\underline{a}\in\mathcal{S}}$ of τ functions by setting $\tau_{\underline{a}}(t) = \xi_{\underline{a}}(t)$ for each $\underline{a} \in \mathcal{S}$.

Before going further, we need to refer to an analogue of the

Cauchy Lemma. Let $\underline{u} = (u_0,\dots,u_{m-1})$ be an m-tuple of r-vectors $u_i = (u_i(0),\dots,u_i(r-1))$ of indeterminates. Then, for each m-tuple $\underline{\alpha} = (\alpha_0,\dots,\alpha_{m-1})$ of multi-indices, we define

$$(4.10) \qquad \Delta_{\underline{\alpha}}(\underline{u}) = \det(u_j^{\alpha_i} \; ; \; 0 \leqq i,j < m).$$

Let $\underline{v} = (v_0,\dots,v_{m-1})$ be another m-tuple of r-vectors $v_i = (v_i(0),\dots,v_i(r-1))$ of indeterminates. Then we have

$$(4.11) \qquad \det\left(\frac{1}{1 - \langle u_i, v_j \rangle} \right) \Pi_{i,j}(1 - \langle u_i, v_j \rangle)$$
$$= \Sigma'_{\underline{a},\underline{b} \in \mathcal{S}} \; C_{\underline{a};\underline{b}} \; \Delta_{\underline{a}}(\underline{u}) \; \Delta_{\underline{b}}(\underline{v}),$$

where $\langle u_i, v_j \rangle = \Sigma_{k=0}^{m-1} u_i(k) v_j(k)$. In the equality (4.11), the coefficients $C_{\underline{a};\underline{b}}$ are integers satisfying $C_{\underline{b};\underline{a}} = C_{\underline{a};\underline{b}}$. Using these coefficients, we set

$$(4.12) \qquad \chi^*_{\underline{a};\underline{\alpha}}(t) = \Sigma'_{\underline{b} \in \mathcal{S}} \; C_{\underline{a};\underline{b}} \; \chi_{\underline{b};\underline{\alpha}}(t)$$

for each $\underline{a} \in \mathcal{S}$ and $\underline{\alpha} \in L^m$.

<u>Proposition 4.2</u>. For each $\underline{\alpha} \in L^m$, one has

$$(4.13) \qquad \frac{|\alpha_0|! \dots |\alpha_{m-1}|!}{\alpha_0! \cdots \alpha_{m-1}!} \; \xi_{\underline{\alpha}}(t) = \Sigma'_{\underline{a} \in \mathcal{S}} \; \chi^*_{\underline{a};\underline{\alpha}}(\tilde{\partial}_t) \; \tau_{\underline{a}}(t),$$

where $\tilde{\partial}_t = \left(\frac{(|\alpha| - 1)!}{\alpha!} \partial_t \right)_{\alpha \in L^*}$.

Recall that the Plücker coordinates $(\xi_{\underline{\alpha}}(t))_{\underline{\alpha} \in L^m}$ satisfy the Plücker relations (4.4). Hence, by Proposition 4.2, we see that our family of τ functions satisfies a system of bilinear differential equations. These bilinear equations can be rewritten into the Hirota form.

Given a differential operator $P(\partial_t)$ and a couple $(f(t),g(t))$ of functions in t, with Prof. Hirota, we write

(4.14) $$P(D_t)\, f(t)\cdot g(t) = P(\partial_s)\, f(t+s)\, g(t-s)\Big|_{s=0}.$$

With this notation, we have

<u>Theorem 4.3</u>. The family $(\tau_{\underline{a}}(t))_{\underline{a}\in\mathcal{S}}$ of τ functions of the higher dimensional KP hierarchy satisfy the following system of Hirota's bilinear equations : For any $\underline{\alpha} = (\alpha_0,\dots,\alpha_{m-2}) \in L^{m-1}$ and $\underline{\beta} = (\beta_0,\dots,\beta_m) \in L^{m+1}$,

(4.15) $$\Sigma'_{\underline{a},\underline{b}\in\mathcal{S}} \{ \Sigma_{i=0}^{m} (-)^i \chi^*_{\underline{a};\alpha_0,\dots,\alpha_{m-2},\beta_i}(\tfrac{1}{2}\tilde{D}_t)\cdot \chi^*_{\underline{b};\beta_0,\dots,\hat{\beta}_i,\dots,\beta_m}(-\tfrac{1}{2}\tilde{D}_t) \}\; \tau_{\underline{a}}(t)\cdot\tau_{\underline{b}}(t) = 0 .$$

<u>References</u>

[1] M. Sato : Soliton equations as dynamical systems on an infinite dimensional Grassmann manifold, RIMS Kokyuroku 439, RIMS Kyoto University, 1981, 30-46.

[2] M. Sato and Y. Sato : Soliton equations as dynamical systems on an infinite dimensional Grassmann manifold, Proc. U.S.-Japan Seminar "Nonlinear Partial Differential Equations in Applied Sciences", Tokyo 1982, edited by H. Fujita, P. D. Lax, and S. Strang, Kinokuniya / North-Holland, 1982, 259-271.

[3] M. Sato : Soliton equations and the universal Grassmann manifold, Sophia Kokyuroku in Math. 18, Sophia University, 1984. (Notes by M. Noumi, in Japanese.)

[4] M. Sato : Lectures at Kyoto University, 1984-.

[5] K. Takasaki : Differential equations and Grassmann manifolds —— from Prof. Sato's lectures ——, RIMS Kokyuroku 597, RIMS Kyoto University, 1986, 109-126.

Higher dimensional analogue of the m-truncated KP hierarchy

上智大・理工　野海正俊
(Masatoshi NOUMI)

[要旨]

KP方程式系の佐藤理論は，"1次元的"な非線型可積分系の普遍的な枠組を与えていると言えよう。この普遍的な理論を自然なやり方で"高次元的"な対象にまで拡張することは重要な課題である。既に高崎氏の論説[5]で紹介されているように，1984年に始まる京大に於ける講義の中で，佐藤先生は，非線型可積分系を，線型微分方程式とその解空間との対応関係から生じる，ある種のGrassmann多様体の同型対応によって把握できることを指摘され，それによって高次元理論を構築するスキームを提出された．この佐藤先生のアイディアに従って，京都では高崎金久，鈴木範男，大山陽介各氏等，東京では上野喜三雄，原田等両氏及び筆者等が，高次元理論へ向けていくつかの試みを行なってきた．この論稿では，その中から特にm切断KP系（原田氏の命名）の高次元化について紹介する．

§1では，高次元でのWronski行列式についての基本的な性質を述べ，それから"象徴的"な意味でのRiemann-Hilbert対応が成立することに言及する．そこからGrassmann多様体の同型対応が生じる．これはある種の非線型方程式系の解空

間がGrassmann多様体によってパラメトライズされることを意味するが，この種の"Grassmann形式"を具体的な形で呈示するための1つの方法論として，§2では巡回Ω加群の標準形 ― Gröbner表示と呼ぶ ― の議論を紹介する．§3では，r変数の$\mathbb{C}$係数形式巾級数環の中のm次元$\mathbb{C}$部分空間全体のつくるGrassmann多様体$GM(m;V)$を考え，そこで"KP的"な時間発展を考察する．これがm切断KP系の高次元化と了解する．この時間発展は，Grassmann多様体の同型対応により（generic cellでは）ある種の微分作用素の族$(W_\mu)_{\mu\in A}$に対する佐藤方程式を生じる．以上§1～§3の内容は，定式化の問題を別にして，京都と東京のグループの共通の了解事項とも言うべきものであった．最後の§4では，この高次元m切断系のτ函数について，上野，原田両氏と筆者が共同で行った研究から紹介する．そこでは，あるクラスのPlücker座標から"τ函数族"を定め，それについて広田型の双線型方程式系が生じることを述べる．（このm切断系はある意味でholonomicな対象であるが，適切な極限移行が実現できるかどうかは定かでない．）

この論稿の本質的な部分は佐藤先生に負うものである．内容的にも，佐藤先生の講義と重複している部分も多いと思われるが，その点は御容赦願いたい．