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Laplace transforms of hyperfunctions

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1. Heaviside's operational calculus.

The Laplace transform

(1) $$\hat{f}(\lambda) = \int_0^\infty e^{-x\lambda} f(x)\, dx$$

is usually defined for a continuous (or measurable) function $f(x)$ on $[0, \infty)$ satisfying the estimate

(2) $$|f(x)| \leqq C e^{Hx}, \quad 0 \leqq x < \infty,$$

with constants H and C. Then $\hat{f}(\lambda)$ is a holomorphic function on the half plane $\{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda > H\}$ having the estimate

(3) $$|\hat{f}(\lambda)| \leqq C/(\mathrm{Re}\,\lambda - H), \quad \mathrm{Re}\,\lambda > H,$$

and the original function $f(x)$ is restored by the Bromwich integral

(4) $$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{x\lambda} \hat{f}(\lambda)\, d\lambda, \quad 0 < x < \infty,$$

almost everywhere, where c is an abscissa greater than H.

The Laplace transforms were employed to justify Heaviside's operational calculus (see G. Doetsch [3]). Let

(5) $$P(d/dx) = a_m(d/dx)^m + \cdots + a_0$$

be a linear differential operator with constant coefficients $a_i \in \mathbb{C}$ and let $f(x)$ be an m times continuously differentiable function on $[0, \infty)$ such that all derivatives $f^{(i)}(x)$, $0 \leqq i \leqq m$, satisfy estimates (2). Then we have by integration by parts

$$(6)\qquad \begin{aligned}(P(d/dx)u)^{\wedge}(\lambda) &= P(\lambda)\hat{f}(\lambda) - (a_m f^{(m-1)}(0) + \cdots + a_1 f(0)) \\ &\quad - \cdots - (a_m f'(0) + a_{m-1} f(0))\lambda^{m-2} - a_m f(0)\lambda^{m-1}.\end{aligned}$$

Hence the initial value problem

$$(7)\qquad \begin{cases} P(d/dx)\, u(x) = f(x) \\ u^{(j)}(0) = g_j, \quad j = 0, \cdots, m-1, \end{cases}$$

may be solved by applying the inversion formula to

$$(8)\qquad \begin{aligned}\hat{u}(\lambda) = P(\lambda)^{-1}\{\hat{f}(\lambda) &+ (a_m\, g_{m-1} + \cdots + a_1\, g_0) \\ &+ \cdots + a_m\, g_0\, \lambda^{m-1}\}.\end{aligned}$$

However, this solution has been believed to have the following disadvantages:

1. The datum $f(x)$ must satisfy the estimates (2) of exponential type.

2. There is no good characterization of the Laplace images of functions satisfying (2), so that we do not know a priori whether or not $\hat{u}(\lambda)$ of (8) is the Laplace transform of a solution.

3. The inversion formula (4) does not necessarily converge absolutely.

For example, the simplest equation

$$(9)\qquad \begin{cases} (d/dx - \alpha)\, u(x) = 0 \\ u(0) = b \end{cases}$$

has the solution

$$(10)\qquad u(x) = \frac{b}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{x\lambda} (\lambda - \alpha)^{-1}\, d\lambda, \quad 0 < x < \infty,$$

but the integral converges only in the sense of Cauchy's principal value.

To avoid these disadvantages J. Mikusiński [7] invented an algebraic foundation of Heaviside's calculus based on Titchmarsh's theorem on convolutions of continuous functions (see also K. Yosida [15]). However, Heaviside's calculus seems to lose its computational character by his approach.

We will develop the theory of Laplace transforms of hyper-functions with defining function of exponential type and overcome these disadvantages in a natural way.

2. Laplace transforms of holomorphic functions of exponential type on a closed sector.

This section is a review of the classical results by E. Borel [2], G. Pólya [9] and A. J. Macintyre [6] (see R. P. Boas [1]). Let

$$(11)\qquad \Sigma = \{z \in \mathbb{C};\ |\arg z| \leq \alpha\}$$

be a closed sector. A holomorphic function $f(z)$ on a neighbor-hood of Σ is said to be of <u>exponential type</u> if there are constants H and C such that

$$|f(z)| \leqq C\, e^{H|z|}, \quad z \in \Sigma.$$

The indicator function $h(\vartheta)$ for $|\vartheta| \leqq \alpha$ is defined by

$$(12) \qquad h(\vartheta) = \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\vartheta})|}{r}.$$

The Laplace transform

$$(13) \qquad \hat{f}(\lambda) = \int_0^\infty e^{-z\lambda} f(z)\, dz$$

can be continued to a holomorphic function on

$$(14) \qquad \rho(f) = \bigcup_{|\vartheta|\leqq\alpha} \{\lambda \in \mathbb{C};\ \mathrm{Re}(\lambda\, e^{i\vartheta}) > h(\vartheta)\}$$

by rotating the path of integration. Its complement

$$(15) \qquad \sigma(f) = \mathbb{C} \setminus \rho(f)$$

is called the conjugate diagram by Pólya. We call it the convex spectrum of f. We have also estimates of $|\hat{f}(\lambda)|$ in $\rho(f)$.

The following proposition is easily proved by the change of order of integrations and Cauchy's integral formula.

Proposition 1. Let $\Gamma : R \longrightarrow \rho(f)$ be a path of integration such that $\arg \Gamma(t) \longrightarrow \pm(\pi/2 + \beta)$ as $t \longrightarrow \pm\infty$ with $0 < \beta < \alpha$. Then we have

$$(16) \qquad f(z) = \frac{1}{2\pi i} \int_\Gamma e^{z\lambda}\, \hat{f}(\lambda)\, d\lambda, \quad |\arg z| < \beta.$$

Since $\hat{f}(\lambda)$ is bounded on Γ, it follows that

$$(17) \qquad h(\vartheta) \leqq \sup \mathrm{Re}(\Gamma\, e^{i\vartheta}).$$

Hence we obtain the following.

Theorem 1 (Pólya). Extreme points of $\sigma(f)$ are singular points of $\hat{f}(\lambda)$. In particular, $f(z)$ and $f(z + a)$ have the same convex spectrum.

In case where $\alpha > \pi/2$, $\rho(f)$ should be understood to be a Riemann domain since the analytic continuations from above and from below may be different.

Theorem 2. Let $\alpha > \pi/2$. Then $\hat{f}(\lambda)$ coincides on the overlaping domain if and only if $f(z)$ can be continued to an entire function of exponential type.

Proof. The "if" part is due to Borel and Pólya. It is proved by the termwise integration of (13) as $f(z)$ is developed into the Taylor series.

The "only if" part holds because the path Γ of integration in (16) can be deformed into a closed curve.

3. Laplace transforms of Laplace hyperfunctions.

Suppose that a hyperfunction $f(x)$ with support in $[a, \infty)$ has a defining function $F(z) \in \mathcal{O}^{\exp}(\mathbb{C} \setminus [a, \infty))$ of exponential type, i. e. on each closed sector $\Sigma = \{z \in \mathbb{C};\ \alpha \leqq \arg(z - z_0) \leqq \beta\} \subset \mathbb{C} \setminus [a, \infty)$ there are constants H and C such that

$$|F(z)| \leqq C\, e^{H|z|}, \quad z \in \Sigma.$$

Then the Laplace transform $\hat{f}(\lambda)$ of $f(x)$ may be defined by the integral

$$\text{(18)} \qquad \hat{f}(\lambda) = \int_\Gamma e^{-\lambda z} F(z)\, dz = \hat{F}_c^-(\lambda) - \hat{F}_c^+(\lambda),$$

where Γ is a path as in Figure 1 and $\hat{F}_c^{\pm}$ are branches of the Laplace transform of $F(z)$ with origin at c.

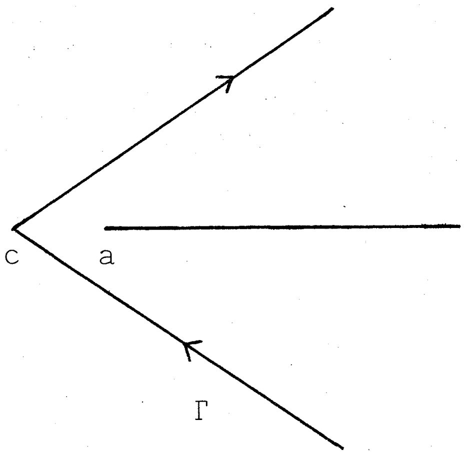

Figure 1

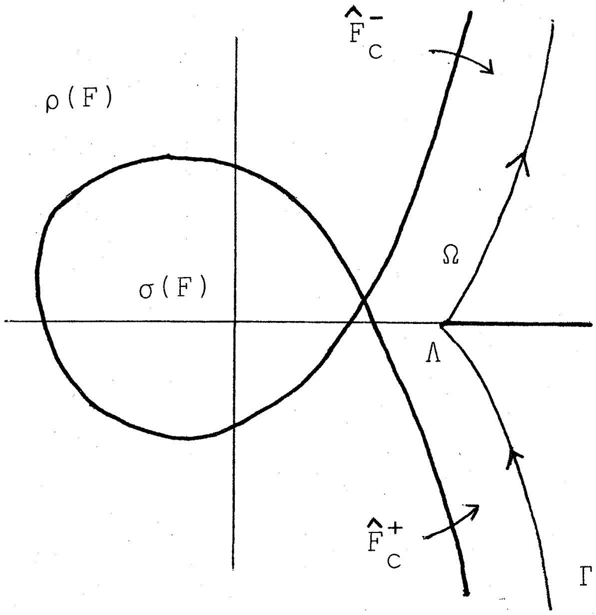

Figure 2

Theorem 2 asserts that the right hand side of (18) vanishes if and only if $F(z)$ is an entire function of exponential type. Therefore, to make the Laplace transform well defined, we should consider f to be an element of

(19) $$B^{\exp}_{[a,\infty]} = \mathcal{O}^{\exp}(\mathbb{C} \setminus [a, \infty))/\mathcal{O}^{\exp}(\mathbb{C})$$

Defintion 1. We call the elements of the above space <u>Laplace hyperfunctions</u> with support in $[a, \infty]$, and the integral (18) the <u>Laplace transform</u> of the Laplace hyperfunction f.

Theorem 3. The Laplace transform $\hat{f}(\lambda)$ of a Laplace hyperfunction $f(x)$ with support in $[a, \infty]$ is a holomorphic function of exponential type on a domain Ω such that all rays $\mathbb{R}^+e^{i\vartheta}$ with $|\vartheta| < \pi/2$ are eventually contained in Ω and its indicator function satisfies

$$h(\vartheta) = \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\vartheta})|}{r}$$

(20)

$$\leqq - a \cos \vartheta, \quad |\vartheta| < \pi/2.$$

Conversely such a holomorphic function $\hat{f}(\lambda)$ is the Laplace transform of a unique Laplace hyperfunction in $B^{exp}_{[a,\infty]}$ with a defining function

(21) $$F(z) = \frac{1}{2\pi i} \int_{\Lambda}^{\infty} e^{z\lambda} \hat{f}(\lambda)\, d\lambda.$$

Thus we have symbolically

(22) $$f(x) = \frac{1}{2\pi i} \int_{\Gamma} e^{z\lambda} \hat{f}(\lambda)\, d\lambda,$$

where Γ is a path as in Figure 2.

Proof. Suppose that $f(x)$ is a Laplace hyperfunction in $B^{exp}_{[a,\infty]}$ with the defining function $G(z)$. In the representation (16) of $G(z)$ in Proposition 1 we can deform the path Γ into the sum of a ray from ∞ to Λ, a closed curve from Λ to Λ and the ray from Λ to ∞. In view of (18) the sum of the integrals over the rays is the right hand side of (21). The interal over the closed curve is clearly an entire function of exponential type. Hence $F(z)$ is a defining fuction of $f(x)$.

The converse is a direct consequence of Proposition 1.

4. The sheaf theoretical formulation.

In the previous section we gave a global defintion of Laplace hyperfunctions. However, except for the difference of the growth order of defining functions and for the support condition Laplace hyperfunctions are not different from Fourier hyperfunctions of M. Sato [12] and T. Kawai [4] and from modified Fourier hyperfunctions of Y. Saburi [10]. Therefore the following sheaf theoretical formulation is more desirable.

Definition 2. Let

$$\mathbb{D} = \mathbb{C} \cup S^1\infty \tag{23}$$

be the radial compactification of the complex plane as in Saburi [10]. We define the sheaf $\mathcal{O}^{exp}$ of holomorphic functions of exponential type as the sheaf on $\mathbb{D}$ whose section space $\mathcal{O}^{exp}(V)$ on an open set V in $\mathbb{D}$ is the space of all holomorphic functions $F(z)$ on $V \cap \mathbb{C}$ satisfying the condition that for each point $e^{i\vartheta}\infty$ at infinity in V there is a cone neighborhood $\Sigma = \{z \in \mathbb{D};\ |\arg(z - c) - \vartheta| \leqq \alpha\}$ and constants H and C such that

$$|F(z)| \leqq C\, e^{H|z|}, \quad z \in \Sigma \cap \mathbb{C}. \tag{24}$$

If Theorem B for the sheaf $\mathcal{O}^{exp}$

$$H^1(V, \mathcal{O}^{exp}) = 0 \tag{25}$$

is true for all (or sufficiently many) open sets V in $\mathbb{D}$, then the sheaf $\mathcal{B}^{exp}$ of Laplace hyperfunctions can be defined by

$$\begin{aligned} \mathcal{B}^{exp}(\Omega) &= \mathcal{O}^{exp}(V \setminus \Omega)/\mathcal{O}^{exp}(V) \\ &= H^1_{\Omega}(V, \mathcal{O}^{exp}) \end{aligned} \tag{26}$$

for any open set Ω in the extended real line $[-\infty, \infty]$ and open neighborhood V of Ω in $\mathbb{D}$ as in the theory of hyperfunctions [12]. Then it follows that B^{exp} is a flabby sheaf on $[-\infty, \infty]$ whose restriction to $\mathbb{R}$ is the sheaf B of hyperfunctions. In particular, we have the following.

Theorem 4. The restriction mapping

$$(27) \qquad \rho: B^{exp}_{[a,\infty]} \longrightarrow B_{[a,\infty)}$$

is surjective, where $B_{[a,\infty)}$ is the space of all hyperfunctions on $\mathbb{R}$ with support in $[a, \infty)$, and we have

$$(28) \qquad \operatorname{supp} \rho(f) = \operatorname{supp} f \cap \mathbb{R}.$$

We note that the mapping ρ is induced from the imbeddings $O^{exp}(\mathbb{D} \setminus [a, \infty]) \to O(\mathbb{C} \setminus [a, \infty))$ and $O^{exp}(\mathbb{D}) \to O(\mathbb{C})$ through the representations

$$(29) \qquad B^{exp}_{[a,\infty]} = O^{exp}(\mathbb{D} \setminus [a, \infty])/O^{exp}(\mathbb{D}),$$

$$(30) \qquad B_{[a,\infty)} = O(\mathbb{C} \setminus [a, \infty))/O(\mathbb{C}).$$

If $f(x)$ is a Laplace hyperfunction in $B^{exp}_{[a,\infty]}$ with the defining function $F(z)$ in $O^{exp}(\mathbb{D} \setminus [a, \infty])$, then the support $\operatorname{supp} f$ of f is the complement in $[-\infty, \infty]$ of the set of all points c near which $F(z)$ can be continued to a function in O^{exp}.

Unfortunately we have not been able to prove (25). However, Theorem 4 is proved by the corresponding result

$$(31) \qquad H^1(V, O^{infexp}) = 0$$

of Saburi [10] for the sheaf O^{infexp} of holomorphic functions of infraexponential type, i. e. holomorphic functions $F(z)$

satisfying estimates (24) for any $H > 0$ with a constant C, and for a restricted class of open sets V in $\mathbb{O}$. For the details see [5].

The following is a schematization of related theories.

Base space	$\mathbb{D}^n + i\,\mathbb{R}^n$	$\mathbb{O}^n$
Growth order		
Infraexponential type	Fourier hyperfunctions	Modified Fourier hyperfunctions
	Sato [12], Kawai [4]	Nagamachi-Mugibayashi, Saburi [10]
Exponential type	Analytic functionals with real non-compact carrier	Laplace hyperfunctions
	Morimoto [8], Sargos-Morimoto [11], Oshima-Saburi-Wakayama	

Here $\mathbb{D}^n$ denotes the radial compactification of $\mathbb{R}^n$, and $\mathbb{O}^n$ the radial compactfication of $\mathbb{C}^n$.

In all other theories the global sections on $\mathbb{D}^n$ are identified with the dual of a locally convex space of analytic functions. In our case it is the space of all sections on $\mathbb{D}^n$ of the sheaf O_{exp} on $\mathbb{O}^n$ whose section space $O_{exp}(V)$ on an open set V in $\mathbb{O}^n$ is the space of all holomorphic functions $\varphi(z)$ on $V \cap \mathbb{C}^n$ such that on each closed cone Σ in V

$$|\varphi(z)| \leqq C\, e^{-H|z|}, \quad z \in \Sigma \cap \mathbb{C}^n, \tag{32}$$

for any H with a constant C.

5. Solution of ordinary differential equations.

We consider the initial value problem (7) when $f(x)$ is an arbitrary continuous function on $[0, \infty)$. By Cauchy's existence theorem there is an m times continuously differentiable solution $u(x)$. Let

$$\vartheta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \tag{33}$$

be the Heaviside function. Then the Green formula

$$\begin{aligned} (d/dx)^i(\vartheta(x)\ u(x)) &= \vartheta(x)\ u^{(i)}(x) \\ &+ u^{(i-1)}(0)\ \delta(x) + \cdots + u(0)\ \delta^{(i-1)}(x) \end{aligned} \tag{34}$$

implies that $v(x) = \vartheta(x)\ u(x)$ is a solution of

$$P(d/dx)\ v(x) = g(x) \tag{35}$$

in the sense of distribution, where

$$\begin{aligned} g(x) &= \vartheta(x)\ f(x) + (a_m\ g_{m-1} + \cdots + a_1\ g_0)\ \delta(x) \\ &+ \cdots + a_m\ g_0\ \delta^{(m-1)}(x). \end{aligned} \tag{36}$$

If, more generally, $f(x)$ is a distribution (resp. a hyperfunction) with support in $[0, \infty)$, we interprete the initial value problem (7) as the problem of finding a distribution (resp. hyperfunction) solution $v(x)$ of (35) with support in $[0, \infty)$.

Theorem 5. For any $g(x) \in B_{[0,\infty)}$ there is a unique solution $v(x) \in B_{[0,\infty)}$ of (35). Let $\tilde{g}(x) \in B^{\exp}_{[0,\infty]}$ be an arbitrary Laplace hyperfunction which extends $g(x)$ in the sense that $\rho(\tilde{g}) = g$, and let $\hat{g}(\lambda)$ be the Laplace transform of $\tilde{g}(x)$.

Then the solution $v(x)$ is obtained as the restiction to $\mathbb{R}$ of the inverse Laplace transform in $B^{exp}_{[0,\infty]}$ of

$$\hat{v}(\lambda) = P(\lambda)^{-1}\,\hat{g}(\lambda). \tag{37}$$

Proof. We define the action of the differential operator $P(d/dx)$ on the Laplace hyperfunction $f(x) = F(x + i0) - F(x - i0)$ by

$$P(d/dx)f(x) = P(d/dz)F(x + i0) - P(d/dz)F(x - i0). \tag{38}$$

Clearly this defines $P(d/dx)f(x)$ independent of the defining function $F(z)$ and we have

$$\rho(P(d/dx)f) = P(d/dx)\,\rho(f). \tag{39}$$

Differentiating (21) under the integral sign, we also have

$$(P(d/dx)f)^{\wedge}(\lambda) = P(\lambda)\,\hat{f}(\lambda). \tag{40}$$

Hence it follows that the equation

$$P(d/dx)\,\tilde{v}(x) = \tilde{f}(x) \tag{41}$$

in $B^{exp}_{[0,\infty]}$ has a unique solution $\tilde{v}(x)$ whose Laplace transform is given by (37). Its restriction $v(x)$ to $\mathbb{R}$ is a solution of (35).

Let $v_1(x) \in B_{[0,\infty)}$ be another solution of (35) and let $\tilde{v}_1(x)$ be an arbitrary extension in $B^{exp}_{[0,\infty]}$. Then $P(d/dx)\{\tilde{v}(x) - \tilde{v}_1(x)\}$ has support only at ∞. Hence $\tilde{v}(x) - \tilde{v}_1(x)$ has also support only at ∞ as the representation of the solution of equation (41) shows.

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佐藤超函数のラプラス変換

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ラプラス変換を用いたヘヴィサイド演算子法の正当化は、データである函数が指数型の増大度をもつものに限られるなど適用範囲に限界があると信じられているが，ラプラス変換の定義を佐藤超函数に拡張することにより，この困難が克服できることを示す。

$\mathcal{O}^{exp}(V)$ でもって $V\subset\mathbb{C}$ 上指数型整型函数全体を表わす．$[a,\infty]$ に台のあるラプラス超函数の空間を

$$\mathcal{B}^{exp}_{[a,\infty]} := \mathcal{O}^{exp}(\mathbb{C}\setminus[a,\infty))/\mathcal{O}^{exp}(\mathbb{C})$$

で定義する．埋込み $\mathcal{O}^{exp}(\mathbb{C}\setminus[a,\infty))\hookrightarrow\mathcal{O}(\mathbb{C}\setminus[a,\infty))$ は全射 $\mathcal{B}^{exp}_{[a,\infty]}\to\mathcal{B}_{[a,\infty)}$ をひきおこす．但し右辺は $[a,\infty)$ に台をもつ通常の佐藤超函数全体の空間を表わす．

$f(x)\in\mathcal{B}^{exp}_{[a,\infty]}$ に対しラプラス変換 $\mathcal{L}f(\lambda)=\hat{f}(\lambda)$ が定義でき，$\mathcal{L}\mathcal{B}^{exp}_{[a,\infty]}$ は，漸近的に $\{e^{i\theta}\infty\,;\,|\theta|<\pi/2\}$ を含む領域上の指数型整型函数で

$$\overline{\lim_{r\to\infty}}\, r^{-1}\log|\hat{f}(re^{i\theta})| \leqq -a\cos\theta,\ |\theta|<\pi/2,$$

をみたすもの全体と一致する．