数理解析研究所講究録
第 640 巻 1988 年 106-118

A Self-Dual Yang-Mills Hierarchy: Periodic Reduction, Ansatz Solutions and Transformation Group

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1. Introduction

M. Sato and Y. Sato [1] introduced the Kadomtsev-Petviashvili (KP) equation hierarchy and completely characterized the solution space of the KP equation being a great interest. Here the KP hierarchy is a $GL(\infty)$-invariant infinite system of integrable nonlinear evolution equations having an infinite number of time variables. They also showed that many other integrable nonlinear equations called soliton equations are derived from the KP hierarchy by periodic reductions. It is noted that there are some integrable nonlinear equations which seem to be outside of the KP hierarchy. They are the stationary axially symmetric Einstein equation, the Bogomolny equation and the self-dual Yang-Mills (SDYM) equation. The first two are derived from the last by specializations. Being inspired by Sato's pioneer work, Takasaki [2] presented a formulation of the $GL(n, \mathbb{C})$ SDYM equation and constructed formal power series solutions by making uses of certain infinite matrices. However, it has not been clear how to define an SDYM equation hierarchy. Recently M. Sato and his coworkers [3] develop a theory of higher dimensional integrable nonlinear equations including the SDYM equation in terms of $\mathfrak{D}$ module. In this note, we shall consider the SDYM hierarchy from an alternative point of view and introduce an infinite system, in which every $GL(k, \mathbb{C})$ SDYM equation ($k\in\mathbb{N}$) is embedded, as a candidate for the SDYM hierarchy (Sect. 2). Secondly, we derive sub-hierarchy found in the previous work [4] from the SDYM hierarchy by a periodic reduction. We call it a $GL(n, \mathbb{C})$ SDYM hierarchy and discuss a class of ansatz solutions taking the form of Toeplitz determinants (Sect. 3). It is shown that an infinite dimensional transformation group acts on the solution space of the ($n\geq2$) sub-hierarchy. We also obtain a parametric solution as a representation of the transformation group (Sect. 4). In appendex we give ansatz solutions and an exponential operator of the stationary axially symmetric Einstein equation which are similar to those of the SDYM equation.

2. A Self-Dual Yang-Mills Hierarchy

Let $A=(a_{ij})_{i,j\in\mathbb{Z}}$ be an $\infty\times\infty$ matrix whose elements are arrayed as

$$A = \begin{bmatrix} & & \cdots & & \\ \cdots & a_{-1-1} & a_{-10} & a_{-11} & \cdots \\ \cdots & a_{0-1} & a_{00} & a_{01} & \cdots \\ \cdots & a_{1-1} & a_{10} & a_{11} & \cdots \\ & & \cdots & & \end{bmatrix}. \qquad (1)$$

Let $\Xi=(\xi_{ij})_{i,j\in\mathbb{Z}}$ be a projection matrix with $\xi_{ii}=1$ for $i\geq0$, $\xi_{ij}=\xi_{ij}(y, \bar{y}, z, \bar{z})$ for $i\leq-1$ and $j\geq0$, $\xi_{ij}=0$ for others, where $y=(y_k)_{k\in\mathbb{N}}$, $\bar{y}=(\bar{y}_k)_{k\in\mathbb{N}}$, $z=(z_k)_{k\in\mathbb{N}}$ and $\bar{z}=(\bar{z}_k)_{k\in\mathbb{N}}$ are sets of infinite complex variables. It is noted that columns for $j\geq 0$ of Ξ span an infinite dimensional subspaces $\mathbb{C}^{\mathbb{N}}$ in $\mathbb{C}^{\mathbb{Z}}$, and Ξ describes affine coordinates of an infinite dimensional Grassmann manifold $GM(\infty)$ [1]. Let Λ be a shift matrix defined by $\Lambda=(\delta_{i\,j+1})_{i,j\in\mathbb{Z}}$. The product of $\infty\times\infty$ matrices is defined by $(a_{ij})_{i,j\in\mathbb{Z}}(b_{ij})_{i,j\in\mathbb{Z}} =(\sum_{k\in\mathbb{Z}} a_{ik}b_{kj})_{i,j\in\mathbb{Z}}$. Let H be an isotropy subgroup of $GL(\infty)$ of matrices $A=(a_{ij})_{i,j\in\mathbb{Z}}$, where $a_{ii}=1$ for $i\in\mathbb{Z}$, $a_{ij}=a_{ij}(y, \bar{y}, z, \bar{z})$ for $i\leq -1$, $j\geq 0$ and $a_{ij}=0$ for others. Solutions to the usual $SU(k)$ SDYM equation are given by solving

$$\partial_{\bar{y}_k}(\partial_{y_k}Q_k\cdot Q_k^{-1}) + \partial_{\bar{z}_k}(\partial_{z_k}Q_k\cdot Q_k^{-1}) = 0, \qquad Q_k\in GL(k, \mathbb{C}) \tag{2}$$

under reality conditions. Here $(y_k, \bar{y}_k, z_k, \bar{z}_k)\in\mathbb{C}^4$, $\partial_{y_k} = \partial/\partial_{y_k}$ and so on. We consider an infinite system of linear equations

$$D_kU = U\Lambda^k\partial_{\bar{z}_k}\Xi, \qquad D_k= \partial_{y_k}+\Lambda^k\partial_{\bar{z}_k}, \tag{3}$$

$$D_\ell^*U = -U\Lambda^\ell\partial_{\bar{y}_\ell}\Xi, \qquad D_\ell^*= \partial_{z_\ell}-\Lambda^\ell\partial_{\bar{y}_\ell}, \tag{4}$$

for $k, \ell\in\mathbb{N}$, where $U\in H$. The integrability conditions of U and Ξ w.r.t. $(y, \bar{y}, z, \bar{z})$ are easily found to be second-order nonlinear equations on $GM(\infty)$

$$\partial_{y_k}(\Lambda^\ell\partial_{\bar{y}_\ell}\Xi) + \partial_{z_\ell}(\Lambda^k\partial_{\bar{z}_k}\Xi) + [\Lambda^k\partial_{\bar{z}_k}\Xi, \Lambda^\ell\partial_{\bar{y}_\ell}\Xi] = 0, \tag{5}$$

$$\partial_{y_\ell}(\Lambda^k\partial_{\bar{z}_k}\Xi) - \partial_{y_k}(\Lambda^\ell\partial_{\bar{z}_\ell}\Xi) - [\Lambda^k\partial_{\bar{z}_k}\Xi, \Lambda^\ell\partial_{\bar{z}_\ell}\Xi] = 0,$$

$$\partial_{z_\ell}(\Lambda^k\partial_{\bar{y}_k}\Xi) - \partial_{z_k}(\Lambda^\ell\partial_{\bar{y}_\ell}\Xi) + [\Lambda^k\partial_{\bar{y}_k}\Xi, \Lambda^\ell\partial_{\bar{y}_\ell}\Xi] = 0, \tag{6}$$

for k, $\ell \in \mathbb{N}$. If there is a $GL(\infty)$-valued matrix Q such that $\Lambda^k \partial_{\bar{z}_k} \Xi = -\partial_{y_k} Q \cdot Q^{-1}$ and $\Lambda^\ell \partial_{\bar{y}_\ell} \Xi = \partial_{z_\ell} Q \cdot Q^{-1}$, then (5) and (6) are automatically satisfied. We note that each $GL(k, \mathbb{C})$ SDYM equation (2) is embedded in (5). To see this let us define k×k matrices

$$\Xi_k^{(i,j)} = \begin{bmatrix} \xi_{-(i+1)k,jk} & \cdots & \xi_{-(i+1)k,(j+1)k-1} \\ \vdots & & \vdots \\ \xi_{-ik-1,jk} & \cdots & \xi_{-ik-1,(j+1)k-1} \end{bmatrix}, \tag{7}$$

for i, $j \in \mathbb{N} \cup \{0\}$. It is not hard to see that $\Xi_k^{(0,0)}$ satisfy

$$\partial_{y_k} \partial_{\bar{y}_k} \Xi_k^{(0,0)} + \partial_{z_k} \partial_{\bar{z}_k} \Xi_k^{(0,0)} + [\partial_{\bar{z}_k} \Xi_k^{(0,0)}, \partial_{\bar{y}_k} \Xi_k^{(0,0)}] = 0. \tag{8}$$

Eqs.(8) are zero curvature conditions, there are $GL(k, \mathbb{C})$-valued functions $Q_k = Q_k(y, \bar{y}, z, \bar{z})$ such that $\partial_{\bar{z}_k} \Xi_k^{(0,0)} = -\partial_{y_k} Q_k \cdot Q_k^{-1}$ and $\partial_{\bar{y}_k} \Xi_k^{(0,0)} = \partial_{z_k} Q_k \cdot Q_k^{-1}$, and consequently we have (2) from (8). Thus solutions to every $GL(k, \mathbb{C})$ SDYM equation ($k \in \mathbb{N}$) are given by solving linear system (3) and (4). We say that the system (3) and (4) defines an *SDYM hierarchy* (5) and (6).

We now discuss transformations of solutions to the SDYM hierarchy. Let G ($\in H$) satisfies $D_k G = 0$ and $D_\ell^* G = 0$ for k, $\ell \in \mathbb{N}$. If U ($\in H$) is a solution to (3) and (4), then U'= GU is also a solution which corresponds to a solution Ξ' to (5) and (6). Solutions Ξ and Ξ' are related as

$$\Lambda^k \partial_{\bar{z}_k} \Xi' = G^{-1} \Lambda^k \partial_{\bar{z}_k} \Xi \cdot G, \qquad \Lambda^\ell \partial_{\bar{y}_\ell} \Xi' = -\, G^{-1} \Lambda^\ell \partial_{\bar{y}_\ell} \Xi \cdot G. \tag{9}$$

We call the transformation from Ξ to Ξ' a gauge transformation for the SDYM hierarchy. There is the following nontrivial symmetry. Let M be a $GL(\infty)$-valued matrix such that $D_k M=0$ and $D^*_\ell M=0$ for k, $\ell\in\mathbb{N}$. We consider a factorization of $U^{-1}MU$ into

$$U^{-1}MU = XY^{-1}, \tag{10}$$

where $X\in H$ and Y is an invertible matrix. If the factor X exists, we have a new solution $\tilde{\Xi}$ to the SDYM hierarchy which is defined by a solution $\tilde{U} = UX = MUY\in H$ to (3) and (4). Solutions Ξ and $\tilde{\Xi}$ are related by the formulae

$$\Lambda^k \partial_{\bar{z}_k} \tilde{\Xi} = X^{-1}\Lambda^k \partial_{\bar{z}_k} \Xi\cdot X + X^{-1}D_k X,$$

$$\Lambda^\ell \partial_{\bar{y}_\ell} \tilde{\Xi} = X^{-1}\Lambda^\ell \partial_{\bar{y}_\ell} \Xi\cdot X - X^{-1}D^*_\ell X. \tag{11}$$

It is concluded that $GL(\infty)$ acts on solutions to the SDYM hierarchy as a symmetry group. We note there always exists the factor X of the factorization problem for M=exp m, where $m\in gl(\infty)$, $D_k m=0$ and $D^*_\ell m=0$. For a given initial value $\overset{o}{U}=U(y, 0, z, 0)\in H$, the linear system (3) and (4) is locally in $\bar{y}$ and $\bar{z}$ solved to

$$U = \exp\{ \sum_{\ell\in\mathbb{N}} (\bar{y}_\ell \Lambda^{-\ell}\partial_{z_\ell} - \bar{z}_\ell \Lambda^{-\ell}\partial_{y_\ell})\}\overset{o}{U}, \tag{12}$$

where $\exp X = \sum_{k=0}^{\infty} X^k/k!$. Since $U\in H$, U gives a formal power series solution to the SDYM hierarchy through (3) and (4).

3. Periodic Reduction

Let us consider a sub-hierarchy by imposing an additional constraint which will be called an n-periodic condition. Let Ξ depend also on infinite real parameters $t=(t_m)_{m\in\mathbb{N}}$ through

$$\partial_{t_m}\Xi = [\Lambda^m\Xi, \Xi], \tag{13}$$

for $m\in\mathbb{N}$. These are locally in t solved to

$$\Xi = \exp(\sum_{m\in\mathbb{N}} t_m\Lambda^m)\overset{o}{\Xi}\{\mathbf{I} - \overset{o}{\Xi} + \exp(\sum_{m\in\mathbb{N}} t_m\Lambda^m)\overset{o}{\Xi}\}^{-1}, \tag{14}$$

where $\mathbf{I}=(\delta_{ij})_{i,j\in\mathbb{Z}}$, $\xi_{ij}=\xi_{ij}(y, \bar{y}, z, \bar{z}; t)$ and $\overset{o}{\xi}_{ij}=\xi_{ij}(y, \bar{y}, z, \bar{z}; 0)$. A set $\{M=\exp(\sum_{m\in\mathbb{N}} t_m\Lambda^m);\ t\in\mathbb{R}^{\mathbb{N}}\}$ forms an abelian subgroup of $GL(\infty)$. We remark here the exponential operator $\exp(\sum_{m\in\mathbb{N}} t_m\Lambda^m)$ describes the time evolution of the KP hierarchy [1]. The parameters $t=(t_m)_{m\in\mathbb{N}}$ have some hidden meaning in the theory of SDYM equation. We assume that Ξ satisfies for a positive integer n

$$[\Lambda^n\Xi, \Xi] = 0, \tag{15}$$

or equivalently, $\partial_{t_n}\Xi=0$. After a calculation we have $[\Lambda^m\Xi, \Xi]=0$ and $\partial_{t_m}\Xi=0$ for $m=0 \pmod n$. We refer to (15) as an n-periodic condition for the SDYM hierarchy. Let us regard the $j\geq 0$ part of $W=(w_{ij})_{i,j\in\mathbb{Z}}$ as homogeneous coordinates of $GM(\infty)$ corresponding to Ξ,

$$(\xi_{ij})_{\substack{i\leq -1\\ j\geq 0}} = (w_{ij})_{\substack{i\leq -1\\ j\geq 0}}(w_{ij})^{-1}_{i,j\geq 0}. \tag{16}$$

For $k=\ell=0 \pmod n$ we can reduce (3), (4) and (15) to

$$D_k W = 0, \qquad D^*_\ell W = 0, \qquad [\Lambda^n, W] = 0, \tag{17}$$

respectively. The last equation implies that W is a (block) Toeplitz matrix; $W=(W_{j-i})_{i,j\in\mathbb{Z}}$, where W_{j-i} are n×n matrices. We note that the system $D_n W = 0$, $D^*_n W = 0$ and $[\Lambda^n, W] = 0$ is an infinite matrix representation of linear system for a single $GL(n, \mathbb{C})$ SDYM equation discussed by Takasaki [2]. The sub-hierarchy characterized by the constraint (15) is regarded as an embedding of the $GL(n, \mathbb{C})$ SDYM equation into $GL(k, \mathbb{C})$ SDYM equations for $k=0$ (mod n). The n=1 sub-hierarchy is a Laplace equation hierarchy. We shall call the system (5), (6) and (15) as a $GL(n, \mathbb{C})$ *SDYM hierarchy*. In Ref. [4], it is shown that an infinite dimensional subgroup of $GL(\infty)$ acts on solutions to the $GL(n, \mathbb{C})$, $n\geq 2$, SDYM hierarchy. Let us remember the fact that the K-dV hierarchy and the Boussinesq hierarchy are derived from the KP hierarchy by 2 and 3-periodic reductions, respectively [1].

From the first two of (17) we have (matrix) Laplace equations $(\partial_{y_k}\partial_{\bar{y}_k} + \partial_{z_k}\partial_{\bar{z}_k})W_{j-i} = 0$ for $k\in\mathbb{N}$, $i,j\in\mathbb{Z}$. The formula (16) implies that solutions to the $GL(n, \mathbb{C})$ SDYM hierarchy are given by a nonlinear superposition of solutions to the Laplace equations. If $W_{j-i}= 0$ for $|j-i|>g\geq 0$, then ξ_{ij} are well-defined and take the form of ratio of Toeplitz determinants. Setting $k=\ell=2$ and $n=1$, we obtain the well-known Atiyah-Ward ansatz solutions $\mathcal{A}_g$ to the $SU(2)$ SDYM equation [5]. Two types of Cauchy problems for (17) are locally solved by using Lie transforms of initial values. The results are

$$W = \exp\{ \sum_{\ell\in\mathbb{N}} (z_{\ell n}\Lambda^{\ell n}\partial_{\bar{y}_{\ell n}} - y_{\ell n}\Lambda^{\ell n}\partial_{\bar{z}_{\ell n}})\} W(0, \bar{y}, 0, \bar{z}),$$

$$W = \exp\{ \sum_{\ell\in\mathbb{N}} (\bar{y}_{\ell n}\Lambda^{-\ell n}\partial_{z_{\ell n}} - \bar{z}_{\ell n}\Lambda^{-\ell n}\partial_{y_{\ell n}})\} W(y, 0, z, 0). \qquad (18)$$

The corresponding power series solutions to the $GL(n, \mathbb{C})$ SDYM hierarchy are given through (16). We note that there are two (mutually commuting) independent time evolution operators.

4. Transformation Group

Let $s = (s_j)_{j\in\mathbb{Z}}$ be a set of complex parameters and h_k, $k\in\mathbb{Z}$, be $gl(n, \mathbb{C})$-valued constant matricies. Define

$$H_k = \mathrm{diag}(h_k)\Lambda^{kn}, \qquad (19)$$

for $k\in\mathbb{Z}$, where $\mathrm{diag}(h_k)$ denotes an infinite (block) diagonal Toeplitz matrix. We note $[\Lambda^{\ell n}, H_k] = 0$ for $k, \ell\in\mathbb{Z}$. Let us consider a one-parameter transformation for the matrix W defined by

$$(\partial_{s_k} - H_k) W(y, \bar{y}, z, \bar{z}; s) = 0, \qquad (20)$$

for some $k\in\mathbb{Z}$. A parametric solution $W(s) = W(y, \bar{y}, z, \bar{z}; s)$ to the system (20) gives $\Xi(s) = \Xi(y, \bar{y}, z, \bar{z}; s)$ by (16) which satisfies

$$\partial_{s_k}\Xi(s) = [H_k\Xi(s), \Xi(s)], \qquad \partial_{s_{-\ell}}\Xi(s) = [H_{-\ell}, \Xi(s)], \qquad (21)$$

for $k\in\mathbb{N}$ and $\ell\in\mathbb{N}\cup\{0\}$. If we focus on the $n\times n$ matrices $\Xi_n^{(i,j)}$ defined by (7), we have from the first system of (21)

$$\partial_{s_k}\Xi_n^{(i,j)} = h_k\Xi_n^{(i+k,j)} - \Xi_n^{(i,j+k)}h_k - \sum_{\ell=0}^{k-1}\Xi_n^{(i,\ell)}h_k\Xi_n^{(k-\ell-1,j)}, \qquad (22)$$

for i, $j\in\mathbb{N}\cup\{0\}$, $k\in\mathbb{N}$. The transformation described by (22) is very similar to the Kinnersley-Chitre (KC) transformation of the Geroch group acting on the stationary Einstein equations. See [4]. Analogue of the KC transformation for the SDYM equation has been known as an infinitesimal Riemann-Hilbert transformation [6].

We now exponentiate the linear system (20) for a given data which satisfies the linear system (17). Let $W^0 = W(y, \bar{y}, z, \bar{z}; 0)$ satisfing (17) be an initial data for (3.2). Then a solution

$$W(s_k) = \exp\ (s_k H_k) W^0 \tag{23}$$

to (20) for some $k\in\mathbb{Z}$ also satisfies (17). This implies that $\Xi(s_k)$ corresponding to $W(s_k)$ gives a solution to the $GL(n, \mathbb{C})$ SDYM hierarchy. In other words, the one-parameter transformation (23) for W gives rise to a symmetry of the hierarchy. It is noted that if $(w^0_{ij})_{i,j\geq 0}$, where $w^0_{ij} = w_{ij}(y, \bar{y}, z, \bar{z}; 0)$, is invertible then $(w_{ij})_{i,j\geq 0}$ is so. Let Ξ^0 be the projection matrix corresponding to W^0. Then $\Xi(s_k)$ defined by

$$\Xi(s_k) = \exp(s_k H_k)\Xi^0 \cdot \{I - \Xi^0 + \exp(s_k H_k)\Xi^0\}^{-1}, \tag{24}$$

for some $k\in\mathbb{Z}$ is a parametric solution to the $GL(n, \mathbb{C})$ SDYM hierarchy (5) and (6) with (15). Proof is given in [4].

Next we discuss an algebraic structure of a whole set of one-parameter transformations induced by $\{H_k;\ k\in\mathbb{Z}\}$. We note

$$[H_k,\ H_\ell] = \mathrm{diag}([h_k,\ h_\ell])\Lambda^{(k+\ell)n}, \tag{25}$$

for $k, \ell\in\mathbb{Z}$. Since $[H_{k+m},\ H_{\ell-m}] = [H_k,\ H_\ell]$ providing $[h_{k+m}, h_{\ell-m}] = [h_k,\ h_\ell]$, an algebra which the $\{H_k;\ k\in\mathbb{Z}\}$ action on the matrix W forms is homomorphic to the loop algebra, a subalgebra of $gl(\infty)$,

$$gl(n, \mathbb{C})\times\mathbb{C}[\lambda, \lambda^{-1}], \tag{26}$$

where λ is a complex parameter such that $|\lambda|=1$. Hence it is concluded that the $\{H_k;\ k\in\mathbb{N}\}$ action on Ξ forms a Lie algebra homomorphic to (26). Let us recall the fact that the KP hierarchy and its solutions are obtained from a representation of $gl(\infty)$ and sub-hierarchies such as the K-dV hierarchy and the Boussinesq hierarchy are associated with infinite dimensional subalgebras of $gl(\infty)$ [1]. This infinitesimal property reflects a group theoretical structure of the transformations (24). In Ref. [4], we prove that the transformations (24) form a (Banach Lie) group $\mathfrak{G}$ acting on a solution space to the $GL(n, \mathbb{C})$ SDYM hierarchy. Indeed it is shown that the infinite dimensional transformation group having the representation (24) acts on the $GL(n, \mathbb{C})$ hierarchy in the case that $n\geq2$. If we choose $n=1$ in (19), then we see that a one-parameter transformation group having the representation (24) acts on the hierarchy (a Laplace equation hierarchy).

5. Concluding Remarks

The facts proved in this note and [4] look rather promising and it is reasonable to expect that we can prove a complete integrability of the $GL(n, \mathbb{C})$ SDYM equation. We also have an interesting question to understand how our $GL(\infty)$-invariant SDYM hierarchy can be realized as a dynamical system on $GM(\infty)$.

Appendix

Here we consider ansatz solutions and an exponential operator of the stationary axially symmetric vacuum Einstein equation

$$\partial_\rho(\rho\partial_\rho Q\cdot Q^{-1}) + \partial_z(\rho\partial_z Q\cdot Q^{-1}) = 0, \qquad Q = \frac{1}{f}\begin{bmatrix} 1 & \varphi \\ \varphi & f^2+\varphi^2 \end{bmatrix}, \tag{A1}$$

where $f=f(\rho,z)$ and $\varphi=\varphi(\rho,z)$. Let $W=(w_{ij})_{i,j\in\mathbb{Z}}$, $\Lambda=(\delta_{i-j+1,0})_{i,j\in\mathbb{Z}}$, and $T=((-\rho^2)^j\delta_{i+j+1,0})_{i,j\in\mathbb{Z}}$, where $w_{ij}=w_{ij}(\rho,z)$. The linear system associated with the ansatz solutions is

$$(\partial_\rho-\rho\Lambda\partial_z)W = 0, \qquad [\Lambda, W] = 0, \qquad [T, W] = 0. \tag{A2}$$

See the linear system (17) for $k=\ell=2$, $n=1$. The system (A2) implies

$$\partial_\rho\Delta_k-\rho\partial_z\Delta_{k-1}= 0, \qquad \partial_z\Delta_k+\{\rho\partial_\rho-2(k-1)\}\Delta_{k-1}= 0, \tag{A3}$$

for $k\in\mathbb{N}$, where $\Delta_{j-i}=w_{ij}$. The integrability conditions are

$$\{\rho\partial_\rho{}^2+(3-2k)\partial_\rho+\rho\partial_z{}^2\}\Delta_{k-1}= 0. \tag{A4}$$

We suppose the Δ_0 is a harmonic function on $\mathbb{R}^3$, where $\rho^2=x_1{}^2+x_2{}^2$, $z=x_3$. Then we have a set of C^2-functions $\{\Delta_k;\ 0\le k\le\ell-1\}$, $\ell\in\mathbb{N}$, satisfying (A4). There is a useful lemma proved in [7] that the functions f_1 and φ_2 defined by $f_1=\Delta_0{}^{-1}$ and $\varphi_1=-i\Delta_0{}^{-1}$ satisfy the Einstein equation (A1), where $i^2=-1$. Let us call the family of solutions the ansatz $\mathfrak{P}_1'$. The second and the third ansatze $\mathfrak{P}_2'$ and $\mathfrak{P}_3'$ were also found in [7] by using a Bäcklund transformation: $(f, \varphi) \to (\tilde{f}, \tilde{\varphi})$,

$$\tilde{f} = \rho f^{-1}(f^2+\varphi^2), \quad \partial_\rho\tilde{\varphi} = i\rho f^{-2}(f^2+\varphi^2)^2\partial_z\{\varphi(f^2+\varphi^2)^{-1}\},$$

$$\partial_z\tilde{\varphi} = -i\rho f^{-2}(f^2+\varphi^2)^2\partial_p\{\varphi(f^2+\varphi^2)^{-1}\}. \qquad (A5)$$

The general ansatz $\mathfrak{P}'_\ell$ defined by the same manner is given in [8] as

$$f_\ell = \rho^{1-\ell}D_\ell^{-1}D_{\ell-1}, \qquad D_\ell = \det W_\ell = \begin{vmatrix} \Delta_0 & \Delta_1 & \cdots & \Delta_{\ell-1} \\ -\rho^{-2}\Delta_1 & \Delta_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ -\rho^{2-2\ell}\Delta_{\ell-1} & & \cdots & \Delta_0 \end{vmatrix}, \qquad (A6)$$

$$\varphi_\ell = (-i)^\ell D_\ell^{-1}\tilde{D}_{\ell-1}$$

where $\tilde{D}_{\ell-1}$ is the (1, ℓ) minor of W_ℓ. We see this family of solutions takes the form of ratio of Toeplitz determinants and their minors. Elements of these Toeplitz matrices are solutions to the first order linear equations (A3). It is noted that the celebrated Kerr solution is a special member of the ansatz $\mathfrak{P}'_3$.

Let D(z) be an $\infty\times\infty$ diagonal matrix function of only z satisfying $[\Lambda, D(z)]=0$. Set $W(0, z)= \exp(-\frac{1}{2}\Lambda^{-1}\partial_z)D(z)$, which is lower triangular and Toeplitz. Then $(\partial_\rho-\rho\Lambda\partial_z)W(p, z)=0$ is integrated to be

$$W(\rho, z) = \exp(\tfrac{1}{2}\rho^2\Lambda\partial_z)W(0, z). \qquad (A7)$$

Observe that $W(\rho, z)$ satisfies $[\Lambda, W(p, z)]=0$, and hence the operator $\exp(\frac{1}{2}\rho^2\Lambda\partial_z)$ acts on the space of ansatz solutions. The formula (A7) is analogous to the time evolution formulae for soliton equations [1], the SDYM equation [2] and the $GL(n, \mathbb{C})$ SDYM hierarchy (18). Recently Nagatomo [9] also has obtained the operator $\exp(\frac{1}{2}\rho^2\Lambda\partial_z)$ generating formal power series solutions. It is expected these observations play an important role in the study of solution space to the Einstein equation.

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