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Vertex Operators in the Conformal Field Theory on $\mathbb{P}^1$ and Representations of the Hecke Algebras of type A_N

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0. In this note, we report that our investigation[7] is generalized as follows: The vertex operators(primary fields) are constructed for the conformal field theory on $\mathbb{P}^1$ also by integrable modules of any non- twisted affine Lie algebra $X_n^{(1)}$, and in the case that $X_n^{(1)} = A_n^{(1)}$, the commutation relations of vertex operators induce monodromy representations of the braid group on the spaces of vacuum expectations of compositions of vertex operators which give all unitarizable modules of Hecke algebras of type A_N constructed by H. Wenzl[5].

1. The 2-dimensional conformal field theory is initiated by A.A. Belavin, A.N. Polyakov and A.B. Zamolodchikov[1] and they pointed out the significance of the primary fields for this theory. Since then the theory has been developed by many physicists, e.g. [2,4,6]. V.G. Knizhnik and A.B. Zamolodchikov[4] developed the theory with current algebra symmetry, and proposed a notion of primary fields with gauge symmetry, and gave the differential equations of multipoint correlation functions. Our aim here is to give rigorous mathematical foundations to the work of [4], and to reformulate and develop the operator formalism in the conformal field theory on the complex projective line $\mathbb{P}^1$. Main results are the existence and uniqueness theorem of primary fields and that the monodromies on N-point functions coincide with the representations of the Hecke algebra $H_N(q)$ constructed by H. Wenzl[5] for some roots q of the unity.

2. Let $\hat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c$ be the affine Lie algebra of type $X_n^{(1)}$, where $\mathfrak{g}$ is the classical Lie algebra of type X_n. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, a root basis $\Pi=\{\alpha_1,\cdots,\alpha_n\}$ of the root system Δ for $(\mathfrak{g},\mathfrak{h})$, and the nondegerate $\mathfrak{g}$-invariant bilinear form (,) with the normalized condition $(\theta,\theta)=2$, where θ is the maximal root. Let $\{H_1,\cdots,H_n\}$ be the coroot basis and $\{E_1,\cdots,E_n;\ F_1,\cdots,F_n\}$ be the Chevalley generators of $\mathfrak{g}$. Denote $X(m)= X\otimes t^m$ $(X\in\mathfrak{g})$, then $[X(m),\ Y(n)] = [X,Y](m+n) + m\delta_{m+n,0}(X,Y)\ c$ and $[X(m),c] = 0$ $(X,Y\in\mathfrak{g},\ m,n\in\mathbb{Z})$. The Lie algebra $\hat{\mathfrak{g}}$ has a decomposition $\hat{\mathfrak{g}} = \mathfrak{m}_+\oplus\mathfrak{g}\oplus\mathbb{C}c\oplus\mathfrak{m}_-$, where $\mathfrak{m}_\pm = \mathfrak{g}\otimes\mathbb{C}[t^{\pm1}]t^{\pm1}$.

We fix the value ℓ(positive integer) of the central element c of $\hat{\mathfrak{g}}$ on the space $\mathscr{H}$(defined below) of operands. Denote by P_ℓ the set of all weights $\lambda\in\mathfrak{h}^*$ with $\langle\lambda,H_i\rangle\in\mathbb{Z}_{\geq0}$ and $0\leq(\theta,\lambda)\leq\ell$. For any dominant integral weight $\lambda\in P_\ell$, there is a unique integrable (irreducible) highest weight left $\hat{\mathfrak{g}}$-module $\mathscr{H}_\lambda$ with the highest weight vector $|\lambda\rangle$, such that the subspace $V_\lambda = \{v\in\mathscr{H}_\lambda;\ \mathfrak{m}_+v=0\}$ is an irreducible $\mathfrak{g}$-module of highest weight λ.

We can define the corresponding irreducible highest weight right $\hat{\mathfrak{g}}$(or $\mathfrak{g}$)-module $\mathscr{H}_\lambda^\dagger$ (or $V_\lambda^\dagger$) (and fix a lowest weight vector $\langle\lambda|$), and the nondegenerate bilinear pairing (called *vacuum expectation value*) $\langle\ |\ \rangle$: $\mathscr{H}_\lambda^\dagger\times\mathscr{H}_\lambda\longrightarrow\mathbb{C}$ such that $\langle\lambda|\lambda\rangle = 1$ and $\langle va|w\rangle = \langle v|aw\rangle$ for any $v\in\mathscr{H}_\lambda^\dagger$, $a\in\hat{\mathfrak{g}}$, $w\in\mathscr{H}_\lambda$. Its restriction on $V_\lambda^\dagger\times V_\lambda$ is also nondegenerate.

The Virasoro algebra $\mathscr{L}$ acts on each $\mathscr{H}_\lambda$ and $\mathscr{H}_\lambda^\dagger$ through the Sugawara form $L(m), m\in\mathbb{Z}$, that is,

$$L(m) = \frac{1}{2(\ell+g)}\sum_{k\in\mathbb{Z}}\left\{\sum_{i=1}^{n} :H^i(-k)H_i(m+k): + \sum_{\gamma\in\Delta} :X^\gamma(-k)X_\gamma(m+k):\right\},$$

where $\{H^i\in\mathfrak{h}\ (1\le i\le n),\ X^{\gamma}\in\mathfrak{g}_{-\gamma}\ (\gamma\in\Delta)\}$ and $\{H_i\in\mathfrak{h}\ (1\le i\le n),\ X_{\gamma}\in\mathfrak{g}_{\gamma}\ (\gamma\in\Delta)\}$ are dual bases of $\mathfrak{g}$ and g is the dual Coxeter number of $X_n^{(1)}$, i.e. g = $n+1$ (for $X_n=A_n$ or B_n), $= 2n-1$ (for $X_n=C_n$), $= 2n-2$ (for $X_n=D_n$), $= 9$ (for $X_n=E_6$ or F_4), $=18$ (for $X_n=E_7$), $= 30$ (for $X_n=E_8$), $= 4$ (for $X_n=G_2$). The normal ordering $:\ :$ is defined by

$$:X(m)Y(n): = X(m)Y(n) \quad \text{if } m<n, \qquad = Y(n)X(m) \quad \text{if } m>n;$$
$$= \frac{1}{2}\{X(m)Y(n)+Y(n)X(m)\} \quad \text{if } m=n.$$

Then $\mathcal{H}_{\lambda}$ and $\mathcal{H}_{\lambda}^{\dagger}$ have the eigenspace decompositions w.r.t. the operator $L(0)$: $\mathcal{H}_{\lambda}=\sum_{d\in\mathbb{Z}_{\ge 0}}\mathcal{H}_{\lambda,d}$ and $\mathcal{H}_{\lambda}^{\dagger}=\sum_{d\in\mathbb{Z}_{\ge 0}}\mathcal{H}_{\lambda,d}^{\dagger}$, where $\mathcal{H}_{\lambda,d}$ and $\mathcal{H}_{\lambda,d}^{\dagger}$ are the eigenspaces with the same eigenvalue $\Delta_{\lambda}+d$, where $\Delta_{\lambda}=\dfrac{\{(\lambda,\lambda)+2(\lambda,\rho)\}}{2(\ell+g)}$ and ρ is the half sum of positove roots of $(\mathfrak{g},\mathfrak{h},\Pi)$.

Introduce the spaces $\mathcal{H}$ and $\mathcal{H}^{\dagger}$ defined by $\mathcal{H}=\sum_{\lambda\in P_{\ell}}\mathcal{H}_{\lambda}$ and $\mathcal{H}^{\dagger}=\sum_{\lambda\in P_{\ell}}\mathcal{H}_{\lambda}^{\dagger}$, and extend $\langle\ |\ \rangle$ to $\langle\ |\ \rangle$: $\mathcal{H}^{\dagger}\times\mathcal{H}\longrightarrow\mathbb{C}$ by $\langle\mathcal{H}_{\lambda}^{\dagger}|\mathcal{H}_{\lambda'}\rangle = 0$ for $\lambda\ne\lambda'$.

By an *operator*, we mean a linear mapping $\Phi:\mathcal{H}\longrightarrow\hat{\mathcal{H}}$, where $\hat{\mathcal{H}}$ is a completion of $\mathcal{H}$(see [7] for the definition). Note that an operator Φ is characterized by bilinear mapping $\hat{\Phi}$: $\mathcal{H}^{\dagger}\times\mathcal{H}\longrightarrow\mathbb{C}$ defined by $\langle v|\hat{\Phi}|w\rangle = \langle v|\Phi(w)\rangle$ for any $v\in\mathcal{H}^{\dagger}$ and $w\in\mathcal{H}$. An operator-valued function $\Phi(z)$ on a complex manifold M is called *holomorphic*, if the function $\langle v|\Phi(z)|w\rangle$ is holomorphic in $z\in M$ for any $\langle v|\in\mathcal{H}^{\dagger}$ and $|w\rangle\in\mathcal{H}$.

An ordered pair $\{\Phi,\ \Psi\}$ of operators on $\mathcal{H}$ are called *composable*, if

$$\sum_{d\ge 0}\left|\sum_{j=1}^{m_d}\langle v|\Phi|u_{d,j}\rangle\langle u_{d,j}|\Psi|w\rangle\right| < \infty$$

for any vectors $\langle v|\in\mathcal{H}^{\dagger}$ and $|w\rangle\in\mathcal{H}$, where $\{|u_{d,i}\rangle;\ 1\le i\le m_d\}$ and $\{\langle u_{d,j}|;\ 1\le i\le m_d\}$ are dual bases in $\sum_{\lambda\in P_{\ell}}\mathcal{H}_{\lambda,d}$ and $\sum_{\lambda\in P_{\ell}}\mathcal{H}_{\lambda,d}^{\dagger}$, and $m_d=\dim\sum_{\lambda\in P_{\ell}}\mathcal{H}_{\lambda,d}$. In this case, the composed operator $\Phi\Psi$ is defined by

$$\langle v|\Phi\Psi|w\rangle = \sum_{d\ge 0}\sum_{j=1}^{m_d}\langle v|\Phi|u_{d,j}\rangle\langle u_{d,j}|\Psi|w\rangle .$$

Note that two operators may not always be composable.

3. For each $X \in \mathfrak{g}$, the field operator $X(z) = \sum_{m \in \mathbb{Z}} X(m) z^{-m-1}$ obeys the equations of motions:

$$[L(m),\ X(z)] = z^m \left[z\frac{d}{dz} + m+1 \right] X(z) \qquad (m \in \mathbb{Z}).$$

The currents $X(z), X \in \mathfrak{g}$ and the energy momentum tensor $T(z) = \sum_{m \in \mathbb{Z}} L(m) z^{-m-2}$ preserve each space $\mathcal{H}_\lambda$, which can be considered as a free theory. In order to introduce operators describing the interactions in the theory, we define the vertex operators due to [4].

A triple $\mathbf{v} = \begin{bmatrix} & \lambda & \\ \lambda_2 & & \lambda_1 \end{bmatrix}$ of dominant integral weights λ_2, λ_1 and λ is called a *vertex*. Introduce the space $\mathcal{V}(\mathbf{v}) = \mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_{\lambda_1}, V_{\lambda_2})$.

A multi-valued, holomorphic, operator-valued function $\Phi(u;z)$ on the manifold $M_1 = \{z \in \mathbb{C};\ z \neq 0\}$ linearly parametrized by $u \in V_\lambda$ is called a *vertex operator of weight* λ, if for any $u \in V_\lambda$ and $z \in M_1$, an operator $\Phi(u;z): \mathcal{H} \longrightarrow \hat{\mathcal{H}}$ satisfies the conditions:

(Gauge Condition) $\quad [X(m),\ \Phi(u;z)] = z^m\, \Phi(Xu;z) \qquad (X \in \mathfrak{g}, m \in \mathbb{Z})$;

(Equation of Motion) $\quad [L(m),\ \Phi(u;z)] = z^m \{ z\frac{d}{dz} + (m+1)\Delta_\lambda \} \Phi(u;z) \qquad (m \in \mathbb{Z})$,

where the number Δ_λ is called *conformal dimension* of the vertex operator $\Phi(z)$.

A vertex operator $\Phi(z)$ of weight λ is called *of type* $\mathbf{v}$ for a vertex $\mathbf{v} = \begin{bmatrix} & \lambda & \\ \lambda_2 & & \lambda_1 \end{bmatrix}$ with $\lambda_i \in P_\ell\ (i=1,2)$, if $\Phi(u;z) = \Pi_{\lambda_2} \Phi(u;z) \Pi_{\lambda_1}$, where Π_λ is the projection of $\mathcal{H}$ (or $\hat{\mathcal{H}}$) onto $\mathcal{H}_\lambda$ (or $\hat{\mathcal{H}}_\lambda$ respectively).

Then we get the condition for the existence of vertex operators:

Theorem 1.

i) A vertex operator $\Phi(z)$ of type $\mathbf{v} = \begin{bmatrix} & \lambda & \\ \lambda_2 & & \lambda_1 \end{bmatrix}$ is uniquely determined by the form (initial term) $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_2}^{\dagger} \otimes V_\lambda \otimes V_{\lambda_1}, \mathbb{C})$ defined by

$$\varphi(v,u,w) = \left[z^{\hat{\Delta}(\mathtt{v})}\langle v|\Phi(u;z)|w\rangle\right]\Big|_{z=0} \qquad (v\in V^{\dagger}_{\lambda_2},\ u\in V_{\lambda},\ w\in V_{\lambda_1}),$$

where $\hat{\Delta}(\mathtt{v}) = \Delta_{\lambda}+\Delta_{\lambda_1}-\Delta_{\lambda_2}$.

ii) There exists a nonzero vertex operator Φ of type $\mathtt{v}$ on $\mathscr{H}$, if and only if the vertex $\mathtt{v}$ satisfies the ℓ-constrained Clebsch-Gordan condition $(CG)_{\ell}$:

$(CG)_{\ell}$ $\qquad \mathscr{V}(\mathtt{v}) \neq 0$ and $(\lambda_1+\lambda_2+\lambda,\theta) \le 2\ell$.

Remark that the existence of a nontrivial vertex operator of weight λ implies that $\lambda\in P_{\ell}$.

For each vertex $\mathtt{v}\in(CG)_{\ell}$, we choose and fix a basis $\mathscr{BV}(\mathtt{v}) = \{\varphi^{i}_{\mathtt{v}};\ 1\le i\le m(\mathtt{v})\}$ of $\mathscr{V}(\mathtt{v})$ where $m(\mathtt{v}) = \dim\mathscr{V}(\mathtt{v})$, and denote by $\Phi_{\varphi}(z)$ the associated vertex operator of type $\mathtt{v}$ with the initial term $\varphi\in\mathscr{V}(\mathtt{v})$.

Let $\Phi(z)$ be a vertex operator of weight λ. Define the actions of the Lie algebras $\hat{\mathfrak{g}}$ and $\mathscr{L}$ on $\Phi(z)$ by

$$\hat{X}(m)\Phi(z) = \frac{1}{2\pi\sqrt{-1}}\int_C d\xi\ (\xi-z)^m X(\xi)\Phi(z) \qquad (X\in\mathfrak{g},\ m\in\mathbb{Z}),$$

and

$$\hat{L}(m)\Phi(z) = \frac{1}{2\pi\sqrt{-1}}\int_C d\xi(\xi-z)^{m+1}\ T(\xi)\Phi(z) \qquad (m\in\mathbb{Z})$$

for some contour C around z such that 0 is outside C. Then

$$\hat{X}(m)\Phi(u;z) = 0 \qquad (m\ge 1,\ X\in\mathfrak{g},\ u\in V_{\lambda});$$

$$\hat{X}(0)\Phi(u;z) = [X(0),\ \Phi(u;z)] = \Phi(Xu;z) \qquad (X\in\mathfrak{g},\ u\in V_{\lambda});$$

$$\hat{L}(m)\Phi(u;z) = 0 \qquad (m\ge 1,\ u\in V_{\lambda});$$

$$\hat{L}(0)\Phi(u;z) = \Delta_{\lambda}\Phi(u;z) \qquad (u\in V_{\lambda});$$

$$\hat{L}(-1)\Phi(u;z) = \frac{\partial}{\partial z}\Phi(u;z) \qquad (u\in V_{\lambda}).$$

From the relation of the irreducible $\hat{\mathfrak{g}}$-module $\mathscr{H}_{\lambda}$, we get

Theorem 2.

$$\hat{X}_\theta(-1)^{\ell-(\lambda,\theta)+1}\quad \Phi(|\lambda>;z) = 0 .$$

4. Now we call the vectors $|0>\in\mathcal{H}_0$ and $<0|\in\mathcal{H}_0^\dagger$ the *Virasoro vacuum*. They satisfies the equalities

$$X(m)|0>= L(n)|0>= 0 \;;\; <0|X(-m)= <0|L(-n)= 0 \qquad (X\in\mathfrak{g}, m\geq 0, n\geq -1).$$

For an N-ple $\Lambda = (\lambda_N,\cdots,\lambda_1)$ of weights λ_i with $\lambda_i\in P_\ell$, let $V^\vee(\Lambda) = V^\vee_{\lambda_N}\otimes\cdots\otimes V^\vee_{\lambda_1}$, and let $V^\vee_0(\Lambda)= \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_N}\otimes\cdots\otimes V_{\lambda_1},\mathbb{C})$ denote the invariant subspace of $V^\vee(\Lambda)$ under the diagonal $\mathfrak{g}$-action, where $V^\vee_\lambda$ denotes the dual $\mathfrak{g}$-module of V_λ. Let $\Phi_i(z_i)$ be a vertex operator of weight λ_i $(1\leq i\leq N)$. Then the vacuum expectation value of the composed operator

$$<\Phi_N(z_N)\cdots\Phi_1(z_1)> = <0|\Phi_N(z_N)\cdots\Phi_1(z_1)|0>$$

is considered as a $V^\vee(\Lambda)$-valued, formal Laurent series on $(z_N,\cdots,z_1)$ and is called an N-*point function* (of weight Λ). If $\Phi_i(z_i)$ is of type v_i $(1\leq i\leq N)$,

$$<\Phi_N(z_N)\cdots\Phi_1(z_1)> = \prod_{i=1}^{N} z_i^{-\hat{\Delta}(v_i)} \sum C_{m_N\cdots m_1} z_N^{-m_N}\cdots z_1^{-m_1} ,$$

where $C_{m_N\cdots m_1}\in V^\vee(\Lambda)$ and the sum is taken over integers $m_k\in\mathbb{Z}$ $(1\leq k\leq N)$ with $m_N\geq 0$ and $m_1\leq 0$.

Let π_i denote the $\mathfrak{g}$-action on the i-th component of $V^\vee(\Lambda)$ and introduce the operator Ω_{ik} defined by

$$\Omega_{ik} = \sum_{j=1}^{n} \pi_i(H^j)\pi_k(H_j) + \sum_{\gamma\in\Delta}\pi_i(X^\gamma)\pi_k(X_\gamma)$$

and denote $\Omega_i=\Omega_{ii}$, then $\Omega_i = \{(\lambda_i,\lambda_i)+2(\lambda_i,\rho)\}id$ on $V^\vee(\Lambda)$.

Then we get a system of differential equations and a system of algebraic equations for N-point functions:

Theorem 3.

Let $\Phi_i(z_i)$ be a vertex operator of weight λ_i $(1\leq i\leq N)$, then the N-point function $\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle$ satisfies the following equations:

(I) **(projective invariance)** For m=-1,0 and 1,

$$\sum_{i=1}^{N} z_i^m \left(z_i\frac{\partial}{\partial z_i} + (m+1)\Delta_{\lambda_i}\right)\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle = 0 .$$

(II) **(gauge invariance)** For any $X\in\mathfrak{g}$,

$$\sum_{i=1}^{N} \pi_i(X)\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle = 0 .$$

(III) For each $i=1,\cdots,N$,

$$\left[(\ell+g)\frac{\partial}{\partial z_i} - \sum_{\substack{k=1\\k\neq i}}^{N}\frac{\Omega_{ik}}{z_i-z_k}\right]\langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle = 0 .$$

(IV) For each i $(1\leq i\leq N)$ and any $u_k\in V_{\lambda_k}$ $(k\neq i)$,

$$\sum_{\mathbf{m}_i}\binom{L_i}{\mathbf{m}_i}\prod_{k\neq i}(z_k-z_i)^{-m_k}\langle\Phi_N(X_\theta^{m_N}u_N;z_N)\cdot\cdot\Phi_i(|\lambda_i\rangle;z_i)\cdot\cdot\Phi_1(X_\theta^{m_1}u_1;z_1)\rangle = 0 ,$$

where $\mathbf{m}_i = (m_N,\cdots,\hat{m}_i,\cdot\cdot,m_1) \in (\mathbb{Z}_{\geq 0})^{N-1}$ with $\sum_{k\neq i} m_k = L_i = \ell-(\lambda_i,\theta)+1$.

Remark that the equations (I)~(III) are obtained in [4] and the equations (IV) are obtained by Theorem 2. The equations (II) and (III) imply (I), and the system (III) of differential equations is completely integrable.

5. Consider the systems $E(\Lambda)$ of differential equations and $B(\Lambda)$ of algebraic equations for $V_0^\vee(\Lambda)$-valued functions $\Phi(z_N,\cdots,z_1)$ on the manifold $X_N = \{(z_N,\cdots,z_1)\in\mathbb{C}^N;\ z_i\neq z_k\ (i\neq k)\}$:

$$E(\Lambda)\qquad \left[(\ell+g)\frac{\partial}{\partial z_i} - \sum_{\substack{k=1\\k\neq i}}^{N}\frac{\Omega_{ik}}{z_i-z_k}\right]\Phi(z_N,\cdots,z_1) = 0 \qquad (1\leq i\leq N)$$

and for each i $(1\leq i\leq N)$ and any $u_k\in V_{\lambda_k}$ $(k\neq i)$,

$$B(\Lambda) \quad \sum_{\mathbb{m}_i} \binom{L_i}{\mathbb{m}_i} \prod_{k\neq i} (z_k - z_i)^{-m_k} \Phi(z_N, \cdots, z_1)(X_\theta^{m_N} u_N, \cdots, |\lambda_i\rangle, \cdots, X_\theta^{m_1} u_1) = 0 ,$$

where $\mathbb{m}_i = (m_N, \cdots, \hat{m}_i, \cdots, m_1) \in (\mathbb{Z}_{\geq 0})^{N-1}$ with $\sum_{k\neq i} m_k = L_i = \ell - (\lambda_i, \theta) + 1$.

Introduce the set $\mathcal{P}_\ell(\Lambda)$ defined by

$$\mathcal{P}_\ell(\Lambda) = \{\mathbb{p} = (\mu_N, \cdots, \mu_1, \mu_0; \varphi_N, \cdots, \varphi_1, \varphi_0) ; \ \mu_i \in P_\ell \ , \ \mu_N = \mu_0 = 0, \quad \mathbb{v}_i = \begin{bmatrix} \lambda_i \\ \mu_i \mu_{i-1} \end{bmatrix} \in (CG)_\ell, \ \varphi_i \in \mathcal{BV}(\mathbb{v}_i) \} .$$

For each $\mathbb{p} \in \mathcal{P}_\ell(\Lambda)$, the N-point function

$$\Phi_{\mathbb{p}}(z_N, \cdots, z_1) = \langle \Phi_{\varphi_N}(z_N) \cdots \Phi_{\varphi_1}(z_1) \rangle$$

of type $\mathbb{p}$ is a formal Laurent series solution of the joint system $E(\Lambda)$ and $B(\Lambda)$. By the theory of partial differential equations with regular singularities, we get

Theorem 4.

i) for any $\mathbb{p} \in \mathcal{P}_\ell(\Lambda)$, the Laurent series $\Phi_{\mathbb{p}}(z_N, \cdots, z_1)$ is absolutely convergent in the region $\mathcal{R}_z = \{(z_N, \cdots, z_1) \in \mathbb{C}^N; \ |z_N| > \cdots > |z_1|\}$ and is analytically continuated to a multivalued holomorphic function on X_N.

ii) $\{\Phi_{\mathbb{p}}(z_N, \cdots, z_1); \ \mathbb{p} \in \mathcal{P}_\ell(\Lambda)\}$ gives a basis of the solution space of joint system $E(\Lambda)$ and $B(\Lambda)$.

As a corollary of Theorem 4, we get

Theorem 5.

Let $\Phi_i(z_i)$ be the vertex operator of weight λ_i and $u_i \in V_{\lambda_i}$ $(1 \leq i \leq N)$. Then $\{\Phi_N(u_N; z_N), \cdots, \Phi_1(u_1; z_1)\}$ is composable in the region $\mathcal{R}_{z,0} = \{(z_N, \cdots, z_1) \in \mathbb{C}^N; \ |z_N| > \cdots > |z_1| > 0\}$ and the composed operator $\Phi_N(u_N; z_N) \cdots \Phi_1(u_1; z_1)$ is analytically continuated to a multivalued holomorphic function on the manifold $M_N = \{(z_N, \cdots, z_1) \in X_N;\ z_i \neq 0\}$.

6. For vertices $v_2 = \begin{bmatrix} & \lambda_3 & \\ \lambda_4 & & \mu \end{bmatrix}$ and $v_1 = \begin{bmatrix} & \lambda_2 & \\ \mu & & \lambda_1 \end{bmatrix}$ satisfying $(CG)_\ell$ and mappings $\varphi_i \in \mathcal{BV}(v_i)$ $(i=1,2)$, the composed operator $\Phi_{\varphi_2}(w)\Phi_{\varphi_1}(z)$ of the vertex operators $\Phi_{\varphi_2}(w)$ and $\Phi_{\varphi_1}(z)$ is multi-valuedly holomorphic on the manifold M_2.

For a quadruple $\Lambda = (\lambda_4, \lambda_3, \lambda_2, \lambda_1)$ of weights λ_i with $\lambda_i \in P_\ell$, introduce the set $I_\ell(\Lambda)$ of *intermediate edges*, defined by

$$I_\ell(\Lambda) = \left\{ \mu = (\mu, \varphi_2, \varphi_1);\ \mu \in P_\ell,\ v_2(\mu) = \begin{bmatrix} & \lambda_3 & \\ \lambda_4 & a & \mu \end{bmatrix} \in (CG)_\ell,\ v_1(\mu) = \begin{bmatrix} & \lambda_2 & \\ \mu & & \lambda_1 \end{bmatrix} \in (CG)_\ell \right.$$
$$\left. \varphi_i \in \mathcal{BV}(v_i(\mu)) \right\}.$$

Let $\bar{\Lambda} = (\lambda_4, \lambda_2, \lambda_3, \lambda_1)$, then we get the $\mathfrak{g}$-isomorphism $T: V^\vee(\Lambda) \longrightarrow V^\vee(\bar{\Lambda})$ defined by

$$(T\varphi)(u_4 \otimes u_2 \otimes u_3 \otimes u_1) = \varphi(u_4 \otimes u_3 \otimes u_2 \otimes u_1)$$

for $\varphi \in V^\vee(\Lambda) = \mathrm{Hom}(V_{\lambda_4} \otimes V_{\lambda_3} \otimes V_{\lambda_2} \otimes V_{\lambda_1}, \mathbb{C})$ and $u_4 \otimes u_2 \otimes u_3 \otimes u_1 \in V(\bar{\Lambda})$.

For an intermediate edge $\bar{\mu} = (\bar{\mu}, \bar{\varphi}_2, \bar{\varphi}_1) \in I_\ell(\bar{\Lambda})$, similarly define the vertices $\bar{v}_2(\bar{\mu}) = \begin{bmatrix} & \lambda_2 & \\ \lambda_4 & a & \bar{\mu} \end{bmatrix}$, $\bar{v}_1(\bar{\mu}) = \begin{bmatrix} & \lambda_3 & \\ \bar{\mu} & & \lambda_1 \end{bmatrix} \in (CG)_\ell$ and consider the composed operator $\bar{\Phi}_{\bar{\varphi}_2}(w)\bar{\Phi}_{\bar{\varphi}_1}(z)$ of the vertex operators $\bar{\Phi}_{\bar{\varphi}_2}(w)$ and $\bar{\Phi}_{\bar{\varphi}_1}(z)$.

Assume that $I_\ell(\Lambda) \neq \emptyset$ and take $\mu = (\mu, \varphi_2, \varphi_1) \in I_\ell(\Lambda)$. For a point $(w, z) \in I_2 = \{(z_2, z_1) \in \mathbb{R}^2;\ z_2 > z_1 > 0\}$, let $\Phi_{\varphi_2}(z)\Phi_{\varphi_1}(w)$ denote the analytic coninuation of the composition $\Phi_{\varphi_2}(w)\Phi_{\varphi_1}(z)$ of the vertex operators along the path $b(t)$, where the path $b(t) = (\eta(t), \xi(t))$ from the point $(w, z) \in I_2$ to the point $(z, w) \in \bar{I}_2 = \{(z_2, z_1) \in \mathbb{R}^2;\ z_1 > z_2 > 0\}$ on the manifold M_2 is defined by

$$\eta(t) = \frac{w+z}{2} + e^{\pi\sqrt{-1}t}\frac{w-z}{2}, \qquad \xi(t) = \frac{w+z}{2} - e^{\pi\sqrt{-1}t}\frac{w-z}{2} \qquad (t \in [0, 1]).$$

Then

Proposition 6. i) There exists a square constant matrix $C(\Lambda)=\left[C_{\mu}^{\bar{\mu}}(\Lambda)\right]_{\mu\in I_{\ell}(\Lambda),\bar{\mu}\in I_{\ell}(\bar{\Lambda})}$ such that for each intermediate edge $\mu\in I_{\ell}(\Lambda)$,

$$T\,\Phi_{\varphi_2}(z)\Phi_{\varphi_1}(w) = \sum_{\bar{\mu}\in I_{\ell}(\bar{\Lambda})} \bar{\Phi}_{\bar{\varphi}_2}(w)\bar{\Phi}_{\bar{\varphi}_1}(z)\; C_{\mu}^{\bar{\mu}}(\Lambda) .$$

ii) Let $\Lambda = (\tau,\lambda_3,\lambda_2,\lambda_1,\sigma)$, then the braid relation holds:

$$C(\lambda_4,\lambda_3,\lambda_2,\sigma)C(\tau,\lambda_3,\lambda_1,\lambda_2)C(\lambda_1,\lambda_3,\lambda_2,\sigma)$$
$$= C(\tau,\lambda_3,\lambda_2,\lambda_1)C(\lambda_2,\lambda_3,\lambda_1,\sigma)C(\tau,\lambda_2,\lambda_1,\lambda_3) .$$

Now our fundamental problem is

Fundamental Problem.

Determine the matrix $C(\Lambda)=\left[C_{\mu}^{\bar{\mu}}(\Lambda)\right]$ for any quadruple Λ with $I_{\ell}(\Lambda)\neq\emptyset$.

7. Fix a quadruple $\Lambda=(\lambda_4,\lambda_3,\lambda_2,\lambda_1)$ of weights λ_i with $\lambda_i\in P_{\ell}$. By the projective invariance, the system $E(\Lambda)$ of differential equations is reduced to a differential equation $RE(\Lambda)$ for $V_0^{\vee}(\Lambda)$-valued functions of one variable, that is, on the projective line $\mathbb{P}^1$. This equation $RE(\Lambda)$ is of Fuchian type and has regular singurities at 0, 1 and ∞. The fundamental problem is reduced to the problem of how to determine the connection matrix of the equation $RE(\Lambda)$ from 0 to ∞.

We can solve the fundamental problem for the case where $X_n=A_n$, $\lambda_3 = \lambda_2 = \lambda(\square)$ in Λ.

Now we prepare some notations and results on $\mathfrak{g}=\mathfrak{sl}(n+1,\mathbb{C})$ and its representations. Let $\mathfrak{h} = \mathfrak{g}\cap\sum_{j=1}^{n+1}\mathbb{C}E_{jj}$ be a Cartan subalgebra of $\mathfrak{g}$, where E_{ij} denotes the matrix element in $\mathfrak{gl}(n+1;\mathbb{C})$. A coroot basis is given as $\{H_1=E_{11}-E_{22},\cdots,H_n=E_{nn}-E_{n+1,n+1}\}$ and $\{\bar{\Lambda}_1,\cdots,\bar{\Lambda}_n\}$ denotes the fundamental weights, that is, $\langle\bar{\Lambda}_i,\bar{\Lambda}_j\rangle = \delta_{ij}$. Introduce the nondegenerate invariant bilinear form on $\mathfrak{g}$ as $(X,Y)=\mathrm{tr}XY$ (as in $\mathfrak{gl}(n+1)$), then $(\alpha,\alpha)=2$ for any root $\alpha\in\Delta$ for $(\mathfrak{g},\mathfrak{h})$.

Denote by Λ^k the set of all Young diagrams $Y=[f_1,f_2,\cdots,f_k]$ with depth$(Y)\leq k$, where f_j means the number of the j-th row of Y. To any Young diagram $Y=[f_1,\cdots,f_{n+1}]\in\Lambda^{n+1}$, define the dominant integral weight $\lambda(Y)=\sum_{j=1}^{n} b_j\overline{\Lambda}_j\in P_{\ell(Y)}$ and $\ell(Y)=f_1-f_{n+1}=\sum_{j=1}^{n} b_j=(\lambda(Y),\theta)\in\mathbb{Z}_{\geq 0}$ by $b_j=f_j-f_{j+1}$ $(1\leq j\leq n)$. Each weight $\lambda\in P_\ell$ has the Young diagram expression $\lambda=\lambda(Y)$ with $Y\in\Lambda^{n+1}$ and $\ell=\ell(Y)$.

Introduce $\overline{\varepsilon}_j\in\mathfrak{h}^*$ defined by $\lambda(Y)+\overline{\varepsilon}_j=\lambda(Y+\overline{\varepsilon}_j)$, where the Young diagram $Y+\overline{\varepsilon}_j$ is

$$Y+\overline{\varepsilon}_j=[f_1,\cdots,f_{j-1},f_j+1,f_{j+1},\cdots,f_{n+1}].$$

Consider a quadruple $\mathbb{A}=(\lambda_4,\lambda(\square),\lambda(\square),\lambda_1)$ of dominant integral weights with $(\lambda_i,\theta)\leq\ell$ $(i=1,4)$, then we get that $\dim V_0^\vee(\mathbb{A})\leq 2$. Write λ_1 as $\lambda_1=\sum_{k=1}^{n} b_k\overline{\Lambda}_k=\lambda(Y)$, where $Y=[f_1,\cdots,f_{n+1}]$. Then $\dim V_0^\vee(\mathbb{A})=2$ if and only if

(D2) $\lambda_4^a=\lambda_1+\overline{\varepsilon}_i+\overline{\varepsilon}_j$ $(i<j)$, $b_{i-1}\geq 1$ and $b_{j-1}\geq 1$ ($b_0=+\infty$ for conveniece),

where λ^a is the anti-weight of λ, i.e. $-\lambda^a$ is the lowest weight of V_λ. In this case, put $d=d(\mathbb{A})=j-i+f_i-f_j$. The case (D2) is divided into the following cases:

$(D2)_{21}$ $\varepsilon_0=\frac{1}{\kappa}\left\{j-i+1+\sum_{k=i}^{j-1} b_k\right\}<1$ and $j-i<n$; $I_\ell(\mathbb{A})=\{\lambda_1+\overline{\varepsilon}_i,\ \lambda_1+\overline{\varepsilon}_j\}$

$(D2)_{22}$ $\varepsilon_0<1$, $j-i=n$ and $(\theta,\lambda_1)<\ell$; $I_\ell(\mathbb{A})=\{\lambda_1+\overline{\varepsilon}_i,\ \lambda_1+\overline{\varepsilon}_j\}$

$(D2)_1$ $\varepsilon_0=1$, $j-i=n$ and $(\theta,\lambda_1)=\ell$; $I_\ell(\mathbb{A})=\{\lambda_1+\overline{\varepsilon}_{n+1}\}$.

The condition that $\dim V_0^\vee(\mathbb{A})=1$ is divided into the two cases:

$(D1)_1$ $\lambda_4^a=\lambda_1+2\overline{\varepsilon}_i$ and $b_{i-1}\geq 2$; $I_\ell(\mathbb{A})=\{\lambda_1+\overline{\varepsilon}_i\}$

$(D1)_2$ $\lambda_4^a=\lambda_1+\overline{\varepsilon}_i+\overline{\varepsilon}_{i+1}$, $b_{i-1}\geq 1$ and $b_i=0$; $I_\ell(\mathbb{A})=\{\lambda_1+\overline{\varepsilon}_i\}$.

Proposition 7. Let $q = \exp\left(\frac{2\pi\sqrt{-1}}{\kappa}\right)$ $(\kappa=\ell+g=\ell+n+1)$.

i) Cases $(D2)_{21}$ and $(D2)_{22}$:

$$C(\Lambda) = q^{\frac{-(n+2)}{2(n+1)}} \begin{bmatrix} \gamma_+^{-1} & \\ & \gamma_-^{-1} \end{bmatrix} \begin{bmatrix} \frac{-1}{[d]} & \frac{\sqrt{q[d+1][d-1]}}{[d]} \\ \frac{\sqrt{q[d+1][d-1]}}{[d]} & \frac{q^d}{[d]} \end{bmatrix} \begin{bmatrix} \gamma_+ & \\ & \gamma_- \end{bmatrix}$$

where $[\nu]$ denotes the q-integer

$$[\nu] = \frac{q^{\nu}-1}{q-1} \quad \text{and} \quad \gamma_{\pm} = \frac{\Gamma\left(\pm\frac{d}{\kappa}\right)}{\left[\Gamma\left(\pm\frac{d+1}{\kappa}\right)\Gamma\left(\pm\frac{d-1}{\kappa}\right)\right]^{1/2}}.$$

ii) Cases $(D2)_1$ and $(D1)_2$: $\quad C(\Lambda) = -\, q^{\frac{-(n+2)}{2(n+1)}}$.

iii) Case $(D1)_1$: $\quad C(\Lambda) = q\cdot q^{\frac{-(n+2)}{2(n+1)}}$

8. Let $N\geq 2$ and fix a dominant integral weight τ with $(\tau,\theta)\leq\ell$. For the $(N+1)$-ple $\Lambda_\tau = (\tau^a, \lambda(\square), \lambda(\square), \cdots, \lambda(\square))$, introduce the set $\mathcal{P}_\ell(N;\tau)$ defined by

$$\mathcal{P}_\ell(N;\tau) = \Big\{\mathbb{p}=(\lambda_N,\cdots,\lambda_1,\lambda_0);\ \lambda_N=\tau,\ \lambda_0=0,\ \lambda_i\in P_\ell$$
$$\lambda_i=\lambda_{i-1}+\bar{\varepsilon}_j \text{ for some } j\ (1\leq i\leq N)\Big\},$$

and for each $\mathbb{p}\in\mathcal{P}_\ell(N;\tau)$, define the $V_0^\vee(\Lambda_\tau)$-valued, multi-valued holomorphic function $\Psi_{\mathbb{p}}(z_N,\cdots,z_1)$ on X_N by

$$\Psi_{\mathbb{p}}(z_N,\cdots,z_1)(v,u_N,\cdots,u_1) = \langle\nu(v)|\Phi_N(u_N;z_N)\cdots\Phi_1(u_1;z_1)|0\rangle$$

for $v\in V_{\tau^a}$ and $u_i\in V_{\lambda(\square)}$ $(1\leq i\leq N)$, where Φ_i is the vertex operator whose initial term is a unique element of $\mathcal{BV}(\mathrm{v}_i)$ for the vertex $\mathrm{v}_i(\mathbb{p})=\begin{bmatrix} & \lambda(\square) & \\ \lambda_i & & \lambda_{i-1}\end{bmatrix}$ $(1\leq i\leq N)$ and ν is the isomorphism $\nu: V_{\tau^a}\longrightarrow V_\tau^\dagger$ defined by $\nu(|-\tau^a\rangle) = \langle\tau|$ and $\nu(Xv) = \nu(v)\nu_{\mathfrak{g}}(X)$ $(v\in V_{\tau^a},\ X\in\mathfrak{g})$, where $|-\tau^a\rangle$ is the image of $|\lambda\rangle$ by the longest element of the Weyl group and $\nu_{\mathfrak{g}}$ is the anti-atomorphism of $\mathfrak{g}$ determined by $\nu_{\mathfrak{g}}(X)=-X$ $(1\le i\le n)$.

Let $W(N;\tau)$ be the space spanned by $\{\Psi_{\mathbb{p}}(z_N,\cdots,z_1);\ \mathbb{p}\in\mathcal{P}_\ell(N;\tau)\}$, then $\{\Psi_{\mathbb{p}}(\mathbb{z});\ \mathbb{p}\in\mathcal{P}_\ell(N;\tau)\}$ gives a basis of $W(N;\tau)$ and the space $W(N;\tau)$ coincides with the solution space of the analogous equations as $E(\Lambda)$ and $B(\Lambda)$ in §5.

The braid group B_N of N strings of Artin is the fundamental group of the quotient space of X_N by the $\mathfrak{S}_N$-action: $(z_N,\cdots,z_1)\sigma = (z_{(N)\sigma},\cdots,z_{(1)\sigma})$, $\sigma\in\mathfrak{S}_N$. Hence the group B_N acts on the space $W(N;\tau)$ as monodromies. The commutation relations of vertex operators gives a factorization of this monodromy representation $(\pi_{N,\tau},W(N;\tau))$. By the explicit formulae of the representation $\pi_{N,\tau}$ obtained from Proposition 7, we get

Theorem 8. Let $q = \exp(\frac{2\pi\sqrt{-1}}{\kappa})$.

i) The monodromy representation $q^{\frac{n+2}{2(n+1)}}\pi_{N,\tau}$ of the braid group B_N on the space $W(N;\tau)$ gives an irreducible and unitarizable representation of B_N.

ii) This representation factorizes to a representation of the Hecke algebra $H_N(q)$ of type A_{N-1}.

iii) Our representation $(q^{\frac{n+2}{2(n+1)}}\pi_{N,\tau},W(N;\tau))$ of the Hecke algebra $H_N(q)$ is equivalent to the representation $(\pi_Y^{(g,\kappa)},V_Y^{(g,\kappa)})$ constructed by H. Wenzl[5], where $\tau = \lambda(Y)$ for any $Y\in\Lambda_N^{(g,\kappa)}$, that is, for any Young diagram $Y=[f_1,\cdots,f_g]$ on N nodes with $f_1-f_g\le\kappa-g\,(=\ell)$.

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