

Combinatorics on $P_\kappa \lambda$

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It is known that if $P_\kappa \lambda$ has the partition property, then κ is λ -ineffable. (Magidor [2]) We shall discuss the converse direction under the existence of stationary codings.

Definition. Let $X \subset P_\kappa \lambda = \{x \subset \lambda : |x| < \kappa\}$.

- (i) We call a function $F: \{(x,y) : x, y \in X \text{ and } x \subseteq y\} \rightarrow 2$ a partition of X . $H \subset X$ is homogeneous for F if there is a $k < 2$ so that $F((x,y)) = k$ for every x, y in H with $x \subseteq y$.
- (ii) X has the partition property ($\text{Part}^*(X)$) if every partition of X has a stationary homogeneous set.
- (iii) X is λ -ineffable if for any $f: X \rightarrow P_\kappa \lambda$ with $f(x) \subset x$ for every $x \in X$ there is an $A \subset \lambda$ so that $\{x \in X : f(x) = x \cap A\}$ is stationary.
- (iv) Let $t: \lambda \rightarrow P(X)$. t' is a flip of t if $t'(\alpha) = t(\alpha)$ or $X - t(\alpha)$ for all $\alpha < \lambda$.
- (v) X has the flipping property if every $t: \lambda \rightarrow P(X)$ has a flip t' so that $\bigtriangleup_{\alpha < \lambda} t'(\alpha) = \{x \in X : x \in t'(\alpha) \text{ for all } \alpha \in x\}$ is stationary.
- (vi) X has a stationary coding if X is stationary and there is a injective $c: X \rightarrow \lambda$ such that $c(x) \in y$ whenever $x \subseteq y$.

Lemma (DiPrisco, Zwicker [1]). (iii) and (v) are equivalent.

Combining the flipping property and SC(stationary coding), we get;

Proposition. If X is λ -ineffable with SC, then $\text{Part}^*(X)$.

Proof. Suppose that X is λ -ineffable and $c: X \rightarrow \lambda$ is a stationary coding. For a partition F of X , we define $t: \lambda \rightarrow P(X)$ as follows.

$$t(\alpha) = \begin{cases} \{x: F(c^{-1}(\alpha), x) = 1\} & \text{if } \alpha \in \text{range}(c) \\ X & \text{otherwise} \end{cases}$$

Since X has the flipping property, there is a flip t' of t so that $\Delta_{\alpha < \lambda} t'(\alpha) = S$ is stationary.

Let $S_1 = \{x \in S: t'(c(x)) = t(c(x))\}$ and $S_2 = S - S_1$. Either S_1 or S_2 is stationary.

We shall show that both of them are homogeneous for F .

Suppose that $x \xi y \in S_1$. Since $c(x) \in y$ and $y \in S$, $y \in t'(c(x)) = t(c(x))$. Hence $F(x, y) = 1$.

If $x \xi y \in S_2$, $y \in t'(c(x)) = X - t(c(x))$. Thus $F(x, y) = 0$. \square

But it is open whether a λ -ineffable set with SC exists whenever κ is λ -ineffable. If κ is λ -supercompact, the answer is of course "Yes". In fact, every X with normal measure one is λ -ineffable and there is a Y with normal measure one on which $\{\langle x, \text{sup}(x) \rangle: x \in Y\}$ is a stationary coding. ([2] and [4])

Shelah proved the following.

Proposition (Shelah [3]). If κ is ineffable, $\lambda \xrightarrow{w} (\omega)_{\kappa}^{<\omega}$ and $\lambda^{<\kappa} = \lambda$, then there is a stationary coding set.

Moreover it is well known that;

Proposition. If $\text{Part}^*(X)$, there is a $Y \subset X$ with SC.

Proof. Define F by

$$F(x,y) = \begin{cases} 1 & \text{if } c(x) \in y \\ 0 & \text{otherwise.} \end{cases}$$

(c is any injective map from X to λ .) Let Y be a stationary homogeneous set for F . Pick an $x \in Y$. Since Y is unbounded, there is a $y \in Y$ such that $c(x) \in y$. Hence $F(x,y) = 1$. This means $F(x,y) = 1$ for all $x \zeta y$ in Y . Now we have shown that c is a stationary coding for Y .

The above proposition shows that every injection of $P_\kappa \lambda$ into λ can be a stationary coding if $P_\kappa \lambda$ has the partition property.

References.

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