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# Cardinality restrictions of preradicals
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## 1. Definitions and summary

A preradical $T$ is a subfunctor of the identity for abelian groups, i.e., $TA$ is a subgroup of $A$ for each abelian group $A$ and $hTA$ is a subgroup of $TB$ for any $h \in \mathrm{Hom}(A,B)$. For a cardinal $\kappa$, let $T^{[\kappa]}A = \Sigma\{TX\colon X$ is a subgroup of $A$ and $X$ is $<\kappa$-generated$\}$. ($X$ is $<\kappa$-generated, if there exists a set of generators for $X$ whose cardinality is strictly smaller than $\kappa$.) Then, $T^{[\kappa]}$ is also a preradical. It is a subfunctor of $T$ and $T^{[\kappa][\kappa]} = T^{[\kappa]}$ holds. We say that $T$ satisfies the cardinality condition (abbreviated by the c.c.), if there exists a cardinal $\kappa$ such that $T = T^{[\kappa]}$.

In the present paper we investigate the notion $T^{[\kappa]}$ for preradicals $T$. Though some results also hold for R-modules over any ring $R$, others need some restrictions. Since the main interest of this paper is around abelian groups, we confine ourselves only to abelian groups. (Except the finitely generated case, the restrictions are only related to the cardinality of the ring $R$.) To state the main results some definitions are necessary. For preradicals $S$ and $T$, $S{\cdot}T$ is the composition and $S{:}T$ is the cocomposition, i.e. $S{\cdot}T\,A = S(TA)$ and $S{:}TA = \sigma^{-1}S(A/TA)$ where $\sigma\colon A \to A/TA$ is the canonical homomorphism. A preradical is socle, if $T{\cdot}T = T$. $T$ is a radical, if $T{:}T = T$. Let $T^{\alpha+1} = T^{\alpha}{\cdot}T$ for an ordinal $\alpha$, $T^{\alpha}A = \cap_{\beta<\alpha}T^{\beta}A$ for a limit ordinal $\alpha$ and $T^{\infty}A = T^{\alpha}A$, where $T^{\alpha}A = T^{\alpha+1}A$. Dually, let $T^{(\alpha+1)} = T{:}T^{(\alpha)}$, $T^{(\alpha)}A = \Sigma\{T^{(\beta)}A\colon \beta < \alpha\}$ for a limit ordinal $\alpha$ and $T^{(\infty)}A = T^{(\alpha)}A$, where $T^{(\alpha)}A = T^{(\alpha+1)}A$. Though we shall state the definition of Vopénka's principle shortly in Section 2, we refer the reader to [11, 13] for more information and logical and set theoretical background. A cardinal $\kappa$ is regular, if its cofinality is $\kappa$ itself and $\kappa$ is singular, otherwise. $\kappa$ is a strongly limit cardinal, if $2^{\lambda} < \kappa$ for any cardinal $\lambda < \kappa$. Undefined notion and notation is standard [10, 11] and all groups in this paper are abelian.

Theorem 1.1. Under Vopénka's principle, any preradical satisfies the cardinality condition.

Theorem 1.2. Let $\kappa$ be a regular or finite cardinal. For preradicals $S$ and $T$, $(S{\cdot}T)^{[\kappa]} = S^{[\kappa]}{\cdot}T^{[\kappa]}$ and $(S{:}T)^{[\kappa]} = S^{[\kappa]}{:}T^{[\kappa]}$. Hence, if $T$ is a socle, so is $T^{[\kappa]}$ and if $T$ is a radical, so is $T^{[\kappa]}$.

Corollary 1.3. (The first half is in [9]) If preradicals $S$ and $T$ satisfy the c.c., then both $S \cdot T$ and $S{:}T$ also satisfy the c.c..

Corollary 1.4. Let $\kappa$ be a regular or finite cardinal and $T$ a preradical. Then, $T^{[\kappa]\alpha} = T^{\alpha[\kappa]}$, $T^{[\kappa](\alpha)} = T^{(\alpha)[\kappa]}$ for an ordinal $\alpha$ and consequently $T^{[\kappa]\infty} = T^{\infty[\kappa]}$, $T^{[\kappa](\infty)} = T^{(\infty)[\kappa]}$.

These answer a few questions in [9]. In the second half of this paper we shall investigate the preradicals $R_Z{}^{[\kappa]}A$, where $R_Z A = \bigcap\{\mathrm{Ker}(h): h \in \mathrm{Hom}(A,Z)\}$.

Theorem 1.5. $R_Z{}^{[\aleph_1]}$ is not equal to $R_Z{}^{[\aleph_2]}$; $(R_Z{}^{[\kappa]})^{\alpha}$ is not equal to $(R_Z{}^{[\kappa]})^{\beta}$ for any $\kappa > \aleph_1$ and distinct ordinals $\alpha, \beta$; $(R_Z{}^{[\aleph_1]})^{\infty}$ is not equal to $(R_Z{}^{[\aleph_2]})^{\infty}$.

Theorem 1.6. If $\kappa$ is a singular strongly limit cardinal which is less than the least measurable cardinal, then $R_Z{}^{[\kappa]}$ is not a radical.

## 2. General results

First we state Vopénka's principle: Let $A_i$ $(i \in I)$ be structures for the same 1-st order language and $I$ a proper class. Then, there exist two distinct indexes $i$ and $j$ and an elementary embedding $e: A_i \to A_j$. We use this principle in the following form: Let $(A_i, S_i)$ $(i \in I)$ be pairs of groups and their subsets and $I$ a proper class. Then, there exist two distinct indexes $i$ and $j$ and an injective homomorphism $e$: $A_i \to A_j$ such that $e(S_i) \subseteq e(S_j)$.

Proof of Theorem 1.1. We suppose the negation of the conclusion and define cardinals $\kappa_\alpha$ and groups $A_\alpha$ for each ordinal $\alpha$ inductively. Let $\kappa_0 = 0$, $\kappa_\alpha = \sup\{\kappa_\beta: \beta < \alpha\}$ for a limit $\alpha$ and $|A_\alpha| < \kappa_{\alpha+1}$. Let $A_\alpha$ be the direct sum of all groups $A$ such that $TA \neq T^{[\kappa_\alpha]}A$ and $A$ have the minimal set theoretical rank among such groups. (The set theoretical rank $\rho(x) = \sup\{\rho(x): y \in x\}$.) Since $T$ commutes with direct sums, $TA_\alpha \neq T^{[\kappa_\alpha]}A_\alpha$. Now, apply Vopénka's principle to the sequence of pairs $(A_\alpha, TA_\alpha - T^{[\kappa_\alpha]}A_\alpha)$. Then, there exist distinct ordinals $\alpha, \beta$ and an injective homomorphism $e: A_\alpha \to A_\beta$ such that $e(TA_\alpha - T^{[\kappa_\alpha]}A_\alpha) \subseteq e(TA_\beta - T^{[\kappa_\beta]}A_\beta)$. The construction shows $\alpha < \beta$. $\phi \neq e(TA_\alpha - T^{[\kappa_\alpha]}A_\alpha) \subseteq (TA_\beta - T^{[\kappa_\beta]}A_\beta) \cap e(TA_\alpha)$, which contradicts to $e(TA_\alpha) \subseteq T^{[\kappa_\beta]}A_\beta$.

For consequences of Theorem 1.1, see [9]. In case $\kappa$ is an uncountable cardinal, a group $A$ is $<\kappa$-generated iff the cardinality of $A$ (denoted by $|A|$) is less than $\kappa$.

Proof of Theorem 1.2. First, we observe that $T^{[\aleph_0]}A = \bigcup\{T\langle a\rangle: a \in A\}$ by the fundamental theorem of finitely generated groups [10, Theorem 15.5]. Therefore, $T^{[2]} = T^{[\kappa]} = T^{[\aleph_0]}$ for $2 \leqq \kappa \leqq \aleph_0$. In that case $(S\cdot T)^{[\kappa]}A = \bigcup\{S\cdot T\langle a\rangle: a \in A\} =$

$S^{[\kappa]}\cdot T^{[\kappa]}A$. Next, let $\kappa$ be uncountable. $(S\cdot T)^{[\kappa]}A = \Sigma\{S\cdot TX : X \leqq A \;\&\; |X| < \kappa\} = \Sigma\{S^{[\kappa]}\cdot T^{[\kappa]}X : X \leqq A \;\&\; |X| < \kappa\} \leqq S^{[\kappa]}\cdot T^{[\kappa]}A$. $S^{[\kappa]}\cdot T^{[\kappa]}A = \Sigma\{SX: X \leqq T^{[\kappa]}A \;\&\; |X| < \kappa\}$. Since $\kappa$ is regular, for any $X \leqq T^{[\kappa]}A$ with $|X| < \kappa$, there exists a subgroup $Y$ of $A$ such that $|Y| < \kappa$ and $X \leqq TY$ and hence $SX \leqq S\cdot TY$. These imply $(S\cdot T)^{[\kappa]} = S^{[\kappa]}\cdot T^{[\kappa]}$.

For the second proposition, let $U = S^{[\kappa]}$ and $V = T^{[\kappa]}$. Then, $(S{:}T)^{[\kappa]}A = \Sigma\{S{:}TX : X \leqq A \;\&\; X$ is $<\kappa$-generated$\} = \Sigma\{S^{[\kappa]}{:}T^{[\kappa]}X : X \leqq A \;\&\; X$ is $<\kappa$-generated$\} = (U{:}V)^{[\kappa]}A$. What we must show is $U{:}VA \leqq (U{:}V)^{[\kappa]}A$. Let $\sigma: A \to A/VA$ be the canonical homomorphism. Let $2 \leqq \kappa \leqq \aleph_0$, $\sigma(a) \in S<\sigma(b)>$, $k\sigma(b) = \sigma(a)$ and $m$ be the order of $\sigma(b)$. Then, there exist elements $c, d \in A$ such that $a-kb \in T<c>$ and $mb \in T<d>$. (If $<\sigma(b)>$ is infinite cyclic, we let $d = 0$.) Let $X = <a, b, c, d>$ and $\tau: X \to X/TX$ be the canonical homomorphism. Since $TX \leqq TA$, $k\tau(b) = \tau(a)$ and $m$ is the order of $\tau(b)$. Hence, $a \in U{:}VX$. Next let $\kappa$ be uncountable, $X$ a subgroup of $A/VA$ of cardinality less than $\kappa$ and $a^* \in \sigma^{-1}UX$. For an $a \in VA$, there exists a subgroup $Y_a$ of $A$ such that $|Y_a| < \kappa$ and $a \in VY_a$. Let $Y_0$ be a subgroup of $A$ such that $|Y_0| < \kappa$, $X \leqq \sigma(Y_0)$ and $a^* \in Y_0$ and let $Y_{n+1} = Y_n + \Sigma\{Y_a: a \in VA \cap Y_n\}$. Then, $|Y_{n+1}| < \kappa$ and $VA \cap Y_n \leqq VY_{n+1}$ for every $n < \omega$. Let $Y^* = \Sigma\{Y_n: n < \omega\}$ $(=\bigcup\{Y_n: n < \omega\})$. Then, $|Y^*| < \kappa$ and $VA \cap Y^* = VY^*$. Hence, there exists an injective homomorphism $i: Y^*/VY^* \to A/VA$ such that $\sigma|Y^* = i\cdot\tau$ where $\tau: Y^* \to Y^*/VY^*$ is the canonical homomorphism. Since $X \leqq i\cdot\tau(Y^*)$, $\tau(a^*) \in U\cdot Y^*/VY^*$. Therefore, $U{:}VA = \sigma^{-1}U(A/VA) = \sigma^{-1}(\Sigma\{UX: X \leqq A/VA \;\&\; |X| < \kappa\}) = \Sigma\{\sigma^{-1}UX: X \leqq A/VA \;\&\; |X| < \kappa\} \leqq (U{:}V)^{[\kappa]}A$.

Since any radical $T$ is of the form $R_{\mathcal{X}}$ for some class of abelian groups $\mathcal{X}$, i.e. $TA = \bigcap\{\mathrm{Ker}(h): h \in \mathrm{Hom}(A,X), X \in \mathcal{X}\}$, $T^{[\kappa]} = R_{\mathcal{Y}}$ for some $\mathcal{Y}$, in case $\kappa$ is a regular cardinal. Next, we show that $\mathcal{Y}$ can be gotten from $\mathcal{X}$ by using reduced products. We introduce $\kappa$-complete reduced products [2, 4]. Let $A_i$ $(i \in I)$ be abelian groups and $F$ a filter on $I$. The reduced products $\Pi_{i\in I}A_i/F$ is the quotient group $\Pi_{i\in I}A_i/K_F$, where $K_F = \{f \in \Pi_{i\in I}A_i: \{i: f(i) = 0\} \in F\}$. When $F$ is $\kappa$-complete, i.e. $X_\alpha \in F$ $(\alpha < \lambda < \kappa)$, imply $\bigcap_{\alpha<\lambda}X_\alpha \in F$, $\Pi_{i\in I}A_i/F$ is said to be a $\kappa$-complete reduced product of $A_i$ $(i \in I)$.

Theorem 2.1. Let $\kappa$ be an uncountable regular cardinal. Then, $R_{\mathcal{X}}^{[\kappa]} = R_{\mathcal{Y}}$, where $\mathcal{Y}$ is the class of all $\kappa$-complete reduced products of elements of $\mathcal{X}$.

Proof. Suppose that $a \notin R_{\mathcal{X}}^{[\kappa]}A$ for an $a \in A$. Let $S$ be a subset of $A$ of cardinality less than $\kappa$ which contains $a$, then there exist an $X_S \in \mathcal{X}$ and an homomorphism $h_S: <S> \to X_S$ such that $h_S(a) \neq 0$. According to a canonical construction of reduced products, let $P_\kappa A$ be the set of all subsets of cardinality less than $\kappa$ and $F$ the $\kappa$-complete filter generated by all the $U_x$'s where $U_x = \{S$

$\in P_\kappa A: x \in S\}$ $(x \in A)$. We set $X_S = 0$ and $h_S = 0$ for $a \notin S \in P_\kappa A$ and $Y = \Pi_{S \in P_\kappa A} X_S / F$. Then, $(h_S: S \in P_\kappa A)$ naturally defines a homomorphism $h: A \to Y$ such that $h(a) \neq 0$. More precisely, $h(x) = [(h_S': S \in P_\kappa A)]_F$, where $h_S' = h_S$ for $x \in S$ and $h_S' = 0$ otherwise and $[\ \ ]_F : \Pi_{S \in P_\kappa A} X_S \to Y$ is the canonical homomorphism. It is easy to check that $h$ is a homomorphism and $h(a) \neq 0$. Now, we have shown $R_{\mathcal{Y}}A \leqq R_{\mathcal{X}}^{[\kappa]}A$.

For the converse, let $a \in R_{\mathcal{X}}^{[\kappa]}A$, then there exists a subgroup $S$ of $A$ such that $|S| < \kappa$ and $a \in R_{\mathcal{X}}S$. Put a homomorphism $h: A \to Y$ for an $Y \in \mathcal{Y}$ and think of the restriction $h|S$. There exist $A_i$ $(i \in I)$ belonging to $\mathcal{X}$ and a $\kappa$-complete filter $F$ on $I$ such that $Y = \Pi_{i \in I} A_i/F$. Since the cardinality of $S$ is less than $\kappa$ and $\kappa$ is a regular cardinal, there exists a homomorphism $h^*: S \to \Pi_{i \in I} A_i$ such that $h = [\ \ ]_F \cdot h^*$, by [3, Lemma 2.6]. By the assumption, $\pi_i \cdot h^*(a) = 0$ for every projection $\pi_i$ to the i-th component and hence $h^*(a) = 0$ and $h(a) = 0$.

In the rest of this section we think of a dual notion of $T^{[\kappa]}$. For a preradical $T$, let $T_{[\kappa]} = \cap\{h^{-1}TX: h \in \mathrm{Hom}(A,X)$ and $X$ is $<\kappa$-generated$\}$. Then, $T_{[\kappa]}$ is also a preradical.

Proposition 2.2. Let $T$ be a preradical. $T = T_{[\kappa]}$ for some cardinal $\kappa$ iff there exist a group $G$ and its subgroup $H$ such that $TA = \cap\{h^{-1}H: h \in \mathrm{Hom}(A,G)\}$.

Proof. Suppose the second proposition holds. Since $TA \leqq T_{[\kappa]}A$ in general, $T = T_{[\kappa]}$, when $G$ is $<\kappa$-generated. For the other implication, let $\{X_i: i \in I\}$ be a representative set of $<\kappa$-generated groups, i.e. any $<\kappa$-generated group $X$ is isomorphic to some $X_i$. Let $G = \oplus_{i \in I} X_i$ and $H = \oplus_{i \in I} TX_i$ be the subgroup of $G$. Suppose $a \in \cap\{h^{-1}H: h \in \mathrm{Hom}(A,G)\}$. For an $h \in \mathrm{Hom}(A,X)$ where $X$ is $<\kappa$-generated, there exist an $i \in I$ and $h^* \in \mathrm{Hom}(A,G)$ such that $X$ is isomorphic to $X_i$ and $\pi_i \cdot h^* = h$ through this isomorphism, where $\pi_i: G \to X_i$ be the projection. Since $\pi_i \cdot h^*(a) \in TX_i$, $h(a) \in TX$. Hence, $\cap\{h^{-1}H: h \in \mathrm{Hom}(A,G)\} \leqq T_{[\kappa]}A = TA$, and the other inclusion is obvious.

Though Proposition 2.2 answers a question of [9], it does not seem that the notion $T_{[\kappa]}$ works so well as $T^{[\kappa]}$, as we shall see in the next proposition.

Proposition 2.3. If $T$ is a radical, then $T_{[\kappa]}$ is a radical for any cardinal $\kappa$. However, there exists a socle $T$ such that $T_{[\aleph_1]}$ is not a socle.

Let $\sigma: A \to A/T_{[\kappa]}A$ be the canonical homomorphism and $\sigma(a) \neq 0$. Then, there exist a group $X$ and an $h \in \mathrm{Hom}(A,X)$ such that $X$ is $<\kappa$-generated and $h(a) \notin TX$. Let $h^* = \sigma \cdot h$, where $\sigma: X \to X/TX$ is the canonical homomorphism. Then, $h^*(a) \neq 0$, $T \cdot X/TX = 0$ and $X/TX$ is $<\kappa$-generated. Hence, $T_{[\kappa]}A \leqq \mathrm{Ker}h^*$ and so there exists an $h^{**} \in \mathrm{Hom}(A/T_{[\kappa]}A, X/TX)$ such that $h^{**}\cdot\sigma = h^{*}$. Now, $h^{**}\cdot\sigma(a) = h^{*}(a) \neq 0$, which implies that $\sigma(a) \notin T_{[\kappa]}\cdot A/T_{[\kappa]}A$, and so $T_{[\kappa]}\cdot A/T_{[\kappa]}A = 0$.

For the second proposition, let $T$ be the Chase radical $\nu$, i.e. $\nu = R_{\mathcal{X}}$ where $\mathcal{X}$ is the class of $\aleph_1$-free groups, or $R_Z{}^{\infty}$. Then, $T$ is a socle in each case. For a countable group $C$, $R_Z C = R_Z{}^{\infty} C = \nu C$ by Stein's lemma [10, Corollary 19.3]. Since $T_{[\aleph_1]}A = \cap\{h^{-1}R_Z C: h \in \mathrm{Hom}(A,C),\ C \text{ is countable}\}$, $T_{[\aleph_1]} = R_Z$. As well-known and a certain example for it will appear in Section 3, $R_Z$ is not a socle.

## 3. Preradicals $R_Z{}^{[\kappa]}$

In this section we study preradicals $R_Z{}^{[\kappa]}$. A trivial remark is: $R_Z{}^{[\aleph_0]}A$ is the torsion subgroup of $A$ and hence $R_Z{}^{[\aleph_0]}$ is a radical and a socle. After studies of Dugas and Göbel [3, 4], we showed that $R_Z$ satisfies the c.c. (iff $R_Z{}^{\infty}$ satisfies the c.c.) iff there exists a strongly $L_{\omega_1\omega}$-compact cardinal [5]. In another word $R_Z = R_Z{}^{[\kappa]}$ for a strongly $L_{\omega_1\omega}$-compact cardinal $\kappa$. Bergman and Solovay [1] announced a similar result, i.e. The class of all torsionless groups is chracterized by a set of generalized Horn sentences, iff there exists a strongly $L_{\omega_1\omega}$-compact cardinal. They also commented that Magidor showed that the existence of a strongly $L_{\omega_1\omega}$-compact cardinal is strictly weaker than that of a strongly compact cardinal. We showed that the Chase radical $\nu = R_Z{}^{[\aleph_1]}$ [6] and hence $R_Z{}^{[\aleph_1]\infty} = R_Z{}^{[\aleph_1]}$. To investigate $R_Z{}^{[\aleph_2]}$, we need some lemmas and definitions. These are obtained by observing a certain group in [7, 12].

For a subgroup $S$ of $A$, $S^{*A}$ is the subgroup of $A$ defined by: $S^{*A} = \{ a \in A: h(S) = 0 \text{ implies } h(a) = 0 \text{ for any } h \in \mathrm{Hom}(A,Z)\}$. ${}^{\omega}2$ is the set of 0,1-valued functions from $\omega$ and ${}^{<\omega}2$ is the set of 0,1-valued functions from natural numbers, i.e. ${}^{<\omega}2 = \{x\upharpoonright n: n < \omega, x \in {}^{\omega}2\}$. For an element $x\upharpoonright n$ of ${}^{<\omega}2$, $\mathrm{lh}(x\upharpoonright n) = n$. $p_n$ denotes the n-th prime. Let $X$ be a subset of ${}^{\omega}2$ of cardinality $\aleph_1$ and $Y = \{x\upharpoonright n: n < \omega, x \in X\}$. $QX$ and $QY$ are the divisible hull of the free abelian group generated by $X$ and $Y$ respectively. For an element $a$ of a torsionfree group $A$, $Qa + A$ is the subgroup of the divisible hull of $A$ generated by the divisible hull of $\langle a\rangle$ and $A$.

Lemma 3.1. For an element $a$ of a torsionfree group $A$, let $A' = \langle x, y, (x - x\upharpoonright n - x(n)a)/p_n, A: x \in X, y \in Y, n < \omega \rangle$ be the subgroup of $QX \oplus QY \oplus (Qa + A)$. Then, $R_Z A' = (\langle a\rangle + R_Z A)^{*A}$.

Proof. The proof of the fact $a \in R_Z A'$ can be done by the same argument as in [7, 8.8 Theorem], but we present it here. Suppose that $h(a) \neq 0$ for some $h \in \mathrm{Hom}(A',Z)$. Let $p_n$ be a prime so that $|h(a)| < p_n$. Since $|X| = \aleph_1$, there exist distinct $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$, $x_1\upharpoonright m = x_2\upharpoonright m$ and $x_1(m) \neq x_2(m)$ for some $m \geqq n$. Now, $|h(a)| = |h(x_1 - x_1\upharpoonright m - x_1(m)a) - h(x_2 - x_2\upharpoonright m - x_2(m)a)|$ and so

$p_m$ divides $|h(a)|$, which is a contradiction. Hence, $(\langle a\rangle + R_ZA)^{*A} \leqq R_ZA'$. Suppose that $b \in A$ and $b \notin (\langle a\rangle + R_ZA)^{*A}$, then there exists an $h \in \mathrm{Hom}(A,Z)$ such that $h(b) \neq 0$ and $h(a) = 0$. Define $h^*(x) = h^*(y) = 0$ for $x \in X$ and $y \in Y$, we get an extension $h^* \in \mathrm{Hom}(A',Q)$ of $h$. Then, $h^*$ belongs to $\mathrm{Hom}(A',Z)$ and hence $b \notin R_ZA'$. Suppose $b \in A'-A$, then $\sigma(b)\neq 0$ where $\sigma: A \to A/A'$ is the canonical homomorpohism. Since $A' \cap (Qa + A) = A$, $A'/A \simeq \langle x, y, (x - x\restriction n)/p_n$: $x \in X, y \in Y, n < \omega\rangle$. There exist a finite subset $F$ of $X$ and an $n$ such that $x\restriction n \neq x'\restriction n$ for distinct $x, x' \in F$ and $\sigma(b) \in B = \langle x, y, (x-x\restriction k)/p_k$: $x \in F, k \leqq n, lh(y) \leqq n\rangle$. Since $B$ is finitely generated, there exists an $h \in \mathrm{Hom}(B,Z)$ so that $h\cdot\sigma(b) \neq 0$. Extend $h$ to $h^*: QX \oplus QY \to Q$ so that $h^*(x) = h(x\restriction n)$ for any $x \in X-F$; $h^*(y) = h(x)$ if $x\restriction lh(y) = y$ for some $x \in F$ and $lh(y) > n$; $h^*(y) = h(y\restriction n)$ if no $x \in F$ extends $y$ and $lh(y) > n$. Then, $h^*|A'/A \in \mathrm{Hom}(A'/A,Z)$ and $h^*\cdot\sigma(b) \neq 0$.

Lemma 3.2. If $A$ is $\aleph_1$-free, so is $A'$.

Proof. It is enough to show that $A'/A$ is $\aleph_1$-free. Observe that $\langle x, y, (x-x\restriction k)/p_k: x \in F, lh(y) \leqq n, k \leqq n\rangle$ is a pure subgroup of $A'/A$ for a finite $F$ and $n < \omega$. Then, $A'/A$ is $\aleph_1$-free by Pontrjagin's criterion [10, Therem 19.1].

Proof of Theorem 1.5. Let $a = 1$ and $A = Z$ in Lemma 3.1. Then, $R_ZA' = Z$, $|A'| = \aleph_1$ and $A'$ is $\aleph_1$-free by Lemma 3.2. ($A'$ is the same group in [7, 8.8 Theorem].) Since $R_Z^{[\aleph_1]}A' = vA' = 0$ by [6, Theorem 2] and $R_Z^{[\aleph_2]}A' = R_ZA' = Z$, the first proposition holds. By [8, Corollary 3.10] (due to Mines), the second proposition holds. For the third proposition, we show the existence of an $\aleph_1$-free group $A_{\omega_1}$ such that $|A_{\omega_1}| = \aleph_1$ and $\mathrm{Hom}(A_{\omega_1},Z) = 0$. This can be done by iterating the process from $A$ to $A'$ starting from $a = 1$ and $A = Z$. Let $\pi: \omega_1 \times \omega_1 \to \omega_1$ be a bijection so that $\alpha \leqq \pi(\alpha,\beta)$ and $\beta \leqq \pi(\alpha,\beta)$ for $\alpha,\beta < \omega_1$. We inductively define $A_\alpha$'s so that $A_\alpha = \{a_{\alpha\beta}: \beta < \omega_1\}$, $A_\alpha$ is $\aleph_1$-free, $A_\alpha$ is a subgroup of $A_\beta$ for $\alpha < \beta$ and $A_\alpha$ is the union of $\{A_\beta: \beta < \alpha\}$ for a limit $\alpha$. In the stage $\delta = \pi(\alpha,\beta)$, we apply the construction of Lemma 3.1 for $a = a_{\alpha\beta}$ and $A = A_\delta$. It is easy to see that $R_ZA_{\omega_1} = A_{\omega_1}$ and $|A_{\omega_1}| = \aleph_1$. The $\aleph_1$-freeness of $A_{\omega_1}$ follows from the fact that $A$ is pure in $A'$ and Pontrjagin's criterion. Now, $(R_Z^{[\aleph_1]})^\infty A_{\omega_1} = vA_{\omega_1} = 0$, but $(R_Z^{[\aleph_2]})^\infty A_{\omega_1} = (R_Z^\infty)^{[\aleph_2]}A_{\omega_1} = R_Z^\infty A_{\omega_1} = A_{\omega_1}$.

Proof of Theorem 1.6. The $2^\lambda$-$L_{\omega_1\omega}$-compactness of $\lambda$ implies that $\lambda$ is equal to or greater than the least measurable cardinal. Therefore, if $\lambda < \kappa$, then $R_Z^{[\lambda]} \neq R_Z^{[\kappa]}$ by [5, Theorem 1]. (Since the notion $\mu$-$L_{\omega_1\omega}$-compactness is only used here, we refer the reader to [2, 5] for it.) Let $\mu = \mathrm{cf}(\kappa) < \kappa$ and $\kappa_\alpha$ ($\alpha < \mu$) an increasing cofinal sequence for $\kappa$ such that $\kappa_\alpha$ is regular, $R_Z^{[\kappa_\alpha]} \neq R_Z^{[\kappa_{\alpha+1}]}$ and $2^{\kappa_\alpha} < \kappa_{\alpha+1}$. This can be done, because $R_Z^{[\kappa]}A = \Sigma\{R_Z^{[\lambda]}A: \lambda < \kappa\}$. Then, since $R_Z^{[\kappa_\alpha]}$ is a radical, there exist groups $Y_\alpha$ ($\alpha < \mu$) such that $|Y_\alpha| < \kappa_{\alpha+1}$, $R_Z^{[\kappa_\alpha]}Y_\alpha = 0$, $R_Z Y_\alpha \neq 0$ and $Y_\alpha$ is torsionfree. Let $Y = \Pi_{\alpha<\mu}Y_\alpha$. If $\kappa_\alpha > \mu$, $R_Z^{[\kappa_\alpha]}Y = R_Z(\Pi_{\beta<\alpha}Y_\beta) \oplus R_Z^{[\kappa_\alpha]}(\Pi_{\beta\geqq\alpha}Y_\beta)$. Let $X \leqq \Pi_{\beta\geqq\alpha}Y_\beta$ and $|X| < \kappa_\alpha$. Then, $X \leqq \Pi_{\beta\geqq\alpha}\pi_\beta X$, where $\pi_\beta$ is the projection to the $\beta$-th component. Since $R_Z$ commutes with products whose index sets are of cardinality less than the least measurable cardinal [3, Theorem 2.4] and $R_Z\pi_\beta X = 0$ for $\beta \geqq \alpha$, $R_Z^{[\kappa_\alpha]}Y = R_Z(\Pi_{\beta<\alpha}Y_\beta) = \Pi_{\beta<\alpha}R_Z Y_\beta$. Hence, $R_Z^{[\kappa]}Y = \{f \in \Pi_{\alpha<\mu}R_Z Y_\alpha : |\{\alpha : f(\alpha) \neq 0\}| < \mu\}$. $R_Z Y_\alpha$ contains a subgroup isomorphic to $Z$ and so $Y/R_Z^{[\kappa]}Y$ contains a subgroup isomorphic to $Z^\mu/Z^{<\mu}$ where $Z^{<\mu} = \{f \in Z^\mu : |\{\alpha < \mu : f(\alpha) \neq 0\}| < \mu\}$. Since $R_Z^{[\kappa]}(Z^\mu/Z^{<\mu}) = R_Z(Z^\mu/Z^{<\mu}) = Z^\mu/Z^{<\mu}$, $R_Z^{[\kappa]}\cdot Y/R_Z^{[\kappa]}Y \neq 0$.

It seems possible that $R_Z^{[\aleph_\omega]}$ would be a radical. In that case $R_Z^{[\aleph_\omega]} = R_Z^{[\aleph_{n+1}]}$ and $\aleph_\omega < 2^{\aleph_n}$ for some $n$ by an observation of the proof of Theorem 1.6 and [5, Theorem 1].

Problem: Is it consistent with ZFC that $R_Z^{[\aleph_\omega]} = R_Z^{[\aleph_2]}$ or $R_Z^{[\aleph_\omega]}$ is a radical?

Under the scope of [5, Theorem 1], there is a closely related and a little bit stronger question, i.e., Is it consistent with ZFC that $\aleph_2$ is $\aleph_n$-$L_{\omega_1\omega}$-compact for every $n < \omega$? However, during the conference of Logic and its applications at Kyoto in 1987 Hugh Woodin kindly taught me that this does not hold. More precisely, if $\kappa$ is a cardinal less than the least regular limit cardinal, $\kappa$ is not $\kappa$-$L_{\omega_1\omega}$-compact.

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