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A note on nomal form of nonsingular plane quartic curve

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We alredy know the classification of complex projective plane cubic curves. However, so-called "normal form" defining equations were not unique. And a variety of moduli of complex projective nonsingular plane quartic curves has dimension 6. The need for a unique normal form may be questioned.

So, in this paper, we try to impose a condition to constract a unique normal form. With a definition of normal forms, we arrange the classification of plane cubic curves and find the normal form of nonsingular plane quartic curve by using REDUCE.

§ 1. Introduction

1.1 Classification of complex projective plane cubic curves

Let $\mathbb{P}^2$ be a 2-dimensional complex projective space with a coordinate [x,y,z]. Then we can list the types of complex projective plane cubic curves: Nodal curve, Cuspdal curve, Conic and Chord, Conic and tangent, Three general lines, Three concurrent lines, Multiple and single lines, Triple line and Nonsingular elliptic curve. The defining equations are as follows:

Nodal curve　　$x^3+y^3+xyz=0$

Cuspidal curve　　$x^3+yz^2=0$

Conic and chord	$x^3+xyz=0$
Conic and tangent	$x^2z+yz^2=0$
Three general lines	$xyz=0$
Three concurrent lines	$x^3+xz^2=0$
Multiple and single lines	$x^2y=0$
Triple line	$x^3=0$
Nonsingular elliptic curve	$x^3+y^3+z^3+3\lambda xyz=0$, $\lambda^3+1\neq 0$ (see [1])

The defining equation of nonsingular elliptic curve in Weierstrass normal form is as follows:

$$y^2z=x^3+pxz^2+qz^3, \quad 4p^3+27q^2\neq 0.$$

And $$4p^3/(4p^3+27q^2)=(8\lambda-\lambda^4)^3/64(1+\lambda^3)^3 \quad \text{(j-invariant).}$$

This fact is well known. (see [5])

1.2 Variety of moduli of nonsingular elliptic curve

The most important single invariant of a curve is its genus. There are several ways of defining it, all equivalent. For a curve X in projective space, we have the arithmetic genus $p_a(X)$, defined as $1-P_X(0)$, where P_X is the Hilbert polynomial of X. On the other hand, we have the geometric genus $p_g(X)$, defined as $\dim_k \Gamma(X,\omega_X)$, where ω_X is the canonical sheaf.

If X is a curve, then $p_a(X)=p_g(X)=\dim_k(X,\mathcal{O}_X)$, so we call this simply the genus of X, and denote it by g.

For fixed g one would like to endow the set $\mathfrak{M}_g$ of all curves of genus g up to isomorphism with an algebraic structure, in which case we call $\mathfrak{M}_g$ the variety of moduli of curves of genus g.

Let g=3. Then the hyperelliptic curves form an irreducible subvariety of dimension 5 of $\mathfrak{M}_3$. The nonhyperelliptic curves of genus 3 are the nonsingular plane quartic curves. Since the embedding is canonical, two of them are isomorphic as abstract curves if and only if they differ by an automorphism of $\mathbb{P}^2$. The family of all these curves is parametrized by an open set $U \subset \mathbb{P}^N$ with $N=14$, because a form of degree 4 has 15 cofficients. So there is a morphism $U \to \mathfrak{M}_3$, whose fibres are images of the group PGL(2) which has dimension 8. Since any individual curve has only finitely many automorphisms, the fibres have dimension =8, and so the image of U has dimension 14 - 8 = 6. So we confirm that $\mathfrak{M}_3$ has dimension 6. (see [2])

1.3 Elimination method

Let $f_1, \cdots, f_N$ be elements of the polynomial ring $R=I[X_1, \cdots, X_m, Y_1, \cdots, Y_n]$ in m+n variables over an integral domain I. For each maximal ideal $\mathfrak{m}$ of I, let $\varphi_{\mathfrak{m}}$ be the canonical homomorphism with modulus $\mathfrak{m}$, and let $\Omega_{\mathfrak{m}}$ be an algebraically closed field containing $I/\mathfrak{m}$. Let $W_{\mathfrak{m}}$ be the set of points$(a_1, \cdots, a_n)$ of the n-dimensional affine space $\Omega_{\mathfrak{m}}^n$ over $\Omega_{\mathfrak{m}}$ such that the system of equations $\varphi_{\mathfrak{m}}(f_i)(X_1, \cdots, X_m, a_1, \cdots, a_n)=0$ $(i=1,2,\cdots,N)$ has a solution in $\Omega_{\mathfrak{m}}^m$. To eliminate $X_1, \cdots, X_m$ from $f_1, \cdots, f_N$ is to obtain $g(Y_1, \cdots, Y_n) \in I[Y_1, \cdots, Y_n]$ such that every point of $W_{\mathfrak{m}}$ is a zero point of $\varphi_{\mathfrak{m}}(g)$ for every $\mathfrak{m}$; such a g (or an equation g=0) is called a resultant of $f_1, \cdots, f_N$. The set $\mathfrak{a}$ of resultants forms an ideal of $I[Y_1, \cdots, Y_n]$, and $(g_1, \cdots, g_M)$ is called a system of resultants if the radical of the ideal generated by it coincides with $\mathfrak{a}$. If I is finitely generated over a field, then, denoting by $\mathfrak{b}$ the radical of the ideal generated by $f_1, \cdots, f_N$, we have $\mathfrak{a}=\mathfrak{b} \cap I[Y_1, \cdots, Y_n]$. In particular, let I be a field. It is obtain that $W_{(0)}$ is contained in the set V of zero points of $\mathfrak{a}$. However, it is not necessarily true that $V=W_{(0)}$. If every f_i is homogeneous in $X_1,\cdots,X_m$ and also in $Y_1,\cdots,Y_n$, then we have $V=W_{(0)}$.

If we wish to write a system of resultants explicity, we can proceed as follows: Regard the f_i as polynomials in X_1 with cofficients in $I[X_2,\cdots,X_m,Y_1,\cdots,Y_n]$, and obtain resultants $R(f_i,f_j)$ by eliminating X_1 from the pairs f_i,f_j. Then eliminate X_2 from these resultants, and so forth. To obtain $R(f_i,f_j)$, we may use Sylvester's elimination method.

Theorem 1.3.1 Let f and g be polynomials in x with cofficients in I: $f=a_0x^m+a_1x^{m-1}+\cdots+a_m$, $g=b_0x^n+b_1x^{n-1}+\cdots+b_n$. Let D(f,g) be the following determinant of degree m+n:

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_m & 0 & & \cdots & & 0 \\ 0 & a_0 & \cdots & a_{m-1} & a_m & 0 & \cdots & & 0 \\ & & & & \cdots & & & & \\ & & & & \cdots & & & & \\ 0 & & \cdots & 0 & a_0 & a_1 & \cdots & a_{m-1} & a_m \\ b_0 & b_1 & \cdots & & b_{n-1} & b_n & 0 & \cdots & 0 \\ & & & & \cdots & & & & \\ 0 & \cdots & 0 & b_0 & b_1 & & \cdots & b_{n-1} & b_n \end{vmatrix}$$

Then D(f,g)=0 if and only if either f and g have a common root or $a_0=b_0=0$.

(see [4])

1.4 Singular point

We review some theorems and definitions about quasihomogeneous polynomials which are given in [3].

Definition 1.4.1 Let $f(z_0,\cdots,z_n)$ be a polynomial in $\mathbb{C}^{n+1}$ and let V be an analytic set such that $V=\{(z_0,\cdots,z_n)\mid f(z_0,\cdots,z_n)=0\}$. Then a point $(z_0,\cdots,z_n)$ in $\mathbb{C}^{n+1}$ is a singular point if $f(z_0,\cdots,z_n)$ $=0$ and $\partial f(z_0,\cdots,z_n)/\partial z_i=0$, $i=0,\cdots,n$.

Definition 1.4.2 Suppose that $(r_0,\cdots,r_n)$ are fixed positive rational number. A polynomial $f(z_0,\cdots,z_n)$ is said to be quasihomogeneous of type$(r_0,\cdots,r_n)$ if it can be expressed as a linear combination of monomials $z_0^{i_0}z_1^{i_1}\cdots z_n^{i_n}$ for which $i_0r_0+i_1r_1+\cdots+i_nr_n=1$.

Let d denote the smallest positive integer so that

$$r_0=\frac{q_0}{d},\quad r_1=\frac{q_1}{d},\quad \cdots,\quad r_n=\frac{q_n}{d}$$

are integers. Then $f(t^{q_0}z_0,\cdots,t^{q_n}z_n)=t^d f(z_0,\cdots,z_n)$.

Theorem 1.4.3 Let $f(z_0,z_1,z_2)$ be a polynomial in $\mathbb{C}^3$ and let V be an anlytic set such that $V=\{(z_0,z_1,z_2)\mid f(z_0,z_1,z_2)=0\}$ which has an isolated singular point at the origin . Then, for any $i(i=0,1,2)$,

(i) There exists an integer a_i so that $a_i\geq 2$, and f has a monomial $z_i^{a_i}$.

or

(ii) There exists an integer $a_i\geq 1$ and $j(i\neq j)$ and f has a monomial $z_i^{a_i}z_j$.

Corollary 1.4.4 Let $g(z_0,z_1,z_2)$ be a quasihomogeneous polynomial in $\mathbb{C}^3$ and let V be an analytic set such that $V=\{(z_0,z_1,z_2) \mid g(z_0,z_1,z_2)=0\}$ which has an isolated singular point at the origin. Then g has at least one of the followins sets (family I~VIII) of monomials:

family	set of monomials	r_0	r_1	r_2
I	$z_0^{a_0}$ $z_1^{a_1}$ $z_2^{a_2}$	$\frac{1}{a_0}$	$\frac{1}{a_1}$	$\frac{1}{a_2}$
II	$z_0^{a_0}$ $z_1^{a_1}$ $z_1z_2^{a_2}$	$\frac{1}{a_0}$	$\frac{1}{a_1}$	$\frac{a_1-1}{a_1a_2}$
III	$z_0^{a_0}$ $z_1^{a_1}z_2$ $z_1z_2^{a_2}$ $a_1\geq 2, a_2\geq 2$	$\frac{1}{a_0}$	$\frac{a_2-1}{a_1a_2-1}$	$\frac{a_1-1}{a_1a_2-1}$
IV	$z_0^{a_0}$ $z_0z_1^{a_1}$ $z_1z_2^{a_2}$	$\frac{1}{a_0}$	$\frac{a_0-1}{a_0a_1}$	$\frac{a_0a_1-a_0+1}{a_0a_1a_2}$
V	$z_0^{a_0}$ $z_0z_1^{a_1}$ $z_0z_2^{a_2}$	$\frac{1}{a_0}$	$\frac{a_0-1}{a_0a_1}$	$\frac{a_0-1}{a_0a_2}$
VI	$z_0^{a_0}z_1$ $z_0z_1^{a_1}$ $z_0z_2^{a_2}$ $a_0\geq 2, a_1\geq 2$	$\frac{a_1-1}{a_0a_1-1}$	$\frac{a_0-1}{a_0a_1-1}$	$\frac{(a_0-1)a_1}{(a_0a_1-1)a_2}$
VII	$z_0^{a_0}z_1$ $z_1^{a_1}z_2$ $z_0z_2^{a_2}$	$\frac{a_1a_2-a_2+1}{a_0a_1a_2+1}$	$\frac{a_2a_0-a_0+1}{a_0a_1a_2+1}$	$\frac{a_0a_1-a_1+1}{a_0a_1a_2+1}$
VIII	$z_0^{a_0}$ z_1z_2	$\frac{1}{a_0}$	r_1	r_2

§ 2. Normal form

In this section, we try to impose a condition to construct a unique normal form of homogeneous polynomial and classify complex projective cubic curves by using REDUCE.

2.1 Normal form to be unique

Let $f=\sum a_i X^{K_i}$ be a homogeneous polynomial. We give a following order to the monomials of f.

Definition 2.1.1 For the exponents $K_i=k_{i_1},\cdots,k_{i_n}$ and $K_j=k_{j_1},\cdots,k_{j_n}$ ($i\neq j$), X^{K_i} is grater than X^{K_j} if there exists an integer $s(1\leq s\leq n)$ such that $k_{i_\mu}=k_{j_\mu}$ for $\mu=1,\cdots,s-1$ and $k_{i_s}>k_{j_s}$.
(Lexicographic linear order)

Manipulation 2.1.2 We try to a monomial X^{K_i} vanish by suitable linear transformations from the maximal i. Then if we can make the monomial X^{K_i} vanish without generating new monomial X^{K_j} ($K_i<K_j$) of f, we do so. Otherwise, we don't use the linear transformation and go to next manipulation.

Manipulation 2.1.3 If we can make the coefficient of the monomial X^{K_i} equal to 1 without generating new dimension of coefficient of monomial X^{K_j} ($K_i<K_j$), we do so. Otherwise, there is nothing to be done.

Definition 2.1.4 We repeat these manipulations in turn for i. Then f is said to be the normal form if the result of these manipulations is equal to f.

We consider it natural that the normal form should be easy to write and remember; that is, the normal form should have the fewest monomials, and each monomial should be simple. The normal forms defined in Definition 2.1.4 meet the above conditions.

2.2 Arragement of classification

We arrange the classification of complex projective plane cubic curves (cubic forms) in § 1, 1.1.

Let f(x,y,z) be a cubic form in $\mathbb{P}^2$. The cubic form f(x,y,z) takes the following form

$a_1x^3+a_2x^2y+a_3x^2z+a_4xy^2+a_5xyz+a_6xz^2+a_7y^3+a_8y^2z+a_9yz^2+a_{10}z^3=0.$

Step 1

We may choose coordinates so that

$a_1x^3+a_2x^2y+a_3x^2z+a_4xy^2+a_5xyz+a_6xz^2+a_7y^3+a_8y^2z+a_9yz^2+z^3=0.$

Replacing z by $z'+\lambda x$ where λ is a solution of $\lambda^3+a_6\lambda^2+a_3\lambda+a_1=0$, we reduce the form to

$g_2x^2y+g_3x^2z+g_4xy^2+g_5xyz+g_6xz^2+a_7y^3+a_8y^2z+a_9yz^2+z^3=0.$

$g_2\neq0$ and $g_3\neq0$ → Step 2

$g_2\neq0$ and $g_3=0$ → Step 3

$g_2=0$ and $g_3=0$ → Step 4 (see Program list 1)

Step 2

$a_1x^2y+x^2z+a_2xy^2+a_3xyz+a_4xz^2+a_5y^3+a_6y^2z+a_7yz^2+a_8z^3=0$ [By a magnification of the x- y- and z- coordinates we can reduce the form to $a_1x^2y+x^2z+a_2xy^2+a_3xyz+a_4xz^2+a_5y^3+a_6y^2z+a_7yz^2+a_8z^3=0$].

Replacing z by $z'-a_1y$, we reduce the form to

$x^2z+g_4xy^2+g_5xyz+a_4xz^2+g_7y^3+g_8y^2z+g_9yz^2+a_8z^3=0.$

$g_4\neq0$ → Step 5

$g_4=0$ and $g_7\neq0$ → Step 6

$g_4=0$ and $g_7=0$ → Step 7

Step 3

$x^2y+a_1xy^2+a_2xyz+a_3xz^2+a_4y^3+a_5y^2z+a_6yz^2+a_7z^3=0.$

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $x^2z+a_3xy^2+a_2xyz+a_1xz^2+a_7y^3+a_6y^2z+a_5yz^2+a_4z^3=0$.

$a_3\neq0 \to$ Step 5, $a_3=0$ and $a_7\neq0 \to$ Step 6, $a_3=0$ and $a_7=0 \to$ Step 7

Step 4

In this case, the form is as follows: $f_2(y,z)x+f_3(y,z)=0$ where $f_i(y,z)$ denotes a homogeneous polynomial of degree i. $f_2(y,z)=0$ gives two points on a projective line. By the definition 2.1.4 for normal forms we obtain the following classification.

$f_2(y,z)\sim yz \to$ Step 10, $f_2(y,z)\sim z^2 \to$ Step 11, $f_2(y,z)\equiv0 \to$ Step 12

Step 5

$x^2y+xy^2+a_1xyz+a_2xz^2+a_3y^3+a_4y^2z+a_5yz^2+a_6z^3=0.$

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\alpha&\beta&1\\\gamma&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $g_1x^3+g_2x^2y+g_3x^2z+g_4xy^2+g_5xyz+xz^2+g_7y^3+yz^2=0$.

Now, $g_1=\alpha^2+a_1\gamma\alpha+a_2\alpha+\alpha\gamma^2+a_3\gamma^3+a_4\gamma^2+a_5\gamma+a_6$

$g_2=a_1\alpha+a_1\beta\gamma+a_2\beta+3a_3\gamma^2+2a_4\gamma+a_5+2\alpha\beta+2\alpha\gamma+\beta\gamma^2$

$g_4=\beta^2+a_1\beta+2\beta\gamma+3a_3\gamma+a_4+\alpha.$

We solve the equation g_1 for a variable α and solve the equation g_4 for a variable β. And we substitute the solutions to the equation g_2, arrange the equation g_2. Then the equation has the ninth degree for a variable γ, we denote the equation by g_2'.

Here let $c_i:=$[coefficient of a term γ^i of g_2'] $(0\le i\le9)$. Then $c_9=a_1^2a_3-3a_1a_3^2-a_1a_4-a_2a_3+2a_3^3+2a_3a_4+a_5.$

We put $a_5:=-a_1^2a_3+3a_1a_3^2+a_1a_4+a_2a_3-2a_3^3-2a_3a_4$. Then $c_9=0$ and

$c_8=3(a_1^2a_3^2+a_1a_2a_3-2a_1a_3^3-2a_1a_3a_4-a_2a_3^2-a_2a_4+a_3^4+2a_3^2a_4+a_4^2+a_6)$.

We put $a_6:=-a_1^2a_3^2-a_1a_2a_3+2a_1a_3^3+2a_1a_3a_4+a_2a_3^2+a_2a_4-a_3^4-2a_3a_4-a_4$. Then $c_i=0$ $(0\leq i\leq 8)$. Hence, for any parameters $(a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{C}^6$, there exists the solution of g_2'.

We set $g_1=0, g_2=0$ and $g_4=0$. Then we reduce the form to $h_3x^2z+h_5xyz+xz^2+h_7y^3+y^2z:=0$.

$h_3\neq 0$ and $h_7\neq 0$ → Step 6, $h_3\neq 0$ and $h_7=0$ → Step 7, $h_3=0$ → Step 4

(see Program list 2)

Step 6

$x^2y+a_1xyz+a_2xz^2+y^3+a_3y^2z+a_4yz^2+a_5z^3=0$.

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&\alpha&\beta\\0&1&\gamma\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $x^2z+g_1xyz+g_2xz^2+y^3+g_3y^2z+g_4yz^2+g_5z^3=0$.

We set $g_1=g_2=g_3=0$. $g_4\neq 0$ or $g_5\neq 0$ → Step 8, $g_4=0$ and $g_5=0$→ Step 9

Step 7

$x^2z+a_1xyz+a_2xz^2+a_3y^2z+a_4yz^2+a_5z^3=0$.

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\alpha&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $g_1x^2y+g_2xy^2+g_3xyz+a_5y^3+a_2y^2z+yz^2:=0$.

We solve a equation g_1 and set $g_1=0$. Then we reduce the form to $h_2xy^2+h_3xyz+a_5y^3+a_2y^2z+yz^2:=0$. Go to Step 4

Step 8

$x^2z+y^3+pyz^2+qz^3:=0$. Replacing y by $y'+\lambda z$, we reduce the form to $x^2z+y'^3+3\lambda y'^2z+(3\lambda^2+p)y'z^2+(p\lambda+q+\lambda^3)z^3=0$.

$p:=-3\lambda^2$ and $q=2\lambda^3$, that is $4p^3+27q^2=0$ → Step 9

$4p^3+27q^2\neq 0$ → This equation is a defining equation of nonsingular elliptic curve in Weierstrass normal form. An analytic set defined by this equation is a nonsingular plane cubic curve in $\mathbb{P}^2$.

Step 9

$x^2z+y^3+a_1y^2z:=0.$

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $xy^2+a_1xz^2+z^3:=0$. Go to Step 4.

Step 10

$xyz+a_1y^3+a_2y^2z+a_3yz^2+a_4z^3:=0$. Replacing x by $x'-a_2y-a_3z$, we reduce the form to $xyz+a_1y^3+a_4z^3=0$.

$a_1\neq 0$ and $a_4\neq 0$ → Step 13, $a_1\neq 0$ and $a_4=0$ → Step 14,

$a_1=0$ and $a_4\neq 0$ → Step 15, $a_1=0$ and $a_4=0$ → Three general lines

Step 11

$xz^2+a_1y^3+a_2y^2z+a_3yz^2+a_4z^3:=0.$

$a_1\neq 0$ → Step 16, $a_1=0$ and $a_2\neq 0$ → Step 17, $a_1=0$ and $a_2=0$ → Step 18

Step 12

In this case, the form is a homogeneous polynomial of degree 3 for two variables y,z. We denote the form by $f_3(y,z)$. Then $f_3(y,z)=0$ gives three points on a projective line. By the definition 2.1.4 for the normal forms we obtain the following classification.

$f_3(y,z)\sim y^2z+z^3$ → Three concurrent lines

$f_3(y,z)\sim yz^2$ → Multiple and single lines

$f_3(y,z) \sim z^3$ → Triple line

Step 13

$xyz+y^3+z^3:=0$. Nodal curve

Step 14

$xyz+y^3:=0$. Changing the coordinates so that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$. Then we reduce the form to $xyz+z^3=0$. Conic and chord

Step 15

$xyz+z^3:=0$. Conic and chord

Step 16

$xz^2+y^3+a_1y^2z+a_2yz^2+a_3z^3:=0$.

Changing the coordinates so that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$. We reduce the form to $xz^2+y^3+g_1y^2z+g_2yz^2+g_3z^3=0$. Here, we set as folloes:

$$\alpha = \frac{a_1{}^2-3a_2}{3}, \quad \beta = \frac{-2a_1{}^3+9a_1a_2-27a_3}{27}, \quad \gamma = \frac{-a_1}{3}.$$

Then we reduce the form to $xz^2+y^3=0$. Cuspidal curve

Step 17

$xz^2+y^2z+a_1^2yz+a_2^3z:=0$. Replacing x by $x'-a_1y-a_2z$, we reduce the form to $xz^2+y^2z=0$. Conic and tangent

Step 18

$xz^2+a_1yz^2+a_2z^3:=0$. Replacing x by $x'-a_1y-a_2z$, we reduce the form to

$xz^2=0$. Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $yz^2=0$. Multiple and single lines

We arrange the classification of cubic curves in $\mathbb{P}^2$. The defining equations are as folloes:

Nodal curve	$xyz+y^3+z^3=0$
Cuspidal curve	$xz^2+y^3=0$
Conic and chord	$xyz+z^3=0$
Conic and tangent	$xz^2+y^2z=0$
Three general lines	$xyz=0$
Three concurrent lines	$y^2z+z^3=0$
Multiple and single lines	$yz^2=0$
Triple line	$z^3=0$
Nonsingular elliptic curve	$x^2z+y^3+pyz^2+qz^3=0$, $4p^3+27q^2\neq 0$

We call these defining equations the normal forms of cubic curves in $\mathbb{P}^2$.

§ 3. Nonsingular plane quartic curve

We consider the normal form of nonsingular plane quartic curve by using REDUCE. Let $f(x,y,z)$ be a homogeneous polynomial of degree 4 in $\mathbb{C}^3$. Then $f(0,0,0)=\frac{\partial f(0,0,0)}{\partial x}=\frac{\partial f(0,0,0)}{\partial y}=\frac{\partial f(0,0,0)}{\partial z}=0$. Hence, an analytic set defined by $f(x,y,z)$ has a singular point at the origin in $\mathbb{C}^3$. The analytic set is a nonsingular quartic curve in $\mathbb{P}^2$ if it has only isolated singular point at origin in $\mathbb{C}^3$.

A variety of moduli of nonsingular plane quartic curve in $\mathbb{P}^2$ has dimension 6 (§1,1.2). Therefore, we can take the defining equation which has 6 parameters.

3.1 Computation

The quartic form f(x,y,z) in $\mathbb{P}^2$ takes the following form

$a_1x^4+(a_2y+a_3z)x^3+(a_4y^2+a_5yz+a_6z^2)x^2+(a_7y^3+a_8y^2z+a_9yz^2+a_{10}z^3)x$
$+a_{11}y^4+a_{12}y^3z+a_{13}y^2z^2+a_{14}yz^3+a_{15}z^4=0.$

Step 1

We may choose coordinate so that

$a_1x^4+(a_2y+a_3z)x^3+(a_4y^2+a_5yz+a_6z^2)x^2+(a_7y^3+a_8y^2z+a_9yz^2+a_{10}z^3)x$
$+a_{11}y^4+a_{12}y^3z+a_{13}y^2z^2+a_{14}yz^3+z^4=0.$

Replacing z by $z'+\lambda x$ where λ is a solution of $\lambda^4+a_{10}\lambda^3+a_6\lambda^2+a_3\lambda+a_1=0$, we reduce the form to

$g_2x^3y+g_3x^3z+g_4x^2y^2+g_5x^2yz+g_6x^2z^2+g_7xy^3+g_8xy^2z+g_9xyz^2+g_{10}xz^3$
$+a_{11}y^4+a_{12}y^3z+a_{13}y^2z^2+a_{14}yz^3+z^4=0.$

$g_3\neq0$ → Step 2, $g_2\neq0$ and $g_3=0$ → Step 3, $g_2=0$ and $g_3=0$ → Step 4

Step 2

$a_1x^3y+x^3z+a_2x^2y^2+a_3x^2yz+a_4x^2z^2+a_5xy^3+a_6xy^2z+a_7xyz^2+a_8xz^3+a_9y^4+a_{10}y^3z$
$+a_{11}y^2z^2+a_{12}yz^3+a_{13}z^4:=0$ [By a magnification of the x- y- and z-coordinates we can reduce the form to $a_1x^3y+x^3z+a_2x^2y^2+a_3x^2yz+a_4x^2z^2$
$+a_5xy^3+a_6xy^2z+a_7xyz^2+a_8xz^3+a_9y^4+a_{10}y^3z+a_{11}y^2z^2+a_{12}yz^3+a_{13}z^4:=0$].

Replacing z by $z'-a_1y$, we reduce the form to

$x^3z+g_4x^2y^2+g_5x^2yz+a_4x^2z^2+g_7xy^3+g_8xy^2z+g_9xyz^2+a_{10}xz^3+g_{11}y^4+g_{12}y^3z$
$+g_{13}y^2z^2+g_{14}yz^3+a_{13}z^4=0.$

$g_4\neq0$ → Step 5, $g_4=0$ and $g_7\neq0$ → Step 6,

$g_4=0$ and $g_7=0$ and $g_{11}\neq0$ → Step 7,

$g_4=0$ and $g_7=0$ and $g_{11}=0 \rightarrow$ Step 8

Step 3

$x^3y+a_1x^2y^2+a_2x^2yz+a_3x^2z^2+a_4xy^3+a_5xy^2z+a_6xyz^2+a_7xz^3+a_8y^4+a_9y^3z+a_{10}y^2z^2+a_{11}yz^3+a_{12}z^4:=0.$

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $x^3z+a_3x^2y^2+a_2x^2yz+a_1x^2z^2+a_7xy^3+a_6xy^2z+a_5xyz^2+a_4xz^3+a_{12}y^4+a_{11}y^3z+a_{10}y^2z^2+a_9yz^3+a_8z^4=0.$

$a_3\neq 0 \rightarrow$ Step 5, $a_3=0$ and $a_7\neq 0 \rightarrow$ Step 6,

$a_3=0$ and $a_7=0$ and $a_{12}\neq 0 \rightarrow$ Step 7,

$a_3=0$ and $a_7=0$ and $a_{12}=0 \rightarrow$ Step 8

Step 4

In this case, the form is as follows:

$f_2(y,z)x^2+f_3(y,z)x+f_4(y,z)=0$ where f_i denotes a homogeneous polynomial of degree i $(2\leq i\leq 4)$. By the Corollary 1.4.4 the above form give a singular curve in $\mathbb{P}^2$.

Step 5

$x^3z+x^2y^2+a_1x^2yz+a_2x^2z^2+a_3xy^3+a_4xy^2z+a_5xyz^2+a_6xz^3+a_7y^4+a_8y^3z+a_9y^2z^2+a_{10}yz^3+a_{11}z^4:=0.$ Here, we try to vanish the monomial x^2y^2.

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\alpha&\beta&1\\\gamma&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $g_1x^4+g_2x^3y+g_3x^3z+g_4x^2y^2+g_5x^2yz+g_6x^2z^2+g_7xy^3+g_8xy^2z+g_9xyz^2+xz^3+g_{11}y^4+g_{12}y^3z+y^2z^2=0.$

Now, $g_{11}=\beta^2+a_3\beta+a_7$

$g_7=a_1\beta^2+a_3\alpha+3a_3\beta\gamma+a_4\beta+4a_7\gamma+a_8+2\alpha\beta+\beta^3+2\beta^2\gamma$

$$g_4=2a_1\alpha\beta+a_1\beta^2\gamma+a_2\beta^2+3a_3\alpha\gamma+3a_3\beta\gamma^2+a_4\alpha+2a_4\beta\gamma+a_5\beta+6a_7\gamma^2+3a_8\gamma+a_9 +\alpha^2+3\alpha\beta^2+4\alpha\beta\gamma+\beta^2\gamma^2.$$

We solve the equation g_{11} for a variable β and solve the equation g_4 for a variable α. And we substitute the solutions to the equation g_7, arrange the equation g_7. Then the degree of the equation is zero. We can not vanish the monomial x^2y^2.

(see Program list 3)

Now, $g_1=a_1\alpha^2\gamma+a_{10}\gamma+a_{11}+a_2\alpha^2+a_3\alpha\gamma^3+a_4\alpha\gamma^2+a_5\alpha\gamma+a_6\alpha+a_7\gamma^4+a_8\gamma^3+a_9\gamma^2 +\alpha^3+\alpha^2\gamma^2$,

$$g_2=a_1\alpha^2+2a_1\alpha\beta\gamma+a_{10}+2a_2\alpha\beta+3a_2\alpha\gamma^2+a_3\beta\gamma^3+2a_4\alpha\gamma+a_4\beta\gamma^2+a_5\alpha +a_5\beta\gamma+a_6\beta+4a_7\gamma^3+3a_8\gamma^2+2a_9\gamma+3\alpha^2\beta+2\alpha^2\gamma+2\alpha\beta\gamma^2.$$

g_1 is a polynomial with two variables (α and γ), g_2 and g_4 are polynomials with three variables (α, β and γ). We reduce the forms to

$$g_4=s_2\beta^2+s_1\beta+s_0,\quad g_2=t_1\beta+t_0$$

where $s_2=3\alpha+\gamma^2+a_1\gamma+a_2$, $s_1=4\alpha\gamma+2a_1\alpha+3a_3\gamma^2+2a_4\gamma+a_5$,

$s_0=\alpha^2+3a_3\alpha\gamma+\alpha\gamma+6a_7\gamma^2+3a_8\gamma+a_9$,

$t_1=3\alpha^2+2\alpha\gamma^2+2a_1\alpha\gamma+2a_2\alpha+a_3\gamma^3+a_4\gamma^2+a_5\gamma+a_6$,

$t_0=2\alpha^2\gamma+a_1\alpha^2+3a_3\alpha\gamma^2+2a_4\alpha\gamma+a_5\alpha+4a_7\gamma^3+3a_8\gamma^2+2a_9\gamma+a_{10}$.

We use Sylvester's elimination method(§1,1.3), eliminate γ, obtain the resultant. Let $R_1(g_2,g_4)$ be the resultant. Then $R_1(g_2,g_4)$ is a polynomial with two variables (α,β). We eliminate α for s_2 and t_1. Then we obtain the resultant. Let $R_2(s_2,t_1)$ be the resultant. And we eliminate α for g_1 and $R_1(g_2,g_4)$. Then we obtain the resultant. Let $R_3(g_1,R_1(g_2,g_3))$ be the resultant. $R_2(s_2,t_1)$ and $R_3(g_1,R_1(g_2,g_3))$ are polynomials with a variable α. Let α_0 be a solution of $R_3(g_1,R_1(g_2,g_3))=0$. If $R_2(s_2,t_1)(\alpha_0)\neq 0$, The simultaneous system of

algebraic equations ($g_1=g_2=g_4=0$) has a common root.
If $R_3(g_1,R_1(g_2,g_3))\neq 0$, there exists no common root.
(see Program list 4)

Hence, for some parameter $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11})$ we can vanish the monomial x^2y^2. We consider it in Step 6,7,8.

We assume that there exists no common root for $g_1=g_2=g_4=0$.

Changing the coordinates so that $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$. We reduce the form to $x^3z+x^2y^2+g_1x^2yz+g_2x^2z^2+g_3xy^3+g_4xy^2z+g_5xyz^2+g_6xz^3+g_7y^4+g_8y^3z$ $+g_9y^2z^2+g_{10}yz^3+g_{11}z^4=0$. Here, we set as folloes:

$$\alpha=\frac{-a_3}{2},\quad \beta=\frac{4a_1{}^2-16a_2-9a_3{}^2}{48},\quad \gamma=\frac{-2a_1+3a_3}{4}.$$

Then we reduce the form to $x^3z+x^2y^2+g_4xy^2z+g_5xz^3+g_6xz^3+g_7y^4+g_8y^3z$ $+g_9y^2z^2+g_{10}yz^3+g_{11}z^4=0$.

If $g_7\neq 0$, we reduce the form to
$x^3z+x^2y^2+a_1xy^2z+a_2xyz^2+a_3xz^3+y^4+a_4y^3z+a_5y^2z^2+a_6yz^3+a_7z^4=0$.
A dimension of parameters space of this form is equal to 7. And we can not take it less than or equal to 6. This is a contradiction (§1,1.2).
If $g_7=0$ and $g_8\neq 0$, we reduce the form to
$x^3z+x^2y^2+a_1xy^2z+a_2xyz^2+a_3xz^3+y^3z+a_4y^2z^2+a_5yz^3+a_6z^4=0$,
$(a_3,a_5,a_6)\neq(0,0,0)$.

(From Corollary 1.4.4,
(coefficient of the monomial y^3z = 0) or ($(a_3,a_5,a_6)=(0,0,0)$)
→ the analytic set defined this form is singular curve in $\mathbb{P}^2$)

A dimension of parameters space of this form is equal to 6.

Step 6

$x^3z+a_1x^2yz+a_2x^2z^2+xy^3+a_3xy^2z+a_4xyz^2+a_5xz^3+a_6y^4+a_7y^3z+a_8y^2z^2+a_9yz^3+a_{10}z^4:=0$.

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&\alpha&\beta\\0&1&\gamma\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $x^3z+g_1x^2yz+g_2x^2z^2+xy^3+g_3xy^2z+g_4xyz^2+g_5xz^3+g_6y^4+g_7y^3z+a_8y^2z^2+a_9yz^3+a_{10}z^4=0$. Here, we set as follows:

$$\alpha=\frac{-a_1}{3},\ \beta=\frac{-a_1^3+3a_1a_3-9a_2}{27},\ \gamma=\frac{a_1^2-3a_3}{9}.$$

Then we reduce the form to

$x^3z+xy^3+a_1xyz^2+a_2xz^3+a_3y^4+a_4y^3z+a_5y^2z^2+a_6yz^3+a_7z^4=0$,

$(a_2,a_6,a_7)\neq(0,0,0)$.

A dimension of parameters space of this form is equal to 6.

Step 7

$x^3z+a_1x^2yz+a_2x^2z^2+a_3xy^2z+a_4xyz^2+a_5xz^3+y^4+a_6y^3z+a_7y^2z^2+a_8yz^3+a_9z^4:=0$.

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&\alpha&\beta\\0&1&\gamma\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $x^3z+g_1x^2yz+g_2x^2z^2+g_3xy^2z+g_4xyz^2+g_5xz^3+y^4+g_6y^3z+a_7y^2z^2+g_8yz^3+a_9z^4=0$. Here, we set as follows:

$$\alpha=\frac{-a_1}{3},\ \beta=\frac{2a_1^4-9a_1^2a_3+27a_1a_6-108a_2}{324},\ \gamma=\frac{-2a_1^3+9a_1a_3-27a_6}{108}$$

Then we reduce the form to

$x^3z+a_1xy^2z+a_2xyz^2+a_3xz^3+y^4+a_4y^2z^2+a_5yz^3+a_6z^4=0$, $(a_3,a_5,a_6)\neq(0,0,0)$.

A dimension of parameters space of this form is equal to 5.

Step 8

$x^3z+a_1x^2yz+a_2x^2z^2+a_3xy^2z+a_4xyz^2+a_5xz^3+a_6y^3z+a_7y^2z^2+a_8yz^3+a_9z^4:=0$.

Replacing x by $x'+\lambda y$ where λ is a solution of $\lambda^3+a_1\lambda^2+a_3\lambda+a_6=0$, we reduce the form to

$$x^3z+g_1x^2yz+g_2x^2z^2+g_3xy^2z+g_4xyz^2+g_5xz^3+g_7y^2z^2+g_8yz^3+g_9z^4=0.$$

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $g_3x^2yz+g_7x^2z^2+g_1xy^2z+g_4xyz^2+g_8xz^3+y^3z+a_2y^2z^2+g_5yz^3+g_9z^4=0$. From Corollary 1.4.4, the analytic set defined this form is singular curve in $\mathbb{P}^2$.

3.2 Normal form of nonsingular plane quartic curve

From the results in 3.1, we obtain a following lemma.

Lemma 3.2.1 There exists the following three types forms as the normal form of nonsingular quartic curve in $\mathbb{P}^2$.

Type I $:x^3z+xy^3+a_1xyz^2+a_2xz^3+a_3y^4+a_4y^3z+a_5y^2z^2+a_6yz^3+a_7z^4=0$

$(a_2,a_6,a_7)\neq(0,0,0)$.

Type II $:x^3z+a_1xy^2z+a_2xyz^2+a_3xz^3+y^4+a_4y^2z^2+a_5yz^3+a_6z^4=0$

$(a_3,a_5,a_6)\neq(0,0,0)$.

Type III$:x^3z+x^2y^2+a_1xy^2z+a_2xyz^2+a_3xz^3+y^3z+a_4y^2z^2+a_5yz^3+a_6z^4=0$

$(a_3,a_5,a_6)\neq(0,0,0)$.

There exists the relations of parameters for the above type I~III. The relations determine a structure of moduli space. The structure of moduli space is important to mathematics.

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