

On Weak and Classical Solutions of the Two-Dimensional
Magnetohydrodynamic Equations

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Let Ω be a bounded domain in \mathbb{R}^2 with smooth boundary $\partial\Omega$. In $Q_T := \Omega \times (0, T)$, we consider the following magnetohydrodynamic equations for an ideal incompressible fluid coupled with magnetic field:

$$\begin{aligned} \partial_t u + (u, \nabla)u - (B, \nabla)B + \nabla((1/2)|B|^2) + \nabla\pi &= f \quad \text{in } Q_T, \\ \partial_t B - \Delta B + (u, \nabla)B - (B, \nabla)u &= 0 \quad \text{in } Q_T, \\ (*) \quad \operatorname{div} u = 0, \quad \operatorname{div} B = 0 &\quad \text{in } Q_T, \\ u \cdot \nu = 0, \quad B \cdot \nu = 0, \quad \operatorname{rot} B = 0, &\quad \text{on } \partial\Omega \times (0, T), \\ u|_{t=0} = u_0, \quad B|_{t=0} = B_0. \end{aligned}$$

Here $u = u(x, t) = (u^1(x, t), u^2(x, t))$, $B = B(x, t) = (B^1(x, t), B^2(x, t))$ and $\pi = \pi(x, t)$ denote the unknown velocity field of the fluid, magnetic field and pressure of the fluid, respectively; $f = f(x, t) = (f^1(x, t), f^2(x, t))$ denotes the given external force, $u_0 = u_0(x) = (u_0^1(x), u_0^2(x))$ and B_0

$= B_0(x) = (B_0^1(x), B_0^2(x))$ denote the given initial data and ν denotes the unit outward normal on $\partial\Omega$.

The first purpose of this paper is to state the existence and uniqueness of a global weak solution of (*) without restriction on the data. In case B is identically equal to zero, i.e., in the case of the Euler equations, such a problem for global weak and classical solutions was solved by Bardos [1] and Kato [4], respectively. (Kikuchi [5] extended the result of Kato [4] in an exterior domain.) Using the energy method developed by Bardos [1], we can obtain a global weak solution in our case.

Our second purpose is to state the existence and uniqueness of a local classical solution of (*). Although the method of characteristic curves for the vorticity equation plays an important role in a global classical solution of the two-dimensional Euler equations, such a method seems to give us only a local classical solution of (*) because of the additional terms $(B, \nabla)B$ and $(u, \nabla)B - (B, \nabla)u$. Our result on classical solution is considered, however, a generalization of that of Kato [4] in some sense.

1. Notation.

Let us introduce some function spaces. $C_{0,\sigma}^\infty(\Omega)$ denotes the set of all C^∞ -real vector-valued functions $\phi = (\phi^1, \phi^2)$ with compact support in Ω such that $\operatorname{div} \phi = 0$. H is the

completion of $C_{0,\sigma}^\infty(\Omega)$ with respect to the L^2 -norm $\|\cdot\|$; (\cdot, \cdot)

denotes the L^2 -inner product. V denotes the set of all vector-valued functions u in $H^1(\Omega)$ with $\operatorname{div} u = 0$ in Ω and $u \cdot \nu = 0$ on $\partial\Omega$. Equipped with the norm $\|\cdot\|$:

$$\|u\|^2 = \|\operatorname{rot} u\|^2 + \|u\|^2,$$

V is a Hilbert space. By Duvaut-Lions [2, Chapter 7 Theorem 6.11], we have

$$\|u\|_{H^1(\Omega)} \leq C(\Omega) \|u\| \quad \text{for all } u \in V.$$

Hence the norm $\|\cdot\|$ is equivalent to the one usually induced from $H^1(\Omega)$ and V is compactly imbedded into H .

If X is a Hilbert space, then $L^p(0, T; X)$ ($1 \leq p < \infty$) denotes the set of all measurable functions $u(t)$ with values in X such that $\int_0^T \|u(t)\|_X^p dt < \infty$ ($\|\cdot\|_X$ is the norm in X).

$L^\infty(0, T; X)$ denotes the set of all essentially bounded (in the norm of X) measurable functions of t with values in X .

Let $C^m([0, T]; X)$ denote the set of all X -valued m -times continuously differentiable functions of t ($0 \leq t \leq T$).

$C_0^m([0, T]; X)$ is the set of all X -valued m -times continuously

differentiable functions on $[0, T)$ with compact support in $[0, T)$.

$C^{k+\alpha}(\bar{\Omega})$ with integer $k \geq 0$ and $0 \leq \alpha < 1$ denotes the usual Hölder space of continuous functions on $\bar{\Omega}$. $\|\cdot\|_{k+\alpha}$ denotes the norm in $C^{k+\alpha}(\bar{\Omega})$. $C^{k,j}(\bar{Q}_T)$ with integers $k, j \geq 0$ is the set of all functions ϕ for which all the $\partial_x^q \partial_t^r \phi$ exist and are continuous on \bar{Q}_T for $0 \leq |q| \leq k$, $0 \leq r \leq j$. $C^{k+\alpha, j+\beta}(\bar{Q}_T)$ with integers $k, j \geq 0$ and $0 \leq \alpha, \beta < 1$ is the subset of $C^{k,j}(\bar{Q}_T)$ containing all functions ϕ for which all the $\partial_x^q \partial_t^r \phi$, $0 \leq |q| \leq k$, $0 \leq r \leq j$, are Hölder continuous with exponents α in x and β in t . If

$$K^{\alpha, \beta}(\phi) = \sup_{\substack{(x, t) \in \bar{Q}_T \\ (x', t) \in \bar{Q}_T \\ |x-x'| < 1}} |\phi(x, t) - \phi(x', t)| / |x - x'|^\alpha + \\ + \sup_{\substack{(x, t) \in \bar{Q}_T \\ (x, t') \in \bar{Q}_T \\ |t-t'| < 1}} |\phi(x, t) - \phi(x, t')| / |t - t'|^\beta,$$

we define the norm $\|\cdot\|_{k+\alpha, j+\beta}$ in $C^{k+\alpha, j+\beta}(\bar{Q}_T)$ by

$$\|\phi\|_{k+\alpha, j+\beta} = \sup_{(x, t) \in \bar{Q}_T} \sum_{\substack{|q| \leq k \\ r \leq j}} |D_x^q \partial_t^r \phi(x, t)| + \sum_{|q|=k} K^{\alpha, \beta}(D_x^q \partial_t^j \phi).$$

For the spaces of vector-valued functions, we shall use the bold-faced letters analogously.

2. Definitions and results.

Our definition of a weak solution of (*) is as follows.

Definition. Let $u_0 \in H$, $B_0 \in H$ and $f \in L^2(0, T; L^2(\Omega))$.

A pair of measurable vector functions u and B on Q_T is called a weak solution of (*) if

$$(i) \ u \in L^\infty(0, T; H) \cap L^2(0, T; V), \quad B \in L^\infty(0, T; H) \cap L^2(0, T; V);$$

$$(ii) \ \int_0^T \{-(u, \partial_t \Phi) + ((u, \nabla)u - (B, \nabla)B, \Phi)\} dt$$

$$= (u_0, \Phi(0)) + \int_0^T (f, \Phi) dt$$

$$\int_0^T \{-(B, \partial_t \Phi) + (\text{rot } B, \text{rot } \Phi) + ((u, \nabla)B - (B, \nabla)u, \Phi)\} dt$$

$$= (B_0, \Phi(0))$$

for all $\Phi \in C_0^1([0, T]; V)$.

Concerning the uniqueness of weak solutions of (*), we have

Proposition. There exists at most one weak solution of (*).

If (u, B) is the weak solution of (*), after a suitable redefinition of $u(t)$ and $B(t)$ on a set of measure zero of the

time interval $[0, T]$, we have that $u \in C([0, T]; H)$ and
 $B \in C([0, T]; H)$.

For the proof of this proposition, see Temam [6, Chapter 3 Theorem 3.2], we omit it.

Our result on the existence of a weak solution now reads:

Theorem 1. Let $u_0 \in V$, $B_0 \in V$ and $f \in L^2(0, T; L^2(\Omega))$
with $\text{rot } f \in L^2(0, T; L^2(\Omega))$. Then there exists a weak solution
 (u, B) of (*) such that $u \in L^\infty(0, T; V) \cap C([0, T]; H)$ and
 $B \in L^2(0, T; H^2(\Omega)) \cap C([0, T]; V)$.

We next proceed to our result on classical solutions. To this end, we make the following assumptions on the domain Ω and the given data u_0, B_0 and f .

Assumption 1. The boundary $\partial\Omega$ of Ω consists of $m + 1$ sufficiently smooth, simple closed curves S_0, S_1, \dots, S_m , where S_j ($j = 1, \dots, m$) are inside of S_0 and outside of one another.

Günter [3, p.22] refers to the above assumption as "Case J".

Assumption 2. For some $0 < \theta < 1$, $u_0 \in C^{1+\theta}(\bar{\Omega})$, $B_0 \in C^{2+\theta}(\bar{\Omega})$ and $f \in C^{1+\theta, 0}(\bar{Q}_T)$. Moreover, u_0 and B_0 satisfy the

conditions $\operatorname{div} u_0 = 0$, $\operatorname{div} B_0 = 0$ in Ω , $u_0 \cdot \nu = 0$, $B_0 \cdot \nu = 0$ on $\partial\Omega$.

Our result on the existence and uniqueness of classical solutions reads:

Theorem 2. Let the assumptions 1 and 2 hold. Then there is a positive number $C_* = C_*(\Omega, T, |u_0|_{1+\theta}, |f|_{1+\theta,0})$ such that if $|B_0|_{2+\theta} \leq C_*$, there exists a solution $(u, B, \pi) \in C^{1,1}(\bar{Q}_T) \times C^{2,1}(\bar{Q}_T) \times C^{1,0}(\bar{Q}_T)$ of (*). Such a solution is unique up to adding an arbitrary function of t to π .

Remark. Taking $B_0 = 0$ in Ω , we may cover the result of Kato [4].

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