

Nonlinear evolution equations with
nonmonotonic perturbations

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1. Introduction. Let V be a Banach space densely and continuously imbedded in a real Hilbert space H . Our purpose in this paper is to consider the existence of solutions of the initial value problem

$$\begin{aligned} \frac{du}{dt} + Au + G(u) &= f, & 0 < t < T, \\ (1.1) \quad u(0) &= u_0, \end{aligned}$$

where A is a monotone operator from V into V' , $G:V \rightarrow H$ is a continuous mapping and $f:(0,T) \rightarrow V'$ is a measurable function.

Problems of this kind has been studied by many authors. The case A is linear was studied by Browder (5) and Pazy (14). The nonlinear case was studied by Attouch & Damlamian (1), Crandall & Nohel (7), Hirano (10), and Vrabie (15, 16). In (15) and (16), Vrabie studied the problem (1.1) under the assumption that A generates a compact semigroup on H , and satisfies

$$(1.2) \quad (Ax - Ay, x - y) + c|x - y|^2 \geq \omega \|x - y\|^p \quad \text{for } x, y \in V,$$

where $c, \omega > 0$, $p \geq 2$ and $\|\cdot\|, |\cdot|$ denotes the norms of V and H , respectively.

In this paper, we consider the case G is a compactly continuous mapping from V into V' . Our argument is based on the existence results for pseudo-monotone mappings (cf. (4, 6)).

2. Statement of main results. Let p, q and T be constants such that $T > 0$, $p \geq 2$ and $1/p + 1/q = 1$. V will denote a reflexive Banach space densely and continuously imbedded in a real Hilbert space H . Identifying H with its dual, we have that $V \subset H \subset V'$, where V' is the dual space of V . The norms of V, H and V' are denoted by $\|\cdot\|, |\cdot|$ and $\|\cdot\|_*$, respectively. Let (x, y) denote the pairing of an element $x \in V$ and an element $y \in V'$. If $x, y \in H$, then (x, y) is the ordinary inner product of H . Let A be a mapping from V into V' . Then A is called monotone if $(Ax - Ay, x - y) \geq 0$ for $x, y \in V$. The mapping A is said to be hemicontinuous if for each $x, y \in V$, $A(u+tv)$ converges to Au weakly in V' , as $t \rightarrow 0$. A is called pseudo-monotone if A satisfies the following condition:

(2.2) If $\{u_n\}$ is a sequence such that u_n converges weakly to $u \in V$ and $\limsup (Au_n, u_n - u) \leq 0$, then

$$(Au, u - v) \leq \liminf_{n \rightarrow \infty} (Au_n, u_n - v) \quad \text{for each } v \in V.$$

Let E, F be Banach spaces, and let g be a mapping from E into F . We denote by E_w and F_w the spaces E and F endowed with their weak topologies, respectively. Then g is said to be weakly continuous if g is a continuous mapping from E_w into F_w . The mapping g is called demicontinuous if g is a continuous mapping from E into F_w . For each $r \geq 1$. We denote by $L^r(0, T; E)$ the space of E -valued measurable functions $u: (0, T) \rightarrow E$ such that $\int_0^T \|u(t)\|^r dt < \infty$. The pairing between $L^p(0, T; V)$ and $L^q(0, T; V')$ is denoted by $\langle \cdot, \cdot \rangle$. Then for each $u, v \in L^2(0, T; H)$, $\langle u, v \rangle$ is the ordinary inner product

of u and v in $L^2(0, T; H)$. The norms of $L^p(0, T; V)$, $L^2(0, T; H)$, $L^q(0, T; V')$ are again denoted by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$. We denote by J the duality mapping from $L^q(0, T; V')$ onto $L^p(0, T; V)$, i. e.,

$$(2.1) \quad J(u) = \{v \in L^p(0, T; V) : \langle v, u \rangle = \|v\|^2 = \|u\|_*^2\}$$

for each $u \in L^q(0, T; V')$. By using the Asplund's renorming theorem, we may assume that J is a single valued monotone and demi-continuous mapping (cf. Proposition 2.14 of (3)). We will denote by L the operator defined by

$$(Lf)(t) = \int_0^t f(s) ds \quad \text{for each } f \in L^2((0, T))$$

The adjoint operator L^* of L is given by

$$(L^*f)(t) = \int_t^T f(s) ds \quad \text{for each } f \in L^2((0, T)).$$

Then L and L^* are positive operators on $L^2((0, T))$.

In the following we will assume that the mapping $A: V \rightarrow V'$ satisfies the following conditions:

(A1) A is a monotone hemicontinuous mapping from V into V' ;

(A2) there exist positive constants C_1 , C_2 and C_3 such that

$$(2.3) \quad \|Ax\|_* \leq C_1(\|x\|^{p-1} + 1), \quad \text{for each } x \in V$$

and

$$(2.4) \quad C_2\|x\|^p \leq C_3 + (Ax, x) \quad \text{for each } x \in V.$$

We impose the following conditions on G :

(G1) G is a completely continuous mapping from V to V' ;

(G2) There exist positive constants a , b and C such that

$$(2.5) \quad (G(x), x) \geq -C \quad \text{for all } x \in V;$$

$$(2.6) \quad \|G(x)\|_x \leq a\|x\|^{p-1} + b \quad \text{for all } x \in V.$$

We now state our result:

Theorem. Suppose that (A1), (A2), (G1) and (G2) hold. Then for each $u_0 \in H$ and $f \in L^q(0, T; V')$, there exists a solution u of (1.1) such that

$$(2.7) \quad u \in C(0, T; H) \cap L^p(0, T; V)$$

$$(2.8) \quad \frac{du}{dt} \in L^q(0, T; V').$$

3. Propositions. Throughout this section, we assume that $u_0 \in V$, $f \in L^q(0, T; V')$, and that (A1), (A2), (G1) and (G2) hold. We denote by \tilde{V} , \tilde{H} , and \tilde{V}' the spaces $L^p(0, T; V)$, $L^2(0, T; H)$ and $L^q(0, T; V')$, respectively. \tilde{A} denote the operator defined by

$$(\tilde{A}u)(t) = A(u(t) + u_0) - f(t), \quad \text{for each } u \in \tilde{V} \text{ and } t \in (0, T).$$

Also we denote by \tilde{G} the mapping defined by

$$(\tilde{G}u)(t) = G(u(t) + u_0) \quad \text{for each } u \in \tilde{V} \text{ and } t \in (0, T).$$

Then it is easy to see that \tilde{A} is a monotone hemicontinuous mapping satisfying the following conditions:

$$(3.1) \quad \|\tilde{A}u\|_* \leq c_1(1 + \|u\|^{p-1}) \quad \text{for } u \in \tilde{V};$$

$$(3.2) \quad c_2\|u\|^p \leq c_3 + \langle \tilde{A}u, u \rangle \quad \text{for } u \in \tilde{V},$$

where c_1 , c_2 and c_3 are positive constants depending on C_1 , C_2 , C_3 , T , u_0 and f . It is also easy to see that G is a continuous mapping from \tilde{V} into \tilde{H} satisfying that

$$(3.3) \quad \langle \tilde{G}u, u \rangle \geq c \quad \text{for all } u \in \tilde{V};$$

$$(3.4) \quad \|\tilde{G}u\| \leq \alpha \|u\|^{p-1} + \beta \quad \text{for all } u \in \tilde{V},$$

where c , α , β are constants depending on C , a , b and T and u_0 .

We now consider the equation of the form

$$(3.5) \quad v + (\tilde{A} + \tilde{G})Lv = 0$$

Let $v \in \tilde{V}'$ be a solution of (3.5). Then it is easy to see that $u = Lv + u_0$ is a solution of (1.1). On the other hand, if u is a solution of (1.1), we have that $v = \frac{du}{dt}$ is a solution of (3.4).

Since L^* is injective, the equation (3.5) is equivalent to

$$(3.6) \quad L^*v + L^*(\tilde{A} + \tilde{G})Lv = 0.$$

Then we will show the existence of the solutions of (3.6) instead of (1.1). In the rest of this section, we assume, for simplicity, that $u_0 = 0$ and $f = 0$. The proofs remains valid for each $u_0 \in V$ and $f \in \tilde{V}'$ with minor changes.

Proposition 1. the mapping $L^* + L^*(\tilde{A} + \tilde{G})L$ is a pseudo-monotone mapping from \tilde{V} into \tilde{V}' .

Proof. From (A1), it is easily verified that $L^* + L^* \tilde{A}L: \tilde{V} \rightarrow \tilde{V}'$ is a monotone hemicontinuous mapping. Let $\{v_n\} \subset \tilde{V}$ be a sequence such that v_n converges to v weakly in \tilde{V} and

$$(3.7) \quad \limsup \langle L^* v_n + L^* (\tilde{A} + \tilde{G})Lv_n, v_n - v \rangle \leq 0.$$

Since v_n converges to v weakly in \tilde{V} , we have that for each $t \in (0, T)$, $(Lv_n)(t)$ converges to $(Lv)(t)$ weakly in V . Since G is completely continuous, we find that $G((Lv_n)(t))$ converges to $G((Lv)(t))$ strongly in V' for all $t \in (0, T)$. Then noting that

$$\|G((Lv_n)(t))\|_* \leq a \| (Lv_n)(t) \|^{p-1} + b \leq a (T^{1/2} \sup \|v_n\|)^{p-1} + b$$

for each $t \in (0, T)$, we obtain by Lebesgue's bounded convergence theorem, that $\tilde{G}(Lv_n)$ converges to $\tilde{G}(Lv)$ strongly in \tilde{V}' . Thus we obtain that

$$(3.8) \quad \langle \tilde{G}Lv, Lv \rangle = \lim \langle \tilde{G}Lv_n, Lv_n \rangle$$

Therefore we have by (3.7) and (3.8) that

$$\limsup \langle L^* v_n + L^* \tilde{A}Lv_n, v_n - v \rangle \leq 0.$$

Then by lemma 1.3 of Chap II of (2), it follows that $L^* v_n + L^* \tilde{A}Lv_n$ converges to $L^* v + L^* \tilde{A}Lv$ weakly in \tilde{V}' and

$$\langle L^* v + L^* \tilde{A}Lv, v \rangle = \lim \langle L^* v_n + L^* \tilde{A}Lv_n, v_n \rangle.$$

Then from (3.7), (3.8) and the equality above, we find that

$$\langle L^* v + L^* (\tilde{A} + \tilde{G})Lv, Lv - z \rangle \leq \liminf \langle L^* v_n + L^* (\tilde{A} + \tilde{G})Lv_n, v_n - z \rangle$$

for each $z \in \tilde{V}$. This completes the proof.

Proposition 2. Let $\{v_n\}$ be a sequence in \tilde{V}' such that v_n converges to v weakly in \tilde{V}' , $\{Lv_n\} \subset \tilde{V}$, Lv_n converges to Lv weakly in \tilde{V} and

$$\limsup \langle (\tilde{A} + \tilde{G})Lv_n, Lv_n - Lv \rangle \leq 0.$$

Then $(\tilde{A} + \tilde{G})Lv_n$ converges to $(\tilde{A} + \tilde{G})Lv$ weakly in \tilde{V}' , and

$$(3.9) \quad \lim \langle (\tilde{A} + \tilde{G})Lv_n, Lv_n \rangle = \langle (\tilde{A} + \tilde{G})Lv, Lv \rangle.$$

Proof. Let $\{v_n\}$ be a sequence in \tilde{V}' satisfying the hypothesis of Proposition 2. Then by using (3.1) and (3.4), we can see that

$$(3.10) \quad \liminf (A((Lv_n)(t)) + G((Lv_n)(t)), (Lv_n - Lv)(t)) \geq 0$$

for all $t \in (0, T)$. Suppose that for some $t \in (0, T)$,

$$(3.11) \quad \liminf (A((Lv_n)(t)) + G((Lv_n)(t)), (Lv_n - Lv)(t)) < 0.$$

From (A2) and (G2), we have that

$$(3.12) \quad (A((Lv_n)(t)) + G((Lv_n)(t)), (Lv_n - Lv)(t)) \\ \geq C_2 \|(Lv_n)(t)\|^p - C_3 - C - C_1 (1 + \|(Lv_n)(t)\|^{p-1}) \|(Lv)(t)\| \\ - (a \|(Lv_n)(t)\|^{p-1} + b) \|(Lv)(t)\|.$$

Then it follows from (3.11) and (3.12) that $\{\|(Lv_n)(t)\|\}$ is bounded. Then since G is completely continuous, $G(Lv_n)(t)$ converges to $G(Lv)(t)$ strongly in V . Therefore we have that

$\lim (G((Lv_n)(t)), (Lv_n)(t) - (Lv)(t)) = 0$. On the other hand, we have from the monotonicity of A that

$$\liminf (A((Lv_n)(t)), (Lv_n - Lv)(t)) \geq 0.$$

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for all $t \in (0, T)$. Then we have that

$$\liminf (A(Lv_n)(t) + G(Lv_n)(t)), (Lv_n - Lv)(t) \geq 0.$$

This contradicts to (3.11). Thus we have shown that (3.10) holds for all $t \in (0, T)$. We can see from (3.12) that

$$\begin{aligned} h_n(t) &= (A(Lv_n)(t) + G(Lv_n)(t)), (Lv_n - Lv)(t) \\ &\geq K_1 \|Lv(t)\|^p + K_2 \end{aligned}$$

for all $t \in (0, T)$ and $n \geq 1$, where K_1, K_2 are constants depending on C, C_1, C_2, C_3, a , and b . Then by Fatou's lemma, we have that

$$\begin{aligned} (3.13) \quad 0 &= \int_0^T \liminf h_n(t) dt \\ &\leq \liminf \int_0^T h_n(t) dt \leq \limsup \langle (\tilde{A} + \tilde{G})Lv_n, Lv_n - Lv \rangle \leq 0. \end{aligned}$$

The inequality above implies that $\lim \int_0^T |h_n| dt = 0$. Then we can choose a subsequence $\{h_{n_i}\}$ of $\{h_n\}$ such that

$$(3.14) \quad \lim (A(Lv_{n_i})(t) + G(Lv_{n_i})(t)), (Lv_{n_i} - Lv)(t) = 0,$$

a.e. $t \in (0, T)$. By (3.12) and (3.14), we find that $\{\|Lv_{n_i}(t)\|\}$ is bounded for a.e. $t \in (0, T)$. Since $(Lv_{n_i})(t)$ converges to $(Lv)(t)$ weakly in V' , we have that $G(Lv_{n_i})(t)$ converges to $G(Lv)(t)$ strongly in H , for a.e. $t \in (0, T)$. Therefore it follows from (2.6) that

$$\lim (G(Lv_{n_i})(t)), (Lv_{n_i})(t) - (Lv)(t) = 0. \text{ Then we have}$$

$$\lim (A(Lv_{n_i})(t)), (Lv_{n_i} - Lv)(t) = 0 \text{ a.e. } t \in (0, T).$$

Then since A is monotone, we have from lemma 1.3 of Chap. II of (2) that $A(Lv_{n_i}(t))$ converges to $A(Lv(t))$ weakly in V' . Here we observe by using (3.1)-(3.4) that for each $z \in \tilde{V}$ and $t \in (0, T)$, there exist real numbers K_3, K_4 such that

$$(3.15) \quad \langle (A + G)(Lv_n(t)), (Lv_n(t) - z(t)) \rangle \geq K_3 \|z(t)\|^p + K_4$$

for each $n \geq 1$. Then from Fatou's lemma, we find that

$$(3.16) \quad \begin{aligned} & \langle (\tilde{A} + \tilde{G})Lv, Lv - z \rangle \\ &= \int_0^T \liminf (A(Lv_{n_i}(t)) + G(Lv_{n_i}(t)), (Lv_{n_i}(t) - z(t)) dt \\ &\leq \liminf \langle (\tilde{A} + \tilde{G})Lv_{n_i}, Lv_{n_i} - z \rangle \\ &\leq \limsup \langle (\tilde{A} + \tilde{G})Lv_n, (Lv_n - Lv) + (Lv - z) \rangle \\ &\leq \limsup \langle (\tilde{A} + \tilde{G})Lv_n, Lv - z \rangle, \quad \text{for all } z \in \tilde{V}. \end{aligned}$$

The inequality above implies that $(\tilde{A} + \tilde{G})Lv_n$ converges to $(\tilde{A} + \tilde{G})Lv$ weakly in \tilde{V}' . We also obtain from (3.13) that (3.9) holds.

Proposition 3. For each $k > 0$, the equation

$$(3.17) \quad L^*v + kJv + L^*(\tilde{A} + \tilde{G})Lv = 0$$

has a solution $v \in \tilde{V}'$.

Proof. Let $B_r = \{v \in \tilde{V} : \|v\| \leq r\}$ for $r > 0$. Since the mapping $L^* + L^*(\tilde{A} + \tilde{G})L : \tilde{V} \rightarrow \tilde{V}'$ is pseudo-monotone and $J: \tilde{V} \rightarrow \tilde{V}'(\subset \tilde{V}')$ is

monotone hemicontinuous, we can see that the sum $kJ + L^* + L^*(\tilde{A} + \tilde{G})L$ is also pseudo-monotone (cf. Proposition 23 of (4)), for each $k > 0$. Then we have, by using theorem 7.8 of (6), that for each $n \geq 1$, there exists a solution $v_n \in B_n$ of the inequality

$$(3.18) \quad \langle L^*v_n + kJv_n + L^*(\tilde{A} + \tilde{G})Lv_n, z - v_n \rangle \geq 0 \quad \text{for all } z \in B_n.$$

The inequality (3.18) implies that for each $m \geq 1$,

$$(3.19) \quad \begin{aligned} \limsup_{n \rightarrow \infty} (\langle v_n + (\tilde{A} + \tilde{G})Lv_n, Lv_n - Lv_m \rangle + k \langle Jv_n, v_n - v_m \rangle) \\ = \limsup_{n \rightarrow \infty} \langle L^*v_n + kJv_n + L^*(\tilde{A} + \tilde{G})Lv_n, v_n - v_m \rangle \leq 0. \end{aligned}$$

By putting $v = 0$ in (3.18), we have

$$(3.20) \quad k \|v_n\|_*^2 + c_2 \|Lv_n\|^p + c \leq c_3 \quad \text{for all } n \geq 1.$$

The inequality above implies that $\{\|v_n\|_*\}$ and $\{\|Lv_n\|\}$ are bounded. Then we may assume without any loss of generality that v_n converges to $v \in \tilde{V}'$ weakly in \tilde{V}' and Lv_n converges to Lv weakly in \tilde{V} . Then from (3.19), it is easy to see that

$$(3.21) \quad \limsup (\langle v_n + (\tilde{A} + \tilde{G})Lv_n, Lv_n - Lv \rangle + k \langle Jv_n, v_n - v \rangle) \leq 0.$$

Here we choose a sequence $\{z_n\} \subset \tilde{V}'$ such that $z_n \in \text{co}\{v_n\}$, z_n converges to v strongly in \tilde{V}' and Lz_n converges to Lv strongly in \tilde{V} . Then since $\langle Lv_n - Lz_m, v_n - z_m \rangle \geq 0$, we find, by letting $m, n \rightarrow \infty$ that

$$(3.22) \quad \liminf \langle v_n, Lv_n - Lv \rangle \geq 0.$$

Also we have by the monotonicity of J that

$$(3.23) \quad \liminf \langle Jv_n, v_n - v \rangle \geq 0.$$

Combining (3.22) and (3.23) with (3.21), we have

$$(3.23) \quad \limsup \langle (\tilde{A} + \tilde{G})Lv_n, Lv_n - Lv \rangle \leq 0.$$

Then we obtain by Proposition 2 that $(\tilde{A} + \tilde{G})Lv_n$ converges to $(\tilde{A} + \tilde{G})Lv$ weakly in \tilde{V}' and

$$(3.24) \quad \lim \langle (\tilde{A} + \tilde{G})Lv_n, Lv_n \rangle = \langle (\tilde{A} + \tilde{G})Lv, Lv \rangle.$$

Then the inequality (3.18) implies that

$$(3.25) \quad \langle L^*v + kJv + L^*(\tilde{A} + \tilde{G})Lv, z \rangle \geq \langle (\tilde{A} + \tilde{G})Lv, Lv \rangle \quad \text{for all } z \in \tilde{V}.$$

Since $z \in \tilde{V}$ is arbitrary, we find that $L^*v + kJv + L^*(\tilde{A} + \tilde{G})Lv = 0$.

4. Proof of Theorems. In the following, we assume that (A1), (A2), (G1) and (G2) are satisfied. We first show that the assertion of Theorem 1 holds for each $u_0 \in V$ and $f \in \tilde{V}'$.

Let $u_0 \in V$, $f \in \tilde{V}'$ and let \tilde{A} , \tilde{G} be as in section 3. Then by Proposition 3, there exists a solution $v_n \in \tilde{V}'$ of the equation

$$(4.1) \quad L^*v_n + \frac{1}{n}Jv_n + L^*(\tilde{A} + \tilde{G})Lv_n = 0$$

for each $n \geq 1$. Multiplying (4.1) by v_n , we find that

$$(4.2) \quad \frac{1}{n} \|v_n\|_*^2 + c_2 \|Lv_n\|^p + c \leq c_3 \quad \text{for each } n \geq 1.$$

From (4.2), we have that $\{\|Lv_n\|\}$ is bounded. Then it follows from

(3.1) and (3.4) that $\{\|\tilde{A}Lv_n\|_*\}$ and $\{\|\tilde{G}Lv_n\|_*\}$ are bounded. It also

follows from (4.2) that $\lim \|\frac{1}{n}Jv_n\| = 0$. Since L^* is injective in \tilde{V} , the equation (4.1) can be rewritten as

$$(4.3) \quad v_n + \frac{1}{n}(L^*)^{-1}Jv_n + (\tilde{A} + \tilde{G})Lv_n = 0 \quad \text{for each } n \geq 1.$$

Here we note that $\langle (L^*)^{-1}Jv_n, Jv_n \rangle \geq 0$ for $n \geq 1$. Then multiplying (4.3) by Jv_n , we have

$$(4.4) \quad \|v_n\|_*^2 \leq \|(\tilde{A} + \tilde{G})Lv_n\|_* \|Jv_n\| \leq (\|\tilde{A}Lv_n\|_* + \|\tilde{G}Lv_n\|_*) \|v_n\|_*.$$

for $n \geq 1$. Thus we find that $\{\|v_n\|_*\}$ is bounded. Then we may suppose without any loss of generality that v_n converges to $v \in \tilde{V}$ weakly in \tilde{V} and Lv_n converges to Lv weakly in \tilde{V} .

While, we have by multiplying (4.1) by $v_n - v_m$ that

$$(4.5) \quad \langle v_n + (\tilde{A} + \tilde{G})Lv_n, Lv_n - Lv_m \rangle + \langle \frac{1}{n}Jv_n, v_n - v_m \rangle = 0.$$

Then since $\frac{1}{n}Jv_n$ converges to 0 in \tilde{V} as $n \rightarrow \infty$, we find that

$$(4.6) \quad \lim_{n \rightarrow \infty} \langle v_n + (\tilde{A} + \tilde{G})Lv_n, Lv_n - Lv_m \rangle = 0 \quad \text{for each } m \geq 1.$$

Then it is easy to see from (4.6) that

$$(4.7) \quad \lim \langle v_n + (\tilde{A} + \tilde{G})Lv_n, Lv_n - Lv \rangle = 0.$$

Since $\liminf \langle v_n, Lv_n - Lv \rangle \geq 0$, we have by (4.7) that

$$(4.8) \quad \limsup \langle (\tilde{A} + \tilde{G})Lv_n, Lv_n - Lv \rangle \leq 0.$$

Then by Proposition 2, we find that $(\tilde{A} + \tilde{G})Lv_n$ converges to $(\tilde{A} + \tilde{G})Lv$ weakly in \tilde{V} . Therefore we obtain from (4.1) that

$L^*v + L^*(\tilde{A} + \tilde{G})Lv = 0$, i.e., (1.1) has a solution.

Now let $u_0 \in H$ and $\{u_n^0\} \subset V$ be a sequence such that u_n^0 converges to u_0 strongly in H . Then by the argument above, we have that for each $n \geq 1$, there exists a solution u_n of the problem

$$(4.9) \quad \frac{du_n}{dt} + Au_n + G(u_n) = f, \quad 0 < t < T,$$

$$u(0) = u_n^0.$$

Here we assume for simplicity that $f = 0$. Then multiplying (4.9) by u_n and integrating, we have

$$(4.10) \quad \frac{1}{2}|u_n(t)|^2 + C_2 \int_0^T \|u_n(s)\|^p ds < (C + C_3)T + \sup_n |u_n^0|^2.$$

Then $\{u_n\}$ is bounded in \tilde{V} . Also by (2.3) and (2.6), we see that

$\left\{\frac{du_n}{dt}\right\}$ is bounded in \tilde{V}' . Here we put $v_n = \frac{du_n}{dt}$ for each $n \geq 1$. Then

from the observation above, we may suppose that v_n converges to

$v \in \tilde{V}'$ weakly in \tilde{V}' and $u_n = Lv_n + u_n^0$ converges to $u = Lv + u_0$

weakly in \tilde{V} . We set $(\bar{A}z)(t) = A(z(t))$ and $(\bar{G}z)(t) = G(z(t))$ for

each $z \in \tilde{V}$ and $t \in (0, T)$. Now we multiply (4.9) by $u_n - u$ and

integrate. Then we have

$$(4.11) \quad \limsup \langle (\bar{A} + \bar{G})(Lv_n + u_n^0), (Lv_n + u_n^0) - (Lv + u_0) \rangle \\ = \limsup \left\langle \frac{du_n}{dt}, u - u_n \right\rangle \\ \leq \limsup \left(-|u(T) - u_n(T)|^2 + |u_0 - u_n^0|^2 + \left\langle \frac{du}{dt}, u_n - u \right\rangle \right) \leq 0.$$

Therefore the hypothesis of Proposition 2 is satisfied with Lv_n

replaced by $Lv_n + u_n^0$ and Lv replaced by $Lv + u_0$. It is easy to

verify that the proof of Proposition 2 remains valid for A, G, Lv

and Lv_n replaced by \bar{A} , \bar{G} , $Lv + u_0$ and $Lv_n + u_n^0$, respectively. Therefore we find that $(\bar{A} + \bar{G})(Lv_n + u_n^0)$ converges to $(\bar{A} + \bar{G})(Lv + u_0)$ weakly in \tilde{V} . Thus we obtain $v + (\bar{A} + \bar{G})(Lv + u_0) = 0$. This implies that $u = Lv + u_0$ is a solution of (1.1). We can see that $u \in C(0, T; H)$ by the usual argument (cf. theorem 4.5 of (3)),

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