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Periodicity of solutions to some parabolic-elliptic variational inequalities

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§1. Results

In this paper, we report the results of Kenmochi-Kubo [2] and give the outline of proofs. Let $\Omega\subset R^N$ $(N\geqq 1)$ be an open bounded set with smooth boundary Γ. We are interested in periodic behavior of solutions to parabolic-elliptic problems with mixed-type boundary conditions prescribed on time-dependent parts of the boundary. Assume that Γ admits the decomposition: $\Gamma = \Gamma_D(t)\cup\Gamma_N(t)\cup\Gamma_U(t)$, for each $t\in R$, where $\Gamma_i(t)$ $(i=D,N,U)$ are mutually disjoint measurable subsets of Γ. Let $\rho:R\to R$ be a non-decreasing Lipschitz-continuous function. The following system is studied:

$$\rho(v)' - \Delta v = f \qquad \text{in } (0,\infty)\times\Omega,$$

$$\rho(v(0,\cdot)) = u_0 \qquad \text{in } \Omega,$$

$$v = 0 \qquad \text{on } \cup_{t>0}\{t\}\times\Gamma_D(t),$$

$$\partial_\nu v = 0 \qquad \text{on } \cup_{t>0}\{t\}\times\Gamma_N(t),$$

$$v \leqq 0, \quad \partial_\nu v \leqq 0, \quad v\cdot\partial_\nu v = 0 \qquad \text{on } \cup_{t>0}\{t\}\times\Gamma_U(t).$$

Here $\rho(v)'=\frac{\partial}{\partial t}\rho(v)$ and ∂_ν is the outward normal derivative on Γ. These kinds of problems arise from the free boundary problems for satulated-unsatulated flows in porous media. We refer to [3, 4] and their references for related topics. In order to give a notion of weak solutions in variational sense, let us introduce the convex sets

$$K(t) = \{z\in H^1(\Omega);\ z=0 \text{ a.e. on } \Gamma_D(t),\ z\leq 0 \text{ a.e. on } \Gamma_U(t)\}, \quad \text{for } t\in R.$$

<u>Definition</u>. Let $J=R$ or R_+. Let $f\in L^2_{loc}(J;L^2(\Omega))$. Then a function $v\in L^2_{loc}(J;H^1(\Omega))$ is called <u>a weak solution to</u> $E(K(t),\rho,f)$ <u>on</u> J, if $v(t)\in K(t)$ for a.e. $t\in J$, $\rho(v)\in W^{1,2}_{loc}(J;L^2(\Omega))$ and v satisfies the following variational inequality for a.e. $t\in J$:

$$\int_\Omega(\rho(v)'(t)-f(t))(v(t)-z)dx + \int_\Omega \nabla v(t)\cdot\nabla(v(t)-z)dx \leq 0,$$

$$\text{for all } z\in K(t).$$

Let us assume the following geometric condition.

(A.1) For each $t\in R_+$ there is a C^1-diffeomorphism $\theta(t,\cdot):\overline{\Omega} \to \overline{\Omega}$ such that

(i) $\theta(0,\cdot) = Id$;

(ii) $\Gamma_i(t) = \theta(t,\Gamma_i(0))$, $i=D,N,U$, for all $t\in R_+$;

(iii) $\frac{\partial}{\partial x_j}\theta$, $\frac{\partial}{\partial t}\theta$, $\frac{\partial^2}{\partial x_j\partial t}\theta \in C^0(R_+\times\overline{\Omega})$;

(iv) $meas_\Gamma \bigcap_{t\geq 0}\Gamma_D(t) > 0$ ($meas_\Gamma$ denotes the surface measure on Γ).

Lemma 1 (cf. Kenmochi-Pawlow [3]). Assume (A.1) holds as well as

$$f \in W^{1,1}_{loc}(R_+;L^2(\Omega)). \tag{A.2}$$

Let u_0 be such that there is $v_0 \in K(0)$ with $u_0=\rho(v_0)$. Then there is a unique weak solution v to $E(K(t),\rho,f)$ on R_+ satisfying $\rho(v)|_{t=0} = u_0$.

Also the existence of a periodic solution is known.

Lemma 2 (cf. Kenmochi-Kubo [1]). In addition to (A.1) and (A.2) assume that there is a constant $T>0$ such that

$$f(t+T) = f(t) \quad \text{and} \quad \Gamma_i(t+T,\cdot) = \Gamma_i(t) \ (i=D,N,U), \text{ for all } t \in R_+. \tag{A.3}$$

Then there is a weak solution ω to $E(K(t),\rho,f)$ on R_+ such that

$$\omega(t+T) = \omega(t), \qquad \text{for a.e. } t \in R_+.$$

Such a solution ω is called a T-periodic solution. Any T-periodic solution can be extended as a solution on the whole of R by using T-periodicity, provided that we extend the function f and $\Gamma_i(t)$ (i=D,N,U) periodically on R. The main result is stated as follows.

Theorem. Under conditions (A.1), (A.2) and (A.3), T-periodic solution ω to $E(K(t),\rho,f)$ is unique and asymptotically stable in the sense that for any weak solution v to $E(K(t),\rho,f)$ on R_+

$$\rho(v)(t) - \rho(\omega)(t) \longrightarrow 0 \quad \text{in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \text{ as } t \to \infty.$$

Moreover the T-periodic solution ω is the only one weak solution on R such that the trajectry $\{\omega(t);\ t \in R\}$ is bounded in $L^2(\Omega)$.

We shall give the outline of the proof of this theorem in the next section. For the detailed proof, see [2].

Remark (cf. [1, 3, 4]). As far as Lemmas 1 and 2 are concerned, condition (iv) of (A.1) can be replaced by weaker one:

(iv)′ $\quad \mathrm{meas}_\Gamma\ \Gamma_D(t) > 0, \qquad$ for all $t \in R_+$.

§2. Outline of Proof

The proof of Theorem is based on the following two lemmas.

Lemma 3. Assume (A.1), (A.2) and (A.3) hold. Let ω and v be weak solutions to $E(K(t),\rho,f)$ on R_+. Suppose that ω is T-periodic and that $\omega \leq v$ (or $\omega \geq v$) a.e. in $R_+ \times \Omega$. Then we have

(1) $$\rho(v)(t+nT) \longrightarrow \rho(\omega)(t) \quad \text{in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \text{ as } n \to \infty$$

for all $t \in R_+$.

<u>Lemma 4</u>. Let v and $\hat{v}$ be weak solutions such that $v \leqq \hat{v}$ a.e in $R_+ \times \Omega$. Then

(2) $\qquad \partial_\nu v(t) \geqq \partial_\nu \hat{v}(t)$ in the sense of $H^{-1/2}(\Gamma)$ for a.e. $t \in R_+$,

that is $\langle \partial_\nu v(t), z \rangle \geqq \langle \partial_\nu \hat{v}(t), z \rangle$ for all $z \in H^{1/2}(\Gamma)$ with $z \geqq 0$. Here $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

<u>Proof of Lemma 4</u>. Fix $t \in R_+$. For each $\lambda > 0$ and $\mu > 0$ let $v_{\lambda,\mu}(t) \in H^1(\Omega)$ be the solution to

$$\begin{cases} v_{\lambda,\mu}(t) - \lambda \Delta v_{\lambda,\mu}(t) = v(t) & \text{in } \Omega, \\ -\partial_\nu v_{\lambda,\mu}(t) = \frac{1}{\mu} \chi_{\Gamma_D(t)} \cdot v_{\lambda,\mu}(t) + \frac{1}{\mu} \chi_{\Gamma_U(t)} \cdot [v_{\lambda,\mu}(t)]^+ & \text{on } \Gamma, \end{cases}$$

where $\chi_{\Gamma_D(t)}$ and $\chi_{\Gamma_U(t)}$ are the characteristic functions of the sets $\Gamma_D(t)$ and $\Gamma_U(t)$, respectively. And let $\hat{v}_{\lambda,\mu}(t)$ be similarly defined. The boundary conditions imply that $\partial_\nu v_{\lambda,\mu}(t)$, $\partial_\nu \hat{v}_{\lambda,\mu}(t) \in L^2(\Gamma)$. Also it follows from $v(t) \leqq \hat{v}(t)$ that $v_{\lambda,\mu}(t) \leqq \hat{v}_{\lambda,\mu}(t)$. Consequently $-\partial_\nu v_{\lambda,\mu}(t) \leqq -\partial_\nu \hat{v}_{\lambda,\mu}(t)$ on Γ. Since $\partial_\nu v_{\lambda,\mu}(t)$ and $\partial_\nu \hat{v}_{\lambda,\mu}(t)$ converge to $\partial_\nu v(t)$ and $\partial_\nu \hat{v}(t)$, respectively in $H^{-1/2}(\Gamma)$ as $\mu \downarrow 0$ and $\lambda \downarrow 0$, we have (2). See [2; Proposition 4.1] for the detail.

q.e.d.

Proof of Lemma 3. We shall prove in the case $\omega \leqq v$. The case $\omega \geqq v$ is similarly proved.

Since $t \mapsto |[\rho(v)(t)-\rho(\omega)(t)]^+|_{L^1(\Omega)}$ is non-increasing (cf. [3, 4]), we have by $v \geqq \omega$

$$t \mapsto \int_\Omega \{\rho(v)(t)-\rho(\omega)(t)\}dx \quad \text{is non-increasing.}$$

In particular, since ω is T-periodic,

$$\int_\Omega \rho(v)(mT)dx \leqq \int_\Omega \rho(v)(nT)dx \qquad \text{for all } n \leqq m \quad (n, m \in N).$$

Therefore

$$\lim_{n\to\infty} \int_\Omega \rho(v)(nT)dx \quad \text{exists.} \tag{3}$$

Next by virtue of [1; Theorem 1], $\{\rho(v)(t);\ t \in R_+\}$ is bounded in $H^1(\Omega)$. Hence on account of the convergence result [4; Theorem 1.4], there are a subsequence $\{n_k\}$ of $\{n\}$ and a weak solution v^* to $E(K(t),\rho,f)$ on R_+ such that

$$\rho(v)(t+n_kT) \longrightarrow \rho(v^*)(t) \quad \text{in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \text{ as } k\to\infty \text{ for all } t \in R_+. \tag{4}$$

We are going to show that $v^* \equiv \omega$. Then the entire sequence $\rho(v)(t+nT)$ converges to $\rho(\omega)(t)$ and we have (1). First by (3) and

(4) we see that

$$(5)\qquad \int_\Omega \rho(v^*)(nT)dx = \lim_{k\to\infty} \int_\Omega \rho(v)(nT+n_kT)dx$$

$$= \lim_{m\to\infty} \int_\Omega \rho(v)(mT)dx \qquad (\text{put } m = n+n_k)$$

$$= \lim_{k\to\infty} \int_\Omega \rho(v)(n_kT)dx$$

$$= \int_\Omega \rho(v^*)(0)dx, \qquad \text{for all } n\in N.$$

Therefore from the equations for v^* and ω it follows that

$$0 = \int_0^{nT} dt \frac{d}{dt}\int_\Omega \{\rho(v^*)(t)-\rho(\omega)(t)\}dx$$

$$= \int_0^{nT} dt \int_\Omega \Delta(v^*(t)-\omega(t))dx$$

$$= \int_0^{nT} \langle \partial_\nu(v(t)-\omega(t)),1\rangle dt, \qquad \text{for all } n\in N.$$

On the other hand, it is evident that $\omega \leqq v^*$. Therefore by (2)

$$\partial_\nu \omega(t) \geqq \partial_\nu v^*(t) \quad \text{in the sense of } H^{-1/2}(\Gamma) \quad \text{for a.e. } t\in R_+.$$

Hence we have

$$\langle \partial_\nu(v(t)-\omega(t)),1\rangle = 0, \qquad \text{for a.e. } t\in R_+.$$

From this we can conclude that

(6) $\partial_\nu \omega(t) = \partial_\nu v^*(t)$ in $H^{-1/2}(\Gamma)$ for a.e. $t\in R_+$.

Next put $\Gamma_0 = \bigcap_{t\geq 0}\Gamma_D(t)$ and $V \equiv \{z\in H^1(\Omega);\ z=0$ on $\Gamma_0\}$. Since $\mathrm{meas}_\Gamma \Gamma_0 > 0$ by assumption, for each $t\in R_+$ there is a unique solution $u(t)\in V$ of the following variational problem:

(7) $$\int_\Omega \nabla u(t)\cdot\nabla z dx = \int_\Omega \{\rho(v^*)(t)-\rho(\omega)(t)\} z dx \qquad \text{for all } z\in V.$$

It is seen from Poincaré's inequality that there exists a constant $C_1>0$ such that

(8) $$|\nabla u(t)|_{L^2(\Omega)} \leq C_1 |\rho(v^*)(t)-\rho(\omega)(t)|_{L^2(\Omega)} \qquad \text{for all } t\in R_+.$$

From (6) and (7) we observe that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2_{L^2(\Omega)} &= \int_\Omega \nabla u'(t)\cdot\nabla u(t) dx \\ &= \int_\Omega \{\rho(v^*)(t)-\rho(\omega)(t)\}' u(t) dx \\ &= \int_\Omega \Delta(v^*(t)-\omega(t)) u(t) dx \\ &= -\int_\Omega \nabla(v^*(t)-\omega(t))\cdot\nabla u(t) dx \\ &= -\int_\Omega (v^*(t)-\omega(t))\{\rho(v^*)(t)-\rho(\omega)(t)\} dx.\end{aligned}$$

Hence by (8) and the Lipschitz continuity of ρ,

$$\frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2_{L^2(\Omega)} + C_2|\nabla u(t)|^2_{L^2(\Omega)}$$

$$\leq \frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2_{L^2(\Omega)} + C_3|\rho(v^*)(t)-\rho(\omega)(t)|^2_{L^2(\Omega)}$$

$$\leq \frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2_{L^2(\Omega)} + \int_\Omega (v^*(t)-\omega(t))\{\rho(v^*)(t)-\rho(\omega)(t)\}dx$$

$$\leq 0, \qquad \text{for a.e. } t\in R_+.$$

From this inequalitiy we can conclude that

$$\frac{d}{dt}|\nabla u(t)|^2_{L^2(\Omega)} \leq 0 \qquad \text{and} \qquad \int_0^\infty |\nabla u(t)|^2_{L^2(\Omega)} dt < \infty.$$

Consequently

$$|\nabla u(t)|_{L^2(\Omega)} \longrightarrow 0 \qquad \text{as } t \longrightarrow \infty.$$

Combining this with (7) we obtain

$$(9) \qquad \int_\Omega \{\rho(v^*)(t)-\rho(\omega)(t)\}z dx \longrightarrow 0 \quad \text{as } t\longrightarrow\infty \quad \text{for all } z\in V.$$

Since $\{\rho(v^*)(t)-\rho(\omega)(t);\ t\in R_+\}$ is bounded in $L^2(\Omega)$ ([1; Theorem 1]) and V is dense in $L^2(\Omega)$, the convergence (9) holds for all $z\in L^2(\Omega)$. In particular ($z\equiv 1$)

$$\int_\Omega \{\rho(v^*)(nT)-\rho(\omega)(nT)\}dx \longrightarrow 0 \qquad \text{as } n\to\infty.$$

On the other hand, the T-periodicity of ω and (5) imply that

$$\int_\Omega \{\rho(v^*)(nT)-\rho(\omega)(nT)\}dx = \int_\Omega \{\rho(v^*)(0)-\rho(\omega)(0)\}dx \qquad \text{for all } n\in N.$$

Hence

$$\int_\Omega \{\rho(v^*)(0)-\rho(\omega)(0)\}dx = 0.$$

Since $\rho(v^*)(0) \geqq \rho(\omega)(0)$, we have $\rho(v^*)(0) = \rho(\omega)(0)$. This implies $v^* \equiv \omega$. Thus we have proved Lemma 3. q.e.d.

Proof of Theorem. First we shall show the uniqueness of $L^2(\Omega)$-bounded solution on R. Uniqueness of T-periodic solution follows from this. Let ω be a T-periodic solution and let v be a weak solution on R such that $\{v(t);\ t\in R\}$ is bounded in $L^2(\Omega)$. We first assume that $\omega \leqq v$ a.e. in $R\times\Omega$. Since $L^2(\Omega)$-boundedness implies $H^1(\Omega)$-boundedness (cf. [1, 3, 4]), there is a subsequence $\{n_k\}$ of $\{n\}$ and a weak solution v^* on R such that

$$v(t-n_kT) \longrightarrow v^*(t) \quad \text{in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \quad \text{as } k\to\infty.$$

On the other hand it follows from (2) and $\omega \leqq v$ that

(10) $$\frac{d}{dt}\int_\Omega \{\rho(v)(t)-\rho(\omega)(t)\}dx = \int_\Omega \Delta(v(t)-\omega(t))dx$$

$$= \langle \partial_\nu(v(t)-\omega(t)),1\rangle$$

$$\leqq 0.$$

Hence

(11) $$\lim_{t\to-\infty} \int_\Omega \{\rho(v)(t)-\rho(\omega)(t)\}dx \equiv d \quad \text{exists.}$$

Therefore for all $t\in R$

$$d = \lim_{k\to\infty} \int_\Omega \{\rho(v)(t-n_kT)-\rho(\omega)(t-n_kT)\}dx = \int_\Omega \{\rho(v^*)(t)-\rho(\omega)(t)\}dx,$$

By the way, since $\omega \leqq v$, it follows from Lemma 3 that

$$v^*(t+nT) - \omega(t+nT) \longrightarrow 0 \quad \text{in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \quad \text{as } n\to\infty.$$

Consequently

$$0 = \lim_{n\to\infty} \int_\Omega \{\rho(v^*)(t+nT)-\rho(\omega)(t+nT)\}dx = d.$$

Therefore it follows from (10) and (11) that $\int_\Omega \{\rho(v)(t)-\rho(\omega)(t)\}dx$ is non-negative and non-decreasingly converges to $d = 0$ as $t\to-\infty$. Hence $\int_\Omega \{\rho(v)(t)-\rho(\omega)(t)\}dx \equiv 0$ so that $\rho(v) \equiv \rho(\omega)$ by $v \geqq \omega$. Therefore $v \equiv \omega$. Similarly we can show that $v \equiv \omega$ in the case $\omega \geqq v$.

Now let v be an arbitrary $L^2(\Omega)$-bounded solution on R. For each $n \in N$, put $u_{0,n} = \rho(v)(-nT) \vee \rho(\omega)(-nT)$ and let v_n be the weak solution to $E(K(t),\rho,f)$ on $[-nT,\infty)$ satisfying $\rho(v_n)|_{t=-nT} = u_{0,n}$. Comparison result implies that $v_n \geqq v \vee \omega$ on $[-nT,\infty)$. Also $L^2(\Omega)$-boundedness on v implies the uniform $L^2(\Omega)$-boundedness of $\{v_n\}_{n \in N}$. Therefore there is a subsequence $\{n_k\}$ of $\{n\}$ and an $L^2(\Omega)$-bounded solution v^* on R such that

$$v_{n_k}(t) \longrightarrow v^*(t) \quad \text{in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \text{ as } k \longrightarrow \infty$$

$$\text{for all } t \in R.$$

Clearly $v^* \geqq v \vee \omega$ on R. Therefore from the argument before we have $v^* \equiv \omega$. Similarly there is an $L^2(\Omega)$-bounded solution v_* on R such that $v_* \leqq v \wedge \omega$. And $v_* \equiv \omega$. Hence we have $v \equiv \omega$.

Next we shall show the asymptotic stability of the T-periodic solution ω. Let v be any weak solution on R_+. Then as before there are weak solutions $\overline{v}$ and $\underline{v}$ such that $\underline{v} \leqq v \wedge \omega \leqq v \vee \omega \leqq \overline{v}$. By Lemma 3, $\rho(\overline{v})(t+nT) - \rho(\omega)(t+nT) \longrightarrow 0$ $(n \longrightarrow \infty)$. On the other hand (cf. [3; Lemma 5.4]), $|\rho(\overline{v})(t) - \rho(\omega)(t)|_{L^1(\Omega)} \leqq |\rho(\overline{v})(s) - \rho(\omega)(s)|_{L^1(\Omega)}$ for all $0 \leqq s \leqq t < \infty$. So we have $\rho(\overline{v})(t) - \rho(\omega)(t) \longrightarrow 0$ $(t \longrightarrow \infty)$: Similarly $\rho(\underline{v})(t) - \rho(\omega)(t) \longrightarrow 0$ $(t \longrightarrow \infty)$. Since $\rho(\underline{v})(t) - \rho(\omega)(t) \leqq \rho(v)(t) - \rho(\omega)(t) \leqq \rho(\overline{v})(t) - \rho(\omega)(t)$, we obtain $\rho(v)(t) - \rho(\omega)(t) \longrightarrow 0$ $(t \longrightarrow \infty)$. q.e.d.

References

[1] N. Kenmochi and M. Kubo, Periodic solutions to a class of nonlinear variational inequalities with time-dependent constraints, Funkcialaj Ekvacioj, 30 (1987), 333-349.

[2] N. Kenmochi and M. Kubo, Periodic behavior of solutions to parabolic-elliptic free boundary problems, preprint.

[3] N. Kenmochi and I. Pawlow, Parabolic-elliptic free boundary problems with time-dependent obstacles, submitted to Japan J. Appl. Math.

[4] N. Kenmochi and I. Pawlow, A class of nonlinear elliptic-parabolic equations with time-dependent constraints, Nonlinear Analysis T.M.A., 10 (1986), 1181-1202.