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THE DIMENSION OF HYPERSPACES OF CONTINUA

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1. Introduction.

By a *continuum* we mean a compact connected metric space. Let X be a continuum. The *hyperspaces* of a continuum X are the following.

$2^X = \{A \mid A$ is a closed subset of X and $A \neq \phi\}$ and

$C(X) = \{A \in 2^X \mid A$ is connected$\}$.

Let A, B $\in 2^X$. Define a metric $d_H(A,B)$ by

$d_H(A,B) = \inf\{\varepsilon > 0 \mid U(A,\varepsilon) \supset B$ and $U(B,\varepsilon) \supset A\}$, where $U(A,\varepsilon)$ denotes the ε-neighborhood of A in X.

Then the metric d_H is called the *Hausdorff metric*. Then 2^X and C(X) are pathwise connected continua. In this note, we consider the dimension of 2^X and C(X).

2. Dimension of 2^X and C(X).

Mazurkiewics showed the following theorem about the dimension 2^X.

2.1. Theorem. *Let* X *be a nondegenerate continuum. Then* 2^X *contains the Hilbert cube* $Q = [0,1]^\infty$. *Hence* $\dim 2^X = \infty$.

By 2.1, the dimension of 2^X has been completely determined. Next, we discuss the dimension of C(X).

2.2. Theorem (Kelley). *If* X *is a Peano (locally connected) continuum, then* C(X) *has a finite dimension if and only if* X *is a finite linear graph.*

For the case of non Peano continua, we have

2.3. Theorem (Kelley). *If* X *is a hereditarily indecomposable continuum with* $\dim X \geq 2$, *then* $\dim C(X) = \infty$.

A continuum X is *decomposable* if there exist $A, B \in C(X)$ such that $A \cup B = X$ and $A \neq X \neq B$. A continuum X is *hereditarily indecomposable* if any subcontinuum of X is not decomposable. It is well-known that the pseudo-arc is a hereditarily indecomposable continuum.

2.4. Theorem (Bing). *Let* X *be a continuum with* $\dim X = n$. *Then there exists a subcontinuum* A *of* X *such that* A *is hereditarily indecomposable and* $\dim A = n - 1$.

Combining 2.3 and 2.4, we have the following

2.5. Theorem (Eberhart, Nadler and Rogers). *If* X *is a continuum with* $\dim X \geq 3$, *then* $\dim C(X) = \infty$.

Finally, the following interesting problem remains open.

Problem 1. If X is any 2-dimensional continuum, then must $\dim C(X) = \infty$?

In relation to Problem 1, some partial answers have been obtained.

2.6. Theorem (Rogers). *If* X *is an arcwise connected continuum and* $\dim X = 2$, *then* $\dim C(X) = \infty$.
If X *contains a subcontinuum which is homeomorphic to a cartesian product of two (nondegenerate) continua, then* $\dim C(X) = \infty$.

3. Dimension of certain 2-dimensional continua.

In this section, we give a partial answer to above problem. Let D be the closed unit ball in the plane, $D = \{(x,y) \in E^2 \mid x^2+y^2 \leq 1\}$, and let $S^1 = \{(x,y) \in D \mid x^2+y^2=1\}$. A map f from a topological space Z to D is *essential* provided that

$$f|f^{-1}(S^1) \colon f^{-1}(S^1) \to S^1$$

can not be extended to a map defined on all of Z to S^1. A map f: X $\to$ Y between continua is *weakly confluent* provided that for each subcontinuum B of Y there is a subcontinuum A of X such that f(A) = B.

The following fact is very useful in continua theory and we need it.

3.1 (Mazurkiewicz). *Let* X *be a continuum. If a map* f: X $\to$ D *is essential, then* f *is weakly confluent.*

Let $H^1(X)$ denote the *Cech cohomology group* of X. Then we have

3.2. Theorem (Grispolakis, Tymchatyn and Kato).
If X *is a 2-dimensional continuum and* rank $H^1(X) < \infty$, *then* dim C(X) = ∞. *More precisely,* C(X) *contains the Hilbert cube* Q, *hence* C(X) *is strongly infinite-dimensional.*

The first part of 3.2 already has been obtained by combining some theorems of [20] and [9].
Recently, the author learned the existence of the paper [20]. But, the proof of author is different from that of [20] (see [5]) and it is more simple. In [5], We used the following theorem.

3.3. Theorem (Kato). *If* X *is a continuum with* dim X $\geq$ 2, *then there is an uncountable closed subset* A *of* C(X) *such that if* A $\in$ A, *then* rank $H^1(A) = \infty$.

3.4. Corollary (Kato). *If* X *is a continuum and each subcontinuum is an* FANR, *then* $\dim X \le 1$.

3.5. Remark. If there exists a continuum X with $\dim X = 2$ such that $\dim C(X) < \infty$, then the continuum X has a quite strange property. For example, for any closed subset A having countable components and any map $f: A \to S^1$, there is an extension $F: X \to S^1$ of f. It is well-known that a continuum Y is $\dim Y \ge 2$ if and only if there exist a closed subset A of Y and a map $f: A \to S^1$ which cannot be extended over Y. Also, it is known that if Z is a hereditarily indecomposable continuum, then for any closed subset A having countable components and any map $f: A \to S^1$ there is an extension $F: X \to S^1$ of f.

Let $\mathcal{P}$ be a collection of compact polyhedra. A continuum X is *$\mathcal{P}$-like* if for any $\varepsilon > 0$ there is an onto map $f: X \to P$ such that $P \in \mathcal{P}$ and $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each $y \in P$. For a continuum X, we consider the following index I(X) as follows: $I(X) \le n$ if and only if there is a collection $\mathcal{P}$ of compact polyhedra such that X is $\mathcal{P}$-like and rank $H^1(P) \le n$ for each $P \in \mathcal{P}$.

3.6. Lemma. *For a continuum* X, $I(X) \ge$ rank $H^1(X)$.

3.7. Corollary. *Let* X *be a 2-dimensional continuum.* If X *is $\mathcal{P}$-like and $\mathcal{P}$ is a finite collection of compact polyhedra, then* dim C(X) = ∞.

Finally, we give the following ploblem.

Problem 2. Give the direct proof of 2.5 without using the Bing's result 2.4. The author believes that if Problem 2 is solved, Problem 1 would be solved by using the same methods.

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