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Locally nice space における Covering property と集合論

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In this note, all spaces are assumed to be regular T_1. P.Daniels proved that normal locally compact zero-dimensional metacompact spaces are subparacompact, see [Da]. It is also known an example of locally compact metacompact space which is not subparacompact, see [Bu,4.2]. In this note, we shall characterize subparacompactness of locally Lindelöf(or locally ω_1-spread) spaces. As a corollary, it will be shown that locally ω_1-spread(or normal locally Lindelöf) submetacompact spaces are subparacompact.

First, we remind some basic definitions and introduce some notations.

For a regular uncountable cardinal κ, a subset of κ is said to be closed unbounded(abbreviated as *cub*) if it is closed and unbounded in its order topology, and a subset of κ is said to be *stationary* if it intersects with every cub set of κ. For a collection $\mathcal{C}$ of subsets of a set X and $x \in X$, $(\mathcal{C})_x$ denotes the collection $\{ C \in \mathcal{C} : x \in C \}$.

For a pairwise disjoint family $\mathcal{F} = \{ F_\alpha : \alpha \in A \}$ of subsets of a space, an *expansion* $\mathcal{U} = \{ U_\alpha : \alpha \in A \}$ of $\mathcal{F}$ is a family of subsets such that $F_\alpha \subset U_\alpha$ for every $\alpha \in A$ and $U_\alpha \cap F_\beta = 0$ for every $\alpha, \beta \in A$ with $\alpha \neq \beta$. An *open expansion* is an expansion whose elements are open. A *(open) separation* is a pairwise disjoint (open) expansion. A subspace Y of a space is *discrete* if there is an open expansion of $\{ \{y\} : y \in Y \}$. A disjoint family $\mathcal{F}$ of a space is said to be *separated* if it has an open separation. A subspace Y of a space is said to be separated if $\{ \{y\} : y \in Y \}$ has an open separation.

Let κ be a cardinal. A space X is *(strongly) κ-collectionwise Hausdorff*(abbreviated as (strongly) κ-CWH) if every closed discrete subspace of cardinality κ has an (discrete, respectively) open separation. When κ is a regular uncountable, a space X is said to be *(strongly) stationary κ-collectionwise Hausdorff*(abbreviated as (strongly) κ-SCWH) if for every stationary set S of κ and closed discrete subspace $\{ x_\alpha : \alpha \in S \}$ of distinct points indexed by S, there is a stationary subset S' of κ such that $S' \subset S$ and $\{ x_\alpha : \alpha \in S' \}$ has an (discrete, respectively) open separation. Similarly, a space X is said to be *(strongly) cub κ-collectionwise Hausdorff*(abbreviated as (strongly) κ-CCWH) if for every cub set S of κ and closed discrete subspace $\{ x_\alpha : \alpha \in S \}$ of distinct points indexed by S, there is a stationary subset S' of κ such that $S' \subset S$ and $\{ x_\alpha : \alpha \in S' \}$ has an (discrete, respectively) open separation. A space is *(strongly) CWH* if it is (strongly) κ-CWH for every cardinal κ. A space is *(strongly) SCWH* if it is (strongly) κ-SCWH for every *regular uncountable* cardinal. Similarly (strongly) κ-CCWH or (strongly) CCWH.

A space is *countable chain condition* (abbreviated as ccc) if there is no pairwise disjoint family of uncountably many non-empty open sets. A space is ω_1*-compact*(ω_1*-spread*) if there is no closed discrete(discrete) subspace of size ω_1. Then the implications "Lindelöf $\rightarrow \omega_1$-compact $\leftarrow \omega_1$-spread $\rightarrow$ ccc" hold.

Let P be a topological property. A space is said to be *locally P* if every point has an open neighborhood whose closure has the property P. Note that if a space is locally ω_1-spread, then so is every subspace.

Next we list some basic facts and well known results.

THE PRESSING DOWN LEMMA. *Assume that S is a stationary subset of a regular cardinal κ and $f:S \rightarrow \kappa$ satisfies $f(\alpha) < \alpha$ for each α in S. Then there is a stationary set $S' \subset S$ and a $\beta \in \kappa$ such* that $f(\alpha) = \beta$ for every α in S'.

FACT.([Ta]) *$2^{\omega} < 2^{\omega_1}$ iff for every closed discrete subset $\{ x_\alpha : \alpha \in \omega_1 \}$ of normal spaces of character $\leqq 2^{\omega}$, there is a stationary set $S \subset \omega_1$ such that $\{ x_\alpha : \alpha \in S \}$ is separated.*

This Taylor's result can be generalized as follows.

FACT. *$2^{\kappa} < 2^{\kappa^+}$ iff normal spaces of character $\leqq 2^{\kappa}$ are κ^+-CCWH.*

FACT. *Under V = L, normal spaces of character $\leqq 2^{\kappa}$ are κ^+-SCWH.*

But it is known that under V = L, normal spaces of character $\leqq 2^{\omega}$ are CWH([Fl]).

Applying these facts to our results, we can get many consistency results.

Z.Balogh showed that locally Lindelöf(locally ccc), strongly CWH(CWH, respectively), submetaLindelöf spaces are paracompact, see [Ba]. By a similar argument(using the pressing down lemma and induction of the Lindelöf degree), we can prove the next result.

THEOREM. *Locally Lindelöf(locally ccc), strongly SCWH(SCWH, respectively), submetaLindelöf spaces are strongly paracompact.*

DEFINITION. Let m be a natural number. A family $\mathcal{U}$ of subsets of a space X is said to be *point-m* if $|(\mathcal{U})_x| \leq m$ for every x in X.

The following lemmas are essences of our results.

LEMMA. *Let D be a closed discrete subspace of a space X and n be a natural number. If D has a point-n open expansion $\mathcal{U} = \{ U_x : x \in D \}$ such that each member of $\mathcal{U}$ is ccc, then D is separated.*

LEMMA. *Let n be a natural number, X be a normal space and $T = \{ x_\alpha : \alpha \in S \}$ be a closed discrete subspace of X of distinct points, where S is stationary in a regular cardinal κ. Assume T has a point-n open expansion $\mathcal{U} = \{ U_\alpha : \alpha \in S \}$ such that each clU_α is Lindelöf. Then there is a stationary set $S' \subset S$ in κ such that $\{ x_\alpha : \alpha \in S' \}$ is separated.*

Using the above results we can prove:

THEOREM. *Let X be a locally Lindelöf(ω_1-spread) space. Then the following assertions are equivalent.*

1) *X is the countable closed sum of (strongly) paracompact subspaces(i.e. $X = \cup_{n\in\omega}X_n$, where each X_n is closed in X and (strongly) paracompact).*

2) *X is the countable closed sum of normal metacompact subspaces(X is the countable closed sum of metacompact subspaces, respectively).*

3) *X is the countable closed sum of normal submetacompact subspaces(X is submetacompact, respectively).*

4) *X is subparacompact.*

The equivalence 3)⟷4) implies the following corollary.

COROLLARY. *Locally ω_1-spread(or normal, locally Lindelöf) submetacompact spaces are subparacompact.*

The example 4.2 of [Bu] is locally compact 2-boundedly metacompact(for definition, see below), but neither subparacompact nor locally ω_1-spread. The example ii) of 4.9 of [Bu] is normal metacompact but not subparacompact, hence not locally Lindelöf.

In the rest of this note, we shall look at paracompactness of locally nice spaces. It is known that normal, locally compact, boundedly metacompact(or normal, locally Lindelöf, screenable) spaces are paracompact, see [Da]([Ba], respectively).

DEFINITION. Let m be a natural number.

(1) A space is *m-boundedly metacompact* if every open cover has a point-m open refinement.

(2) A space is *boundedly metacompact* if every open cover has a point-m open refinement for some m in ω.

(3) A space is *σ-boundedly metacompact* if for every open cover $\mathcal{U}$, there are a sequence $\{ \mathcal{U}_n : n \in \omega \}$ of weak open refinements of $\mathcal{U}$ and a sequence $\{ m(n) : n \in \omega \}$ of natural numbers such that each $\mathcal{U}_n$ is point-$m(n)$ and $\cup\{ \mathcal{U}_n : n \in \omega \}$ covers X.

Note that bounded metacompactness(or screenability) implies σ-bounded metacompactness and also that σ-bounded metacompactness implies (sub)metaLindelöfness.

THEOREM. *Locally ccc(or normal locally Lindelöf), σ-boundedly metacompact spaces are SCWH(thus strongly paracompact by the* first theorem).

Since it is known that ω_1-compact submetaLindelöf spaces are Lindelöf, we can replace local Lindelöfness by local ω_1-compactness in the above results.

Using this theorem and the Dowker Theorem(in the sense of [En,7.2.3]), we can prove that a locally ccc(or normal, locally Lindelöf) space X is paracompact and $\dim X \leq n - 1$ if and only if X is n-boundedly metacompact.

S.Watson proved that it is consistent that there is a locally compact perfectly normal metaLindelöf space which is not paracompact, see [Wa]. Thus we can not replace σ-bounded metacompactness by metaLindelöfness in the above theorems. Here note that perfectly normal locally compact spaces are locally ccc.

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