

On a question of expansion of closure-preserving families

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Through this paper, all spaces are assumed to be T_1 topological spaces.

We introduce the d -IP-expandability and IP-expandability as follows:

Definition 1. We call a space X d -IP-expandable if for a discrete family $\mathbb{F} = \{ F_\lambda : \lambda \in \Lambda \}$ of closed subsets of X and a family $\mathbb{U} = \{ U_\lambda : \lambda \in \Lambda \}$ of open subsets of X such that $F_\lambda \subset U_\lambda$ for each λ , there exists an interior-preserving family $\mathbb{V} = \{ V_\lambda : \lambda \in \Lambda \}$ of open subsets of X such that $F_\lambda \subset V_\lambda \subset U_\lambda$ for each λ .

Definition 2. We call a space X IP-expandable if for a closure-preserving family $\mathbb{F} = \{ F_\lambda : \lambda \in \Lambda \}$ of closed subsets of X and a family $\mathbb{U} = \{ U_\lambda : \lambda \in \Lambda \}$ of open subsets of X such that $F_\lambda \subset U_\lambda$ for each λ , there exists a family $\mathbb{V} = \{ V_\lambda : \lambda \in \Lambda \}$ of open subsets of X such that $F_\lambda \subset V_\lambda \subset U_\lambda$

for each λ and $\{V_\lambda - F_\lambda : \lambda \in \Lambda\}$ is interior-preserving in X .

These spaces have the following properties:

Proposition 3. If a space X is collectionwise normal, then X is d -IP-expandable.

Proposition 4. If a space X is orthocompact, then X is d -IP-expandable.

Theorem 5. If a space X is submetacompact and d -IP-expandable, then X is orthocompact.

Corollary 6. Let X be a developable space. Then X is d -IP-expandable if and only if X is non-archimedean quasi-metrizable.

Theorem 7. For a space X , the following are equivalent:

- (1) X is an orthocompact developable space.
- (2) X has a development $\{\mathbb{U}_n : n \in \mathbb{N}\}$ such that each \mathbb{U}_n is interior-preserving in X .
- (3) X is a d -IP-expandable developable space.
- (4) X is a semi-stratifiable, non-archimedean quasi-metrizable space.

Corollary 8. If for each $n \in \mathbb{N}$, X_n is an orthocompact developable space, then so is $\prod_{n=1}^{\infty} X_n$.

Theorem 9. If a space X is non-archimedean quasi-metrizable, then X has the property (P):

(P) For a closed G_δ -set F of X , there exists a family \mathcal{U} of open subsets of X satisfying the following:

- (1) $\mathcal{U} / (X - F)$ is interior-preserving in $X - F$.
- (2) For each open set V of X , there exists $U \in \mathcal{U}$ such that

$$V \cap F = U \cap F \subset U \subset V.$$

Corollary 10. If a space X is perfect and non-archimedean quasi-metrizable, then X is d -IP-expandable.

Corollary 11. Under the same hypothesis as above, every closed subset F of X has an outer base \mathcal{U} in X such that \mathcal{U} is interior-preserving in $X - F$.

Theorem 12. Let X be a developable space. Then X is IP-expandable if and only if X is d -IP-expandable.

Theorem 13. Let X be a stratifiable space. Then X is an L -space if and only if X is IP-expandable.

Theorem 14. If a space X is semi-stratifiable, then d -IP--expandability implies D -expandability.

Finally, we propose the following question:

Question. If a space X is developable and quasi--metrizable, then is X d -IP-expandable ?

This is equivalent to the well-known problem due to Junnila whether every developable quasi-metrizable space is non--archimedean quasi-metrizable. For the further details, see [1].

[1] T.Mizokami: Expansion of discrete and closure-preserving families, forthcoming in Proc. Amer. Math. Soc.