

On ANR(stratifiable)-spaces

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In metric spaces, the closed embedding theorem of Kuratowski-Wojdyslawski plays an important role in the development of retract theory. In generalized metric spaces, for a paracomplex (i.e. Hyman's M-space)  $X$ , Hyman [11] proved that  $X$  can be embedded in an AR(paracomplex)-space as a closed subset. For a stratifiable space  $X$ , Cauty [5] constructed a space  $Z(X)$ , and proved that  $Z(X)$  is stratifiable and  $X$  is AR(stratifiable) (resp. ANR (stratifiable)) if and only if  $X$  is a retract (resp. neighborhood retract) of  $Z(X)$ . However, it is not known whether  $Z(X)$  is ANR (stratifiable). In previous paper [16], we constructed a space  $E(X)$  for a space  $X$ , and showed that a stratifiable space  $X$  can be embedded in the AR(stratifiable)-space  $E(X)$  as a closed subset.

In this paper, in section 1 we briefly give some contents in [16], and prove some applications of the closed embedding theorem. In section 2, we consider some problems which arise in and around ANR(stratifiable)-spaces.

Throughout this paper, we assume that all spaces are regular and all maps are continuous. For the definitions of AR, ANR, AE and ANE, see Hu [10].

## 1. The closed embedding theorem and some applications

For a full simplicial complex, we introduced the locally convex topology in [15]; i.e. the strongest topology, which is locally convex, contained in the Whitehead topology. The introduction of this topology is valid by the fact that a full simplicial complex with the Whitehead topology need not be locally convex (cf. [9; pp. 416, 4.3]). Note that a full simplicial complex (with the Whitehead topology) with countable vertices is locally convex (cf. [6; Lemma 4.4]). By using the locally convex topology, we can construct  $E(X)$  for a space  $X$  as follows:

CONSTRUCTION 1.1 ([16; 3.1]). Let  $X$  be a space.  $A(X)$  denotes the full simplicial complex with the locally convex topology which has all points of  $X$  as the set of vertices. Let  $i$  be a canonical bijection from the 0-skeleton of  $A(X)$  onto  $X$ . Then  $E(X)$  is the set  $A(X)$  equipped with the topology generated by sets  $U$  such that

- (C1)  $U$  is open in  $A(X)$  and  $i(U \cap X)$  is open in  $X$ , and
- (C2)  $U$  is convex in  $A(X)$ .

By (C1), it is clear that  $X$  is closed in  $E(X)$ . Note the difference of  $E(X)$  and Cauty's  $Z(X)$  [5]. We obtain the following:

THEOREM 1.2 ([16; Theorem 3.4]). If  $X$  is stratifiable,  $E(X)$  is AR(stratifiable).

Next, the notion of  $L$ -space was introduced in [14]. This is defined as follows: A Hausdorff space is called an  $L$ -space if it can be mapped onto some stratifiable space by a perfect

map. As easily seen by this definition,  $L$ -spaces have some analogous properties as paracompact  $M$ -spaces. For example:

THEOREM 1.3([14;1.3]).  $L$ -spaces are precisely the homeomorphic images of closed subsets of product  $S \times I^\tau$ , where  $S$  is stratifiable,  $I = [0,1]$  and  $\tau$  is an arbitrary cardinal.

Now, we give some applications of the closed embedding theorem. The following theorem is an analogous result of [19; Theorem 4].

THEOREM 1.4. Let  $\mathcal{Q}$  be a closed hereditary class consisting of normal spaces. If every  $\text{ANR}(\text{stratifiable})$ -space is  $\text{ANE}(\mathcal{Q})$ , then every  $\text{ANR}(L)$ -space is  $\text{ANE}(\mathcal{Q})$ .

Proof. Let  $X$  be any  $\text{ANR}(L)$ -space. Then by Theorem 1.3,  $X$  is a closed subset of product  $S \times I^\tau$  where  $S$  is a stratifiable space and  $\tau = w(X)$ . By Theorem 1.2,  $S$  is considered as a closed subset of an  $\text{AR}(\text{stratifiable})$ -space  $E(S)$ . Therefore, we can assume that  $X$  is a closed subset of product  $E(S) \times I^\tau$ . Since  $E(S) \times I^\tau$  is an  $L$ -space,  $X$  is a neighborhood retract of  $E(S) \times I^\tau$ . Since  $E(S)$  is  $\text{ANE}(\mathcal{Q})$  by assumption and  $I^\tau$  is clearly  $\text{ANE}(\mathcal{Q})$ ,  $E(S) \times I^\tau$  is  $\text{ANE}(\mathcal{Q})$ . Therefore  $X$  is  $\text{ANE}(\mathcal{Q})$ . This completes the proof.

For  $\mathcal{Q}$  satisfying the condition in this theorem, see Theorem 2.3 and Problem 2.5 in later section.

For pairs of  $\text{ANR}$ 's, see [13], and for closed embeddings of pairs, see [18].

THEOREM 1.5. Every pair  $(X, X_0)$  of stratifiable spaces, where  $X_0$  is a closed subset of  $X$ , has a closed embedding in a pair of  $\text{AR}(\text{stratifiable})$ 's.

Proof. As easily seen from Construction 1.1,  $(E(X), E(X_0))$  is a required pair of AR(stratifiable)'s.

Note that, in [18;Theorem 1.1], metric case was considered.

## 2. Problems

In general, the following problem naturally arises.

PROBLEM 2.1. For any two classes  $Q$  and  $Q'$  with  $Q \subset Q'$ , is an ANR( $Q$ )-space ANR( $Q'$ )?

For example, the case  $Q = \text{metric}$  has been studied as follows:

THEOREM 2.2. (1) (Dowker [8]) Every ANR(metric)-space is ANR(collectionwise normal and perfectly normal).

(2) (Lisica [12]) Every ANR(metric)-space is ANR(paracompact M).

(3) (Mardešić and Šostak [14]) Every ANR(metric)-space is ANR(L).

For the case  $Q = \text{stratifiable}$ , the following result was shown so far as I know.

THEOREM 2.3 (Borges [1],[2]) Every ANR(stratifiable)-space is ANR(linearly stratifiable).

This theorem is easily seen by [1;Theorem 4.1 and 4.4], [2;Theorem 2.1] and [5;Theorem 1.8].

In connection with Theorem 2.2(2) and (3), the following problem was posed in [14].

PROBLEM 2.4 (Mardešić and Šostak). Is an ANR(stratifiable)-space ANR(L)?

For this problem, a partial answer was proved in [14;Theorem 2.4 and 2.5]. In connection with Theorem 2.3 and Problem 2.4, we pose

PROBLEM 2.5. Find a class  $\mathcal{Q}$  containing stratifiable spaces such that every ANR(stratifiable)-space is ANR( $\mathcal{Q}$ ).

The class of all paracompact  $\sigma$ -spaces is not such a class (cf. [7;Example 2] and [19;Example 2]). In connection with this, Tsuda [19] posed

PROBLEM 2.6(Tsuda). If an ANR(stratifiable)-space  $X$  is ANR(paracompact  $\sigma$ ), then is  $X$  metrizable?

Finally, let  $X$  and  $Y$  be two spaces,  $A$  a closed subset of  $X$  and  $f:A \rightarrow Y$  a map. It is well known [10;pp. 178] that if  $X$ ,  $A$  and  $Y$  are ANR(metric)'s, then the adjunction space  $X \cup_f Y$  is ANR(metric) provided that it is metrizable. This result was essentially proved in successive stages by Borsuk [3], Whitehead [20] and Hanner [9]. For attempt to generalize this theorem, Hyman [11] proved the case of paracomplex spaces. Cauty [4] announced the case of stratifiable spaces, but his proof was false. (This was pointed out by San-nou [17].) So, the case of stratifiable spaces is still open:

PROBLEM 2.7. Let  $X$  and  $Y$  be two stratifiable spaces,  $A$  a closed subset of  $X$  and  $f:A \rightarrow Y$  a map. If  $X$ ,  $A$  and  $Y$  are ANR(stratifiable)'s, is the adjunction space  $X \cup_f Y$  ANR(stratifiable)?

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