数理解析研究所講究録
第 653 巻 1988 年 185-192

或る意味での順序を保存する作用素不等式について

弘前大・理　古 田　孝 之 (Takayuki Furuta)

A capital letter means a bounded linear operator on a Hilbert space. An operator T is said to be positive in case $(Tx,x) \geq 0$ for every x in a Hilbert space. What functions preserve the ordering of positive operators? In other words, what must f satisfy so that

$$A \geq B \geq 0 \quad \text{implies} \quad f(A) \geq f(B)?$$

A function f is said to be operator monotone if a real valued continuous function f satisfies the property stated above. This problem was first studied by K. Löwner, who had given a complete description of operator monotone functions. Also he had shown the following Theorem A.

Theorem A [9][10]. If $A \geq B \geq 0$, then $A^{\alpha} \geq B^{\alpha}$ for each $\alpha \in [0,1]$.

The following result is well known.

Theorem B. $A \geq B \geq 0$ does not always ensure $A^p \geq B^p$ for any $p > 1$.

The purpose of this speech is to show "operator inequalities preserving order in some sense" on A and B in case $A \geq B \geq 0$. Our central results are as follows.

Theorem 1 [3]. If $A \geq B \geq 0$, then for each $r \geq 0$

(i) $$(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$$

(ii) $$A^{(p+2r)/q} \geq (A^r B^p A^r)^{1/q}$$

hold for each p and q such that $p \geq 0$, $q \geq 1$ and $(1+2r)q \geq p+2r$.

Corollary 1 [3]. If $A \geq B \geq 0$, then for each $r \geq 0$

(i) $$(B^r A^p B^r)^{1/p} \geq B^{(p+2r)/p}$$

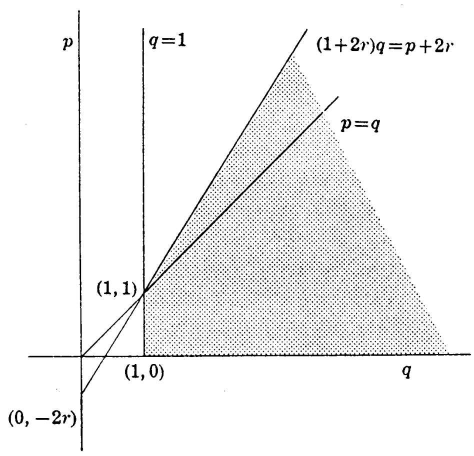
(ii) $$A^{(p+2r)/p} \geq (A^r B^p A^r)^{1/p}$$

hold for each $p \geq 1$.

Corollary 2 [3]. If $A \geq B \geq 0$, then $(BA^2B)^{3/4} \geq B^3$ and $A^3 \geq (AB^2A)^{3/4}$.

Corollary 3 [3]. If $A \geq B \geq 0$, then $(BA^2B)^{1/2} \geq B^2$ and $A^2 \geq (AB^2A)^{1/2}$.

Remark 1. Theorem 1 yields Theorem A when we put $r = 0$ in Theorem 1. Corollary 3 is just an affirmative answer to a conjecture in matrix case [1]. Theorem 1 asserts that although $A^p \geq B^p$ for any $p > 1$ does not always hold even if $A \geq B \geq 0$, $f(A^p) \geq f(B^p)$ and $g(A^p) \geq g(B^p)$ hold where $f(X) = (B^r X B^r)^{1/q}$ and $g(Y) = (A^r Y A^r)^{1/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ and $(1+2r)q \geq p+2r$. (see Figure)



Figure

In order to give a proof to Theorem 1, we show the following Lemma 1.

Lemma 1. If $X \geq 0$ and $\|Y\| \leq 1$, then

(i) $Y^*XY \geq (Y^*X^{\alpha}Y)^{1/\alpha}$ for any α such that $1 \geq \alpha \geq 1/2$

(ii) $(Y^*XY)^{\alpha} \geq Y^*X^{\alpha}Y$ for any α such that $1 \geq \alpha \geq 0$.

Proof. $T = X^{\alpha/2}Y = UH$ be the polar decomposition of T, that is, U is the partial isometry and H is the positive operator such that $H = (T^*T)^{1/2}$. Then

$$Y^*XY = Y^*X^{\alpha/2}X^{(1-\alpha)}X^{\alpha/2}Y = HU^*X^{1-\alpha}UH \tag{1}$$

and

$$H^2 = HU^*UH = Y^*X^{\alpha}Y \tag{2}$$

because U^*U is the initial projection. By the hypothesis $\|Y\| \leq 1$, we have

$$X^{\alpha} \geq X^{\alpha/2}YY^*X^{\alpha/2} = UH^2U^*. \tag{3}$$

The hypothesis $1 \geq \alpha \geq 1/2$ ensures $1 \geq (1-\alpha)/\alpha \geq 0$, so that (3) implies the following (4) by Theorem A

$$X^{1-\alpha} = (X^{\alpha})^{(1-\alpha)/\alpha} \geq (UH^2U^*)^{(1-\alpha)/\alpha} = UH^{2(1-\alpha)/\alpha}U^*. \tag{4}$$

By (1), (4) and (2), we have

$$Y^*XY \geq HU^*UH^{2(1-\alpha)/\alpha}U^*UH = H^{2/\alpha} = (Y^*X^{\alpha}Y)^{1/\alpha}, \tag{5}$$

so that we have (i). Using (i) and by induction we have

$$(Y^*XY)^{\alpha_1\alpha_2\cdots\cdots\alpha_n} \geq Y^*X^{\alpha_1\alpha_2\cdots\cdots\alpha_n}Y$$

for any $\alpha_1, \alpha_2, \cdots\cdots, \alpha_n$ such that $1 \geq \alpha_k \geq 1/2$ for all integer k, so we have (ii).

We give a simple proof to (ii) shown in [6].

Proof of Theorem 1. In the case $1 \geq p \geq 0$, $A \geq B \geq 0$ ensures $A^p \geq B^p$ by Theorem A, so the result is obvious. We have only to show the following for each $r \geq 0$, $p \geq 1$ and $q = (p+2r)/(1+2r)$:

$$(6) \qquad (B^r A^p B^r)^{1/q} \geq B^{1+2r}$$

since (i) of Theorem 1 for values q larger than $(p+2r)/(1+2r)$ follows by Theorem A. If $A \geq B \geq 0$, then $A+\varepsilon \geq B+\varepsilon$ for any $\varepsilon > 0$, so $B+\varepsilon$ and $A+\varepsilon$ are both invertible, therefore we may assume that A and B are invertible. In the case $1/2 \geq r \geq 0$, $A \geq B \geq 0$ ensures $A^{2r} \geq B^{2r}$ by Theorem A, so $B^r A^{-2r} B^r \leq 1$, namely $\|A^{-r}B^r\| \leq 1$. Put $q = (p+2r)/(1+2r) \geq 1$. By (ii) in Lemma 1, we have

$$(B^r A^p B^r)^{1/q} = (B^r A^{-r} A^{p+2r} A^{-r} B^r)^{1/q} \geq B^r A^{-r} A^{(p+2r)/q} A^{-r} B^r$$

$$= B^r A B^r \geq B^{1+2r}.$$

Put $A_1 = (B^r A^p B^r)^{1/q}$ and $B_1 = B^{1+2r}$. Then this inequality $A_1 \geq B_1$ means that (6) holds for $1/2 \geq r \geq 0$. Repeating (6) again for $1/2 \geq r_1 \geq 0$ and $p_1 \geq 1$

$$(B_1^{r_1} A_1^{p_1} B_1^{r_1})^{1/q_1} \geq B_1^{1+2r_1}$$

for $q_1 = (p_1+2r_1)/(1+2r_1)$; that is,

$$\{B^{(1+2r)r_1}(B^r A^p B^r)^{p_1/q} B^{(1+2r)r_1}\}^{1/q_1} \geq B^{(1+2r)(1+2r_1)}.$$

Put $p_1 = q \geq 1$. Then we have

$$(7) \qquad \{B^{(1+2r)r_1+r} A^p B^{r+(1+2r)r_1}\}^{1/q_1} \geq B^{(1+2r)(1+2r_1)}$$

Put $r_2 = (1+2r)r_1+r$. Then $q_1 = (p_1+2r_1)/(1+2r_1) = (p+2r_2)/(1+2r_2)$ since $p_1 = q$ and $(1+2r)(1+2r_1) = 1+2r_2$. Consequently (7) means that (6) holds for $r_2 \in [0, 3/2]$ since $r, r_1 \in [0,1/2]$ and repeating this method, (6) holds for each $r \geq 0$ and (i) is shown.

By hypothesis, $B^{-1} \geq A^{-1} \geq 0$. Then by (i), for each $r \geq 0$, $(A^{-r}B^{-p}A^{-r})^{1/q} \geq A^{-(p+2r)/q}$ holds for each p and q such that $p \geq 0$, $q \geq 1$ and $(1+2r)q \geq p+2r$. Taking inverses gives (ii).

Alternative proof of Theorem 1 [5]. In the case $1 \geq p \geq 0$, the result is obvious by Theorem A. We have only to consider $p \geq 1$ and $q = (p+2r)/(1+2r)$ since (i) of Theorem 1 for values q larger than $(p+2r)/(1+2r)$ follows by Theorem A. We may assume that A and B are invertible without loss of generality. The operator mean XmY is defined by $XmY = X^{1/2}f(X^{-1/2}YX^{-1/2})X^{1/2}$ for invertible positive X and Y where f is an operator monotone function and $f(t) = 1mt$ [8]. In the case $1/2 \geq r \geq 0$, $A^{2r} \geq B^{2r}$ holds by Theorem A, then for $q = (p+2r)/(1+2r)$ and $f(t) = 1mt = t^{1/q}$.

$$B^{-2r}mA^{p} \geq A^{-2r}mA^{p} = A \geq B = B^{-2r}mB^{p}. \tag{8}$$

We have only to show the following (9) for $s = 2r+1/2$, $q_1 = (p+2s)/(1+2s)$ and $f_1(t) = 1m_1t = t^{1/q_1}$

$$B^{-2s}m_1A^{p} \geq B \tag{9}$$

because (9) means that (8) holds for $3/2 \geq s \geq 0$ since $1/2 \geq r \geq 0$ and repeating this method, (8) holds for each $r \geq 0$. Proof of (9) is an immediate consequence of (8) as follows.

$$\begin{aligned}
B^{-2s}m_1A^{p} &= B^{-r}[B^{-(2r+1)}m_1(B^{r}A^{p}B^{r})]B^{-r} \\
&\geq B^{-r}[(B^{r}A^{p}B^{r})^{-1/q}m_1(B^{r}A^{p}B^{r})]B^{-r} \quad \text{by (8)} \\
&= B^{-r}(B^{r}A^{p}B^{r})^{(q+1-q_1)/qq_1}B^{-r} \\
&= B^{-r}(B^{r}A^{p}B^{r})^{1/q}B^{-r} \quad \text{since } q+1 = 2q_1 \\
&\geq B \quad \text{by (8)}
\end{aligned}$$

whence (9) is shown, so the proof is complete.

We remark that there are given proofs via operator means of Theorem 1 for $p = q = 2$, $r = 1$ and $p = 2$, $q = 4/3$, $r = 1$ in [7].

Theorem 2 [2]. Let A, B and C be nonnegative Hermitian matrices such that $C \geq A$ and $C \geq B$. There exist A, B and C such that

$$\sqrt{2}\, C \geq (A^2 + B^2)^{1/2}$$

does not always hold.

There is a counterexample in Theorem 2, but we have the following results related to Theorem 2 by using Theorem 1.

Corollary 4 [4]. If $C \geq A \geq 0$ and $C \geq B \geq 0$, then for each $r \geq 0$

$$2^{p(1+2r)/(p+2r)} C^{1+2r} \geq \{C^r (A+B)^p C^r\}^{(1+2r)/(p+2r)}$$

hold for each $p \geq 1$.

Corollary 5 [4]. If $C \geq A \geq 0$ and $C \geq B \geq 0$, then

$$2C^2 \geq \{C(A+B)^2 C\}^{1/2}.$$

As an application of (i) in Lemma 1, we show the following results because $\|B^r A^{-r}\| \leq 1$ and $1 \geq (p-s+2r)/(p+2r) \geq 1/2$ hold.

Theorem 3 [4]. If $A \geq B \geq 0$, then for each r such that $1/2 \geq r \geq 0$

(i) $$B^r A^p B^r \geq (B^r A^{p-s} B^r)^{(p+2r)/(p-s+2r)}$$

(ii) $$(A^r B^{p-s} A^r)^{(p+2r)/(p-s+2r)} \geq A^r B^p A^r$$

hold for each p and s such that $p \geq s \geq 0$ and $p+2r \geq 2s$.

Corollary 6 [4]. If $A \geq B \geq 0$, then for each r such that $1/2 \geq r \geq 0$

(i) $$B^r A^p B^r \geq (B^r A B^r)^{(p+2r)/(1+2r)}$$

(ii) $$(A^r B A^r)^{(p+2r)/(1+2r)} \geq A^r B^p A^r$$

hold for each p with $2(1+r) \geq p \geq 1$.

Corollary 7 [4]. If $A \geqq B \geqq 0$, then for each r such that $1/2 \geqq r \geqq 0$

(i) $$B^r A^p B^r \geqq (B^r A^{p/2-r} B^r)^2$$

(ii) $$(A^r B^{p/2-r} A^r)^2 \geqq A^r B^p A^r$$

hold for each $p \geqq 2r \geqq 0$.

At the end of my speech, we show an elementary proof to Corollary 3 without use of Corollary 1.

A proof to Corollary 3. For any $r \in [0,1/2]$ we have

$$(B^r A^2 B^r)^{1/2} = (B^r A^{1-r} A^{2r} A^{1-r} B^r)^{1/2}$$

$$\geqq (B^r A^{1-r} B^{2r} A^{1-r} B^r)^{1/2} = B^r A^{1-r} B^r \geqq B^{1+r} \quad \cdots\cdot(*).$$

Put $r = 1/2$ in (*), so $C \equiv (B^{1/2} A^2 B^{1/2})^{1/2} \geqq B^{3/2} \equiv D \geqq 0$.
Then applying (*) to C and D and put $r = 1/3$, $(D^{1/3} C^2 D^{1/3})^{1/2} \geqq D^{4/3}$, that is, $(BA^2B)^{1/2} \geqq B^2$, the second inequality follows by the first one.

References

[1] N.N. Chan and Man Kam Kwong, Hermitian matrix inequalities and a conjecture, Amer. Math. Monthly, 92(1985), 58-71.

[2] T. Furuta, A counterexample to a conjectured hermitian matrix inequality, Amer. Math. Monthly, 94(1987), 271-272.

[3] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc., 101(1987), 85-88.

[4] T. Furuta, $A \geq B \geq 0$ ensures $B^r A^p B^r \geq (B^r A^{p-s} B^r)^{(p+2r)/(p-s+2r)}$ for $1 \geq 2r \geq 0$, $p \geq s \geq 0$ with $p+2r \geq 2s$ (to appear).

[5] T. Furuta, A proof via operator means of an order preserving inequality (to appear).

[6] F. Hansen, An operator inequality, Math. Ann., 246(1980), 249-250.

[7] E. Kamei, Furuta's inequality via operator means, to appear in Math. Japon.

[8] F. Kubo and T. Ando, Means of positive linear operators, Math. Ann., 246(1980), 205-224.

[9] K. Löwner, Über monotone matrixfunctionen, Math. Z., 38(1934), 177-216.

[10] G.K. Pedersen, Some operator monotone function, Proc. Amer. Math. Soc., 36(1972), 309-310.

Department of Mathematics, Faculty of Science, Hirosaki University, Bunkyo-cho 3, Hirosaki 036, Aomori, Japan