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Essential spectrum of linearized operator for MHD plasma in cylindrical region

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§1. Introduction

Concerning the plasma confinement and heating problems in the controlled thermonuclear fusion research, the linearized magnetohydrodynamic (MHD in short) equation:

$$\rho\frac{\partial^2\xi}{\partial t^2} = K\xi \equiv [\mathrm{grad}(\gamma P(\mathrm{div}\ \xi)+(\mathrm{grad}\ P)\cdot\xi)] + \frac{1}{\mu}[B\times\mathrm{rot}(\mathrm{rot}(B\times\xi))-(\mathrm{rot}\ B)\times\mathrm{rot}(B\times\xi)] \qquad (1.1)$$

plays an important role. Here $\xi=\xi(t,x,y,z)$, $(x,y,z) \in \Omega \subset R^3$ denotes the Lagrangian displacement of plasma, and ρ, P and B are the density, the pressure and the magnetic field of the given equilibrium which satisfy:

$$\mathrm{grad}\ P = \frac{1}{\mu}(\mathrm{rot}\ B)\times B \quad (P>0), \quad \mathrm{div}\ B = 0 \qquad (1.2)$$

in Ω with an arbitrary positive function ρ, where γ and μ are positive constants which denote the specific heat ratio and the magnetic permeability respectively.

In this lecture, we consider the above equation in the cylindrical region Ω surrounded by an infinitely conducting wall $\partial\Omega$ and investigate the spectral properties of the force operator

K in (1.1) in some appropriate Hilbert space. Especially, we treat the essential spectrum of K by means of the analysis of it's resolvent.

In §2, we give a special equilibrium which depends only on the radial argument, and then make the Fourier decomposition of the operator K and establish the selfadjoint realization of the decomposed operator by determining it's resolvent. The main point is the adequate treatment of the boundary condition at the magnetic axis, which has not been fully analyzed in mathematically rigorous fashion up to now. The second order differential operator $E_{\lambda\rho}$ (see (2.13)) plays an important role in the analysis.

In §3, we shall study the spectrum of the Fourier decomposed operator and determine it's essential spectrum using the well known theorem that the essential spectrum is invariant under compact perturbations. We work on the resolvent and to treat the (seeming) singularity at the magnetic axis, we introduce an adequate diagonalization of the operator (see (3.8)).
It is shown that there are two types of essential spectra in the negative (stable) part which correspond to the Alfvén and the slow magnetosonic waves, and there are only possible discrete spectra on the positive (unstable) part, i.e. it excludes the possibility of the extra essential spectrum for this case (c.f. Descloux-Geymonat [3]).

Finally, in the last section, we present some results on further problems on the discreteness of the positive part of the spectrum for partially Fourier decomposed operators. But the detailed proofs and discussions will be given in the forthcoming paper.

Here we shall make some comments on the previous works on this problem. The linearized MHD equation in the form (1.1) has been long known (see Bernstein et al. [2]) and, in the case of cylindrical plasma, Hain-Lüst [8] observed that the eigenvalue problem : $K\xi=\lambda\xi$, can be reduced to the so-called Hain-Lüst equation (see the remark in §2.3 after the proof of Lemma 2.1.). Later, H.Grad [6] investigated the spectral properties of K and pointed out the existence of continuous (or essential) spectrum in the stable part of K (see also Appert et al. [1]). J.P.Goedbloed [4] studied the same problem and also the axisymmetric case and determined the essential spectrum by somewhat heuristic arguments (see also [3], [9] and [11] for the mathematically rigorous treatments of the problem). In [7], Grubb and Geymonat treated the problem of the essential spectrum for the elliptic system of mixed order in general situations using the theory of pseudo differential operators which could be applied to the present problem of MHD. In [10], the present author investigated the spectrum of K in the case of slab plasma and obtained the mathematically rigorous results for the continuous spectrum of K. For the general survey and other related topics, see [5], [12],[13] and [14]. Recently, Descloux and the present author treated the approximation problem for this cylindrical case with a magnetic axis [16], where the expression of the resolvent given in this paper plays an essential role.

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§2. Selfadjoint realization of force operator

2.1. Cylindrical plasma equilibrium

Let us find out the equilibrium solution of ideal magnetohydrodynamic system of equations:

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho V)=0, \quad \frac{D}{Dt}(P\rho^{-\gamma})=0, \quad \rho\frac{DV}{Dt}=-\operatorname{grad} P + j\times B, \tag{2.1}$$

$$\frac{\partial B}{\partial t}=-\operatorname{rot} E, \quad \operatorname{div} B=0, \quad E+V\times B=0, \quad j=\frac{1}{\mu}\operatorname{rot} B$$

with $D/Dt = \partial/\partial t + V\cdot\operatorname{grad}$ (convective derivative).

Here V is the velocity field, j the electric current density, E the electric field and other quantities have been already introduced in §1. Replacing μP by P and $\mu\rho$ by ρ, we assume $\mu=1$ from now on. The equilibrium is a static, i.e. time-independent, solution of (2.1) with zero velocity field: $V\equiv 0$. Then (2.1) is reduced to (1.2). If we linearize the equation (2.1) in the vicinity of a given equilibrium and introduce the Lagrangian displacement vector ξ along the given velocity field $V(t,x,y,z)$ as the solution of the ordinary differential equation:

$$\frac{\partial\xi}{\partial t}(t,x,y,z)=V(t,\xi(t,x,y,z)+(x,y,z)), \quad \xi(0,x,y,z)=(0,0,0), \qquad (2.2)$$

we obtain the equation (1.1) for ξ. The other linearized quantities can be expressed by ξ in the way:

$\delta B=-\mathrm{rot}(B\times\xi)$, $\delta\rho=\mathrm{div}(\rho\xi)$ and $\delta P=-(\mathrm{grad}\ P)\cdot\xi-\gamma P(\mathrm{div}\ \xi)$.

Let Ω be the cylindrical region with radius r_0 and $2\pi R_0$ periodicity in z-direction:

$$\Omega=\{(x,y,z)\,|\,x^2+y^2=r^2< r_0{}^2,\ z\in S^1_{R_0}\equiv R/2\pi R_0\}$$
$$=\{(r,\theta,z)\,|\,0<r<r_0,\ 0<\theta<2\pi,\ z\in S^1_{R_0}\},$$

and consider the equilibrium which depends only on the radial coordinate r. Then, the gradient of P is, if it is not zero, parallel to the radial direction and by (1.2) the magnetic field is perpendicular to this direction. Hence, we can write B as

$$B=(0,\ b(r)\sin\phi(r),\ b(r)\cos\phi(r)) \qquad (2.3)$$

in cylindrical coordinates, and (1.2) is reduced to

$$P'(r)=-b(r)b'(r)-\frac{1}{r}b(r)^2\sin^2\phi(r) \qquad (2.4)$$

which is integrated as

$$P(r)+\frac{1}{2}b(r)^2=P(0)+\frac{1}{2}b(0)^2-\int_0^r \frac{1}{s}b(s)^2\sin^2\phi(s)\ ds. \qquad (2.5)$$

This solution is still valid in the case that there exists a point for which grad $P = 0$.
In the following, we assume that the equilibrium quantities are smooth so that $b(r)$ and $\phi(r)$ are smooth functions of r with $b'(0)=\phi(0)=0$, and $P(r)$ is defined by (2.5). We also assume that the pressure $P(r)$ is positive definite: $P(r) > P_0 > 0$. In this case, we have

$$b(r)=b(0)+\frac{1}{2}b''(0)r^2+0(r^3) \quad \text{and} \quad \phi(r)=\phi'(0)r+0(r^2). \tag{2.6}$$

Furthermore, we shall assume that the density ρ is also a smooth positive definite function depending only on r: $\rho=\rho(r)$, and hence $\rho'(0)=0$.

2.2. Fourier mode decomposition

Using the equilibrium defined in §2.1, let us consider the special solution of (1.1) with the form:

$$\xi(t,x,y,z)= e^{im\theta+ikz}\eta(t,r) \tag{2.7}$$

where m and $R_0 k$ are fixed integers. Then (1.1) can be written in cylindrical coordinates as

$$\rho\frac{\partial^2\eta}{\partial t^2} = K(m,k)\eta \tag{2.8}$$

with $\eta={}^t(\eta_r,\eta_\theta,\eta_z)$ and the reduced force operator $K(m,k)$:

$$K(m,k)=U^*K_\phi(m,k)U$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} K_\phi(m,k) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} . \quad (2.9)$$

Namely, $K_\phi(m,k)$ is the representation by the local coordinates e_r, $e_\perp$ and $e_{/\!/}$ with radial direction e_r, direction of magnetic field $e_{/\!/}= B/|B|$ and the remainder direction $e_\perp=e_{/\!/}\times e_r$, and is given as:

$K_\phi(m,k)=$

$$\begin{pmatrix} \partial_r(b^2+\gamma P)\partial_r^\dagger-r(\frac{b^2\sin^2\phi}{r^2})'-bk_\phi^2 & i\partial_r(b^2+\gamma P)m_\phi-2ib^2k\frac{\sin\phi}{r} & i\partial_r\gamma Pk_\phi \\ i(b^2+\gamma P)m_\phi\partial_r^\dagger+2ib^2k\frac{\sin\phi}{r} & -m_\phi^2(b^2+\gamma P)-b^2k_\phi^2 & -m_\phi k_\phi\gamma P \\ i\gamma Pk_\phi\partial_r^\dagger & -m_\phi k_\phi\gamma P & -k_\phi^2\gamma P \end{pmatrix}$$

with $\partial_r=\partial/\partial r='$ and $\partial_r^\dagger=\frac{1}{r}\partial_r r$, (2.10)

and $k_\phi=k(\cos\phi)+\frac{m}{r}(\sin\phi)$ and $m_\phi=\frac{m}{r}(\cos\phi)-k(\sin\phi)$.

We can prove this formula after some tedious but straightforward calculations (c.f. Goedbloed [4]). As is easily seen, $K_\phi(m,k)$ is formally selfadjoint in the Hilbert space $\mathcal{H}=(L^2(0,r_0;rdr))^3$. By changing the weight rdr to dr, the operator $K_\phi(m,k)$ is transformed into $\sqrt{r}K_\phi(m,k)\frac{1}{\sqrt{r}} \equiv H(m,k)$ which is formally selfadjoint in $\mathcal{K}=(L^2(0,r_0;dr))^3$. Further, to clarify the nature of a seeming singularity at a magnetic axis $r=0$, we change the variable r to s defined by $s=\log r$ (or $r=e^s$). The interval $[0,r_0]$ is then mapped to the semi-infinite interval $(-\infty, s_0]$

with $s_0=\log r_0$, and if we define the transformation W from $\mathcal{K}$ to $\mathcal{Y}\equiv(L^2(-\infty,s_0;ds))^3$ as

$$W:\ \zeta(r)=(\zeta_i(r))_{i=1,2,3}\to(W\zeta)(s)=e^{s/2}\zeta(e^s)=((w\zeta)_i(s))_{i=1,2,3},$$

it is unitary and we have $\partial_s=w(\sqrt{r}\partial_r\sqrt{r})w^*$.

Further, H(m,k) is transformed to $WH(m,k)W^*\equiv Y(m,k)$, and using the notations; $\partial_s=d/ds='$, $r=r(s)=e^s$, $b=b(r)=b(e^s)$ and so on, we have

$$Y(m,k)=$$

$$\begin{pmatrix} \partial_s\frac{(b^2+\gamma P)}{r^2}\partial_s-(\frac{b^2\sin^2\phi}{r^2})'-bk_\phi^2 & i\partial_s\frac{(b^2+\gamma P)}{r}m_\phi-2ib^2k\frac{\sin\phi}{r} & i\partial_s\frac{\gamma P}{r}k_\phi \\ i\frac{(b^2+\gamma P)}{r}m_\phi\partial_s+2ib^2k\frac{\sin\phi}{r} & -m_\phi^2(b^2+\gamma P)-b^2k_\phi^2 & -m_\phi k_\phi\gamma P \\ i\frac{\gamma P}{r}k_\phi\partial_s & -m_\phi k_\phi\gamma P & -k_\phi^2\gamma P \end{pmatrix} \tag{2.11}$$

We write Y(m,k) as $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ with 1×1 operator A, 1×2 operator B, 2×1 operator B^* and 2×2 operator C. Here and hereafter we use B as the notation for this operator instead of the previous notation for the magnetic field.

Now, the original equation (2.8) is rewritten as

$$\frac{\partial^2\zeta}{\partial t^2}=\rho^{-1/2}Y(m,k)\rho^{-1/2}\zeta,\quad \zeta(s)=\rho^{1/2}(e^s)W(\sqrt{r}(U\zeta)(r))(s).$$

Hence we shall investigate the operator $\rho^{-1/2}Y(m,k)\rho^{-1/2}\equiv Y_\rho(m,k)$ in the following.

2.3. Resolvent and selfadjoint realization

We shall now determine the selfadjoint operator which corresponds to the formal operator K(m,k) or equivalently $Y_\rho(m,k)$. There is a crucial difference between the case m=0 and the cases m≠0. When m=0, the function $m_\phi(r)$ defined in (2.11) is uniformly bounded on $(0, r_0]$. So the coefficients of the operator $K_\phi(m,k)$ are all bounded and if we follow the same arguments developed in the case of the slab (or the flat torus) plasma (cf. Kako [10]), we can obtain the same results imposing the reasonable boundary conditions on the radial component which are, in the case of the operator K(m,k),

$$\lim_{r\to 0} \sqrt{r}\eta_r(r)=0 \quad \text{and} \quad \eta_r(r_0)=0.$$

Therefore, we shall treat in this paper only the case m≠0. Further, we shall adopt the representation in $\mathcal{Y}$. Let $\mathcal{D}_0$ be the set of all smooth functions in $\mathcal{Y}$ with compact support in $(-\infty, s_0]$, then for sufficiently large positive λ the inverse of $Y_\rho(m,k)-\lambda$ in $\mathcal{D}_0$ is formally expressed as

$$(Y_\rho(m,k)-\lambda)^{-1}=$$
$$\rho^{1/2}\begin{pmatrix} E_{\lambda\rho}^{-1} & -E_{\lambda\rho}^{-1}(BC_{\lambda\rho}^{-1}) \\ -(C_{\lambda\rho}^{-1}B^*)E_{\lambda\rho}^{-1} & C_{\lambda\rho}^{-1}+(C_{\lambda\rho}^{-1}B^*)E_{\lambda\rho}^{-1}(BC_{\lambda\rho}^{-1}) \end{pmatrix}\rho^{1/2} \tag{2.12}$$

with $E_{\lambda\rho}=A_{\lambda\rho}-BC_{\lambda\rho}^{-1}B^*$, $A_{\lambda\rho}=A-\lambda\rho$ and $C_{\lambda\rho}=C-\lambda\rho$.

In fact, since $(Y_\rho(m,k)-\lambda)^{-1}=\rho^{1/2}(Y(m,k)-\lambda\rho)^{-1}\rho^{1/2}$ and the equation;

$$\begin{pmatrix} A_{\lambda\rho} & B \\ B^* & C_{\lambda\rho} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

can be solved, for sufficiently large λ, by the calculations:

$$g=C_{\lambda\rho}^{-1}v-C_{\lambda\rho}^{-1}B^*f$$

and

$$(A_{\lambda\rho}-BC_{\lambda\rho}^{-1}B^*)f=(E_{\lambda\rho}f=)u-C_{\lambda\rho}^{-1}v,$$

we obtain (2.12). Hence to define the operator $Y_\rho(m,k)$, we have only to determine the bounded selfadjoint operator from the formal operator in (2.12) and to show that it is one to one. To this end, we shall prepare the following lemmas.

Lemma 2.1. For positive λ, the operator $E_{\lambda\rho}$ has an expression:

$$E_{\lambda\rho}= \partial_s F(s;\lambda)\partial_s-G(s;\lambda) \tag{2.13}$$

where

$$F(s;\lambda)=\frac{N(s;\lambda)}{D(s;\lambda)}, \quad N(s:\lambda)=(b^2+\gamma P)(\lambda\rho+\omega_1)(\lambda\rho+\omega_2)$$

$$\text{and} \quad D(s;\lambda)=r^2[(\lambda\rho)^2+(k_\phi^2+m_\phi^2)(b^2+\gamma P)(\lambda\rho)+b^2\gamma Pk_\phi^2(k_\phi^2+m_\phi^2)] \tag{2.14}$$

with $\omega_1=b^2k_\phi^2$ (Alfvén frequency) and $\omega_2=\omega_1\frac{\gamma P}{b^2+\gamma P}$ (slow magnetosonic wave frequency), and

$$G(s;\lambda)=\lambda\rho+\omega_1+\left[\frac{b^2\sin^2\phi}{r^2}\right]'$$

$$-\frac{4k^2b^2\sin^2\phi}{D(s;\lambda)}(b^2\lambda\rho+\gamma P\omega_1)+\left[\frac{2b^2km_\phi\sin\phi}{D(s;\lambda)}(\gamma P+b^2)(\lambda\rho+\omega_2)\right]', \tag{2.15}$$

which becomes positive definite for large λ uniformly with respect s.

Further, as $s\to-\infty$, $F(s;\lambda)$ and $G(s;\lambda)$ have limits:

$$\lim_{s\to-\infty} F(s;\lambda) = \frac{1}{m^2}(\lambda\rho(0)+\omega_1(0))\equiv F_\infty, \tag{2.16}$$

$$\lim_{s\to-\infty} G(s;\lambda) = \lambda\rho(0)+\omega_1(0)\equiv G_\infty, \tag{2,17}$$

and these convergences are of exponential decaying order.

<u>Remark.</u> The expression (2.13) for $E_{\lambda\rho}$ was first introduced by Hain & Lust [8] (see also Goedbloed [4]). The operator $E_{\lambda\rho}$ has an essential spectrum $[G_\infty,\infty)$, but as we shall see in the next section, it doesn't cause the extra essential spectrum of the operator $Y_\rho(k,m)$.

<u>Lemma 2.2.</u> For positive λ, the operator $BC_{\lambda\rho}^{-1}$ has an expression:

$$BC_{\lambda\rho}^{-1}=([-i\partial_s rm_\phi(b^2+\gamma P)(\lambda\rho+\omega_2)+2ib^2kr(\sin\phi)(\lambda\rho+\gamma Pk_\phi^2)]/D(s;\lambda),$$
$$[-i\partial_s r\gamma Pk_\phi(\lambda\rho+\omega_1)-2ib^2kr(\sin\phi)k_\phi m_\phi\gamma P]/D(s;\lambda))$$
$$\equiv(-i\partial_s\beta_{11}+\beta_{10},\ -i\partial_s\beta_{21}+\beta_{20}), \tag{2.18}$$

where β_{ij} ($i=1,2$ $j=0,1$) are all uniformly bounded in $(-\infty, s\,]$ and converge respectively to $\beta_{11}(-\infty)=1/m$ and $\beta_{10}(-\infty)=\beta_{21}(-\infty)=\beta_{20}(-\infty)=0$ as $s\to-\infty$ with an exponential decaying order.

Using these lemmas, we can define the selfadjoint operator $E_{\lambda\rho}$

with the domain

$$\mathcal{D}(E_{\lambda\rho}) \equiv H^2(-\infty, s_0) \cap H_0^1(-\infty, s_0),$$

which is invertible for sufficiently large λ. Here $H^p(-\infty, s_0)$ is the Sobolev space of order p in $(-\infty, s_0)$ and $H_0^p(-\infty, s_0)$ is a subspace of $H^p(-\infty, s_0)$ which is the completion in $H^p(-\infty, s_0)$ of the set of all smooth functions with compact support in $(-\infty, s_0)$. We remark that $\mathcal{D}(E_{\lambda\rho})$ includes smooth functions which satisfy the original boundary condition at s_0, and we need not impose any condition at $s=-\infty$ or equivalently at the magnetic axis $r=0$.

Next, using Lemma 2 we can define the operator expressed by the righthand side of (2.12) as a bounded selfadjoint operator for sufficiently large positive λ which will be denoted by the same notation $(Y_\rho(m,k)-\lambda)^{-1}$. Here the operator $E_{\lambda\rho}^{-1}(BC^{-1})$ in (2.12) should be interpreted as the unique bounded extension of the operator restricted to the set of smooth functions with compact support in $(-\infty, s_0)$, and we use this kind of implicit interpretation also in the rest of this paper. Further, the null space of this operator is trivial, because

$$(Y_\rho(m,k)-\lambda)^{-1} \begin{pmatrix} f \\ g \end{pmatrix} = 0$$

implies

$$E_{\lambda\rho}^{-1} f - E_{\lambda\rho}^{-1}(BC_{\lambda\rho}^{-1}) g = 0 \tag{2.19}$$

and

$$-(C_{\lambda\rho}^{-1}B^*)E_{\lambda\rho}^{-1} f + (C_{\lambda\rho}^{-1} + (C_{\lambda\rho}^{-1}B^*)E_{\lambda\rho}^{-1}(BC_{\lambda\rho}^{-1}) g = 0 \tag{2.20}$$

and, multiplying (2.19) by $(C_{\lambda\rho}^{-1}B^{*})$ from the left and adding it to (2.20), we obtain $C_{\lambda\rho}^{-1}g=0$ which implies $g=0$. So, by (2.19), we have $E_{\lambda\rho}^{-1}f=0$ and then $f=0$.

Summarizing these, we obtain the next theorem.

Theorem 2.3. For smooth equilibrium ρ, P and B defined in §2.1 through (2.3) and (2.5), there exists a unique selfadjoint operator $Y_{\rho}(m,k)$ which has the resolvent defined by the righhand side of (2.12). Further, if $Y_{\rho}(m,k)$ is restricted into $\mathcal{S}^{*}$ which is the set of all smooth functions decreasing rapidly as $s\to-\infty$ and vanishing at s_0, the restricted operator is essentially selfadjoint.

§3. Essential spectrum of force operator

In this section, we study the spectrum of the force operator $Y_{\rho}(m,k)$ defined in §2.3. Especially, the essential spectrum of $Y_{\rho}(m,k)$ will be determined by the investigation of the resolvent of $Y_{\rho}(m,k)$ in detail.

Lemma 3.1. For sufficiently large positive λ, the operator E^{-1} is expressed as

$$E_{\lambda\rho}^{-1}= F_{\infty}^{-\alpha}(\partial_s^2-m^2)^{-1}F_{\infty}^{-\beta}+ R_1 \tag{3.1}$$

where $\alpha+\beta=1$ and R_1 is compact, and $\partial_s^2-m^2$ is a selfadjoint operator in $L^2(-\infty,s_0;ds)$ with domain $H^2(-\infty,s_0)\cap H_0^1(-\infty,s_0)$. Further, if $F_\infty^{-\alpha}$ and/or $F_\infty^{-\beta}$ are/is replaced by $F(s;\lambda)^{-\alpha}$ and/or $F(s;\lambda)^{-\beta}$, we have the same types of expression.

<u>Lemma 3.2.</u> For sufficiently large positive λ, the operator $\partial_s E_{\lambda\rho}^{-1}\partial_s$ is expressed as

$$\partial_s E_{\lambda\rho}^{-1}\partial_s = F(s;\lambda)^{-1}[1+\tilde{E}_{\lambda\rho}^{-1}G(s;\lambda)+\tilde{E}_{\lambda\rho}^{-1}[F(s;\lambda)G(s;\lambda)]'E_{\lambda\rho}^{-1}\partial_s]$$

$$= F(s;\lambda)^{-1}+\frac{m^4}{\lambda\rho+\omega_1(0)}(\partial_s^2-m^2)^{-1}+R_2, \tag{3.3}$$

where $\tilde{E}_{\lambda\rho}$ is a differential operator $\partial_s F(s;\lambda)\partial_s - G(s;\lambda)$ with the Neumann boundary condition at s_0: $\partial_s f(s_0)=0$, and R_2 is a compact operator.

<u>Lemma 3.3.</u> For sufficiently large λ, the operator $\partial_s E_{\lambda\rho}^{-1}$ and $E_{\lambda\rho}^{-1}\partial_s$ are expressed as

$$\partial_s E_{\lambda\rho}^{-1} = F_\infty^{-\alpha}\partial_s(\partial_s^2-m^2)^{-1}F_\infty^{-\beta}+R_3, \tag{3.6}$$

$$E_{\lambda\rho}^{-1}\partial_s = F_\infty^{-\beta}(\partial_s^2-m^2)^{-1}\partial_s F_\infty^{-\alpha}+R_3^*, \tag{3.7}$$

where $\alpha+\beta=1$ and R_3 and R_3^* are compact operators.

Now, using these lemmas, we obtain the expression of $(Y_\rho(m,k)-\lambda)^{-1}$ as follows.

Proposition 3.4. For sufficiently large λ, the resolvent of $Y_\rho(m,k)$ is expressed as

$$(Y_\rho(m,k)-\lambda)^{-1} = \begin{pmatrix} \tilde{V} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(\lambda+\omega_{1\rho})^{-1} & \\ 0 & 0 & -(\lambda+\omega_{2\rho})^{-1} \end{pmatrix} \begin{pmatrix} \tilde{V}^* & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} + R. \tag{3.8}$$

Here $\omega_{1\rho} = \omega_1/\rho$ and $\omega_{2\rho} = \omega_1/\rho$, and the operators $\tilde{V}$ and $\tilde{V}^*$ are unitary in $L^2(-\infty, s_0; ds)^2$ and R is a compact operator in $\mathcal{Y}$.

Proof. Using Lemmas 2.2 and 3.2, we have after some calculations that

$$C_{\lambda\rho}^{-1} + (C_{\lambda\rho}^{-1}B^*)E_{\lambda\rho}^{-1}(BC_{\lambda\rho}^{-1})$$

$$= \begin{pmatrix} -(\lambda\rho+\omega_1)^{-1} - (\partial_s^2-m^2)^{-1}m^2(\lambda\rho(-\infty)+\omega_1(-\infty))^{-1} & 0 \\ 0 & -(\lambda\rho+\omega_2)^{-1} \end{pmatrix} + R_4 \tag{3.9}$$

where R_4 is a compact operator. Hence, using the expression (2.12) for the resolvent of $Y_\rho(m,k)$ together with Lemmas 3.1 and 3.3, we have

$$\rho^{-1/2}(Y_{\lambda\rho}(k,m)-\lambda)^{-1}\rho^{-1/2} =$$

$$\begin{pmatrix} F_\infty^{-1/2}(\partial_s^2-m^2)^{-1}F_\infty^{-1/2} & F_\infty^{-1/2}(\partial_s^2-m^2)^{-1}(i\partial_s)m^{-1}F_\infty^{-1/2} & 0 \\ F_\infty^{-1/2}(i\partial_s)m^{-1}(\partial_s^2-m^2)^{-1}F_\infty^{-1/2} & -(\lambda\rho+\omega_1)^{-1} - F_\infty^{-1/2}(\partial_s^2-m^2)^{-1}F_\infty^{-1/2} & 0 \\ 0 & 0 & -(\lambda\rho+\omega_2)^{-1} \end{pmatrix}$$

$$+ R_5 \tag{3.10}$$

with $F_\infty = m^{-2}(\lambda\rho(0)+\omega_1(0))$ (see (2.16)) and the operator R_5 is compact. This formula can be rewritten as

$$\rho^{-1/2}(Y_\rho(k,m)-\lambda)^{-1}\rho^{-1/2}$$

$$= \begin{pmatrix} (\lambda\rho+\omega_1)^{-1/2}\begin{pmatrix} (\partial_s^2-m^2)^{-1}m^2 & (\partial_s^2-m^2)^{-1}(i\partial_s)m \\ i\partial_s m(\partial_s^2-m^2)_*^{-1} & -1-(\partial_s^2-m^2)^{-1}m^2 \end{pmatrix}(\lambda\rho+\omega_1)^{-1/2} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & -(\lambda\rho+\omega_2)^{-1} \end{pmatrix}$$

$$+ R_6 \qquad (3.11)$$

with a compact operator R_6.

Now we diagonalize the part of 2×2 matrix operator with constant coefficiets in (3.11). To this end, we first write this operator in the form:

$$\begin{pmatrix} P_{od}^* & 0 \\ 0 & P_{ev}^* \end{pmatrix}\begin{pmatrix} (\partial_s^2-m^2)_*^{-1}m^2 & (\partial_s^2-m^2)_*^{-1}(i\partial_s m) \\ i\partial_s m(\partial_s^2-m^2)^{-1} & -1-(\partial_s^2-m^2)_*^{-1}m^2 \end{pmatrix}\begin{pmatrix} P_{od} & 0 \\ 0 & P_{ev} \end{pmatrix}$$

$$+ \begin{pmatrix} P_{od}^* & 0 \\ 0 & P_{ev}^* \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -[(\partial_s^2-m^2)_{**}^{-1} - (\partial_s^2-m^2)^{-1}]m^2 \end{pmatrix}\begin{pmatrix} P_{od} & 0 \\ 0 & P_{ev} \end{pmatrix}. \qquad (3.12)$$

Here P_{od} and P_{ev} are isometric operators from $\mathcal{L}^- \equiv L^2(-\infty,s_0;ds)$ to $\mathcal{L} \equiv L^2(-\infty,\infty;ds)$ defined as

$$P_{od}\, f = \begin{cases} (1/\sqrt{2})f(x), & x<s_0 \\ -(1/\sqrt{2})f(s_0-(x-s_0)), & x>s_0 \end{cases}$$

$$P_{ev}\, f = \begin{cases} (1/\sqrt{2})f(x), & x<s_0 \\ (1/\sqrt{2})f(s_0-(x-s_0)), & x>s_0 \end{cases} .$$

The adjoint operator P^*_{od} and P^*_{ev} are then defined as

$$P^*_{od}\, g = (1/\sqrt{2})[g(x) - g(s_0-(x-s_0))]$$

and

$$P^*_{ev} g = (1/\sqrt{2})[g(x) + g(s_0-(x-s_0))].$$

Furthermore, the operators $(\partial_s^2-m^2)_*$ and $(\partial_s^2-m^2)_{**}$ are defined as the differential operators $\partial_s^2-m^2$ in $\mathcal{L}$ with domains $H^2(-\infty,\infty)$ and $H^2_*(-\infty,\infty)$ respectively, where $H^2(-\infty,\infty)$ is a set of functions in $H^2(-\infty,\infty)$ with zero trace at s_0. As well known, the second part of the expression (3.12) is compact. The right-upper component $(\partial_s^2-m^2)_*^{-1}(i\partial_s m)$ is a natural unique extension of the operator with the same form restricted to some dense set of smooth functions.

Introducing the unitary operator in $(\mathcal{L}^-)^2$ as

$$\tilde{V} \equiv \begin{pmatrix} P^*_{od} & 0 \\ 0 & P^*_{ev} \end{pmatrix} \begin{pmatrix} -i\partial_s & m \\ m & i\partial_s \end{pmatrix} (-\partial_s^2+m^2)_*^{-1/2} \begin{pmatrix} P_{ev} & 0 \\ 0 & P_{od} \end{pmatrix}$$

and

$$\tilde{V}^* \equiv \begin{pmatrix} P^*_{ev} & 0 \\ 0 & P^*_{od} \end{pmatrix} \begin{pmatrix} -i\partial_s & m \\ m & i\partial_s \end{pmatrix} (-\partial_s^2+m^2)_*^{-1/2} \begin{pmatrix} P_{od} & 0 \\ 0 & P_{ev} \end{pmatrix},$$

we can diagonalize the 2×2 matrix operator in (3.11) modulo compact operator as follows:

$$\alpha\tilde{V} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \tilde{V}^*\alpha + R_7, \quad R_7 \text{ is compact in } (\mathcal{L}^-)^2. \tag{3.13}$$

Finally, if we remark that $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^2$ and

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tilde{V}^* \alpha =$$

$$= \begin{pmatrix} P_{ev}^* & 0 \\ 0 & P_{od} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ m(-\partial_s^2+m^2)_*^{-1/2}\alpha_{ev} & -i\partial_s(-\partial_s^2+m^2)^{-1/2}\alpha_{ev} \end{pmatrix} \begin{pmatrix} P_{od} & 0 \\ 0 & P_{ev} \end{pmatrix}$$

with $\alpha = (\lambda\rho+\omega_1)^{-1/2}$ and

$$\alpha_{ev}(x) = \begin{cases} \alpha(x), & x<s_0 \\ \alpha(s_0-(x-s_0)), & x>s_0 \end{cases} ,$$

we get the final results. In fact, from the results in Appendix, the commutators

$$m(-\partial_s^2+m^2)_*^{-1/2}\alpha_{ev} - \alpha_{ev} m(-\partial_s^2+m^2)_*^{-1/2}$$

and

$$\partial_s(-\partial_s^2+m^2)_*^{-1/2}\alpha_{ev} - \alpha_{ev}\partial_s(-\partial_s^2+m^2)_*^{-1/2}$$

are compact, since $\alpha_{ev}(x)$ is continuous and tends to respective constants as $x\to\infty$. This is turn admits to put $\alpha(x)$ inside of the expression (3.13) modulo compact operators, and we have the final expression (3.8). q.e.d.

From Proposition 3.4, we deduce the following theorem.

Theorem 3.5. The essential spectrum of $Y_\rho(k,m)$ consists of σ_A and σ_S with

$$\sigma_A=\{\lambda \mid \lambda=-\omega_{1\rho}(r),\ 0\leq r\leq r_0\} \text{ and } \sigma_S=\{\lambda \mid \lambda=-\omega_{2\rho}(r),\ 0\leq r\leq r_o\} .$$

In particular, there is no positive (unstable) essential spectrum of $Y_\rho(k,m)$.

Proof. This is a consequence of the fact that the essential spectrum is invariant by the compact perturbations and the spectrum mapping theorem. q.e.d.

§4. Some results on partially decomposed operators

So far, we have been concerned with the spectral properties of $Y_\rho(m,k)$ which is the Fourier decomposed operator of the original one. Hence, summing up the spectrum of these operators with respect to m and k and taking the closure of the union, we have the total spectrum. In that case, there will be some possibility of the accumulation of eigenvalues and the limit points will be new essential spectra. If the point is in the unstable part of the spectrum, it will be hopeless in the fusion research to controle the whole unstable modes by using the feedback devices.

In this chapter, we show some partial results on this problem which exclude the possible unstable essential spectrum for the operator $Y_\rho(k)$ which is defined as the Fourier decomposed operator with respect to the z-direction k. Namely, we have the following theorem.

<u>Theorem 4.1</u> The positive part of the operator K(k) defined through the formula

$$K(e^{ikz}\eta(r,\theta)) = e^{ikz}K(k)\eta(r,\theta),$$

which is the operator with respect to the arguments r and θ, is discrete under the same condition as before.

<u>Proof.</u> We prove this by showing that ,for any fixed positive $\lambda_0>0$, there exists no eigenvalue of the operator $Y_\rho(m,k)$ for sufficiently large m in the interval (λ_0,∞). The eigenvalue problem $K(m,k)\eta=\lambda\eta$, which is equivalent to $Y_\rho(m,k)\zeta=\lambda\zeta$, can be reduced to the problem

$$E_{\lambda\rho}f \equiv \{\partial_s F(s;\lambda)\partial_s - G(s;\lambda)\}f = 0 \qquad (4.1)$$

at least for positive λ. We can prove this by the same calculation as that just before Lemma 2.1 in § 2.3. Since $\partial_s F(s;\lambda)\partial_s$ is a negative operator, if we can prove that $G(s;\lambda)$ is strictly positive for sufficiently large m and for $\lambda>\lambda_0>0$, we obtain the results. We write $G(s;\lambda)$ as

$$G(s;\lambda) = \lambda\rho + \omega_1 + I + II + III \qquad (4.2)$$

where (see (2.15))

$$I = \left(\frac{b^2\sin^2\phi}{r^2}\right)',$$

$$II = -4k^2b^2\sin^2\phi(b^2\lambda\rho + \gamma P\omega_1)/D(s;\lambda)$$

and

$$III = [2b^2km_\phi \sin\phi(\gamma P + b^2)(\lambda\rho + \omega_2)/D(s;\lambda)]'$$

with

$$D(s;\lambda) = r^2[(\lambda\rho)^2 + (k_\phi^2+m_\phi^2)(b^2+\gamma P)(\lambda\rho) + b^2\gamma P k_\phi^2(k_\phi^2+m_\phi^2)].$$

Then, after some algebraic calculations and estimates, we can prove that

$$|\ I\ | < C(\lambda\rho+\omega_1)/|m| \ , \tag{4.3}$$

$$|\ II\ | < C/m^2 \tag{4.4}$$

and

$$|\ III\ | < C/|m|. \tag{4.5}$$

Here, the constant C may depend on k and $\lambda_0 > 0$, dut does not depend on m and $\lambda(>\lambda_0)$. The essential point to make those estimates is to remember the properties of equilibrium quantities such as b, ρ and ϕ, especially near the magnetic axis r=0 (s=-∞). From the estimates (4.3)-(4.5), taking m large enough, we can prove the non-existence of eigenvalues of $Y_\rho(k,m)$ in the interval $[\lambda_0,\infty)$, which in turn implies that the partially decomposed operator K(k) has only discrete spectrum in $(0,\infty)$. q.e.d.

<u>Remark</u> : Making use of the same kind of estimates for $G(s;\lambda)$, we can prove that the original operator K is itself upper semi-bounded. The details will be given in the forthcoming papers.

Appendix

In this appendix, we present some results which might already be known, concerning the commutators of some operators and their compactness. First, we remark that the operator $(-\partial^2+m^2)^{-1}$ (we omit * in this section), $\partial=d/ds$, maps $\mathcal{L}=L^2(-\infty,\infty;ds)$ into H^2, the Sobolev space of order 2, and $(-\partial^2+m^2)^{-1/2}$ maps $\mathcal{L}$ into H^1. If $\alpha(x)$ is a bounded function which tends to respective constants $\alpha_\pm$ as x tends to $\pm\infty$, taking a smooth function $\tilde{\alpha}(x)$ which also tends to $\alpha_\pm$ as x tends to $\pm\infty$ and satisfies

$$\lim_{|x|\to\infty}\tilde{\alpha}'(x) = \lim_{|x|\to\infty}\tilde{\alpha}''(x) = 0,$$

we have

$$(-\partial^2+m^2)^{-1/2}\alpha(x) - \alpha(x)(-\partial^2+m^2)^{-1/2}$$
$$= (-\partial^2+m^2)^{-1/2}\tilde{\alpha}(x) - \tilde{\alpha}(x)(-\partial^2+m^2)^{-1/2} + R.$$

Here R is a compact operator by the same reasoning as the proof of Lemma 3.1. Next, we remark the following formula (see Yosida [15], p. 260(4)) which can be proved easily by the Fourier transform:

$$(-\partial^2+m^2)^{-1/2} = \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{\lambda}}\left(\lambda + (-\partial^2+m^2)^{-1}\right)^{-1}(-\partial^2+m^2)^{-1}\,d\lambda$$
$$= \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{\lambda}}\left(\lambda(-\partial^2+m^2) + 1\right)^{-1}d\lambda$$

as an identity for operators in $\mathcal{L}$. Especially, the integral is the limit in operator norm in $\mathcal{L}$ of the integral on the interval $[\delta,-\delta]$ as δ tends to zero. Hence, calculating the commutator as

$$(-\partial^2+m^2)^{-1/2}\tilde{\alpha}(x) - \tilde{\alpha}(x)(-\partial^2+m^2)^{-1/2}$$
$$= \lim_{\delta\to 0} \frac{1}{\pi} \int_{\delta}^{\delta^{-1}} \lambda(-\partial^2+m^2) + 1)^{-1}(\sqrt{\lambda}\tilde{\alpha}''(x) + 2\tilde{\alpha}'(x)\sqrt{\lambda}\partial)\times$$
$$\times(\lambda(-\partial^2+m^2) + 1)^{-1}\, d\lambda$$

and remembering the compactness arguments in the proof of Lemma 3.1, we can conclude that the commutator is a compact operator in $\mathcal{L}$. Furthermore, using the same technique, we can prove that the commutator

$$\partial(-\partial^2+m^2)^{-1/2}\tilde{\alpha}(x) - \tilde{\alpha}(x)\partial(-\partial^2+m^2)^{-1/2}$$
$$= \lim_{\delta\to 0} \frac{1}{\pi} \int_{\delta}^{\delta^{-1}} \frac{1}{\sqrt{\lambda}} (\sqrt{\lambda}\partial)(-(\sqrt{\lambda}\partial)^2+\lambda m^2+1)^{-1}[\sqrt{\lambda}\tilde{\alpha}''(x)+2\tilde{\alpha}'(x)(\sqrt{\lambda}\partial)]\times$$
$$\times(-\sqrt{\lambda}\partial)^2+ \lambda m^2+ 1)^{-1}d\lambda - \tilde{\alpha}'(x)(-\partial^2+m^2)^{-1/2}$$

is compact.

References

[1] Appert,K., Gruber,R and Vaclavik,J.: Continuous spectra of a cylindrical magnetohydrodynamic equilibrium, Phys. Fluids 17, 1471-1472 (1974).

[2] Bernstein,I., Frieman,E., Kruskal,M. and Kulsrud: An energy principle for hydrodynamic stability problems, Proc. Royal Soc. A. 244, 17-40 (1958).

[3] Descloux,J. and Geymonat,G.: Sur le spectre essential d'un opérateur relatif à la stabilité d'un plasma en géométrie toroïdale, C.R. Acad. Sc. Paris 290, 795-797 (1980).

[4] Goedbloed,J.P.: Spectrum of ideal magnetohydrodynamics of axisymmetric toroidal systems, Phys. Fluids 18, 1258-1268 (1975).

[5] Goedbloed,J.P.: Lecture notes on ideal magnetohydrodynamics, Rijnhuizen Report 83-145 FOM-INSTITUUT VOOR PLASMAFYSICA (1983).

[6] Grad,H.: Magnetofluid-dynamic spectrum and low shear stability, Proc. Nat. Acad. Sci. U.S.A. 70, 3277-3281 (1973).

[7] Grubb,G and Geymonat,G: The essential spectrum of elliptic systems of mixed order, Math, Ann. 227, 247-276 (1977).

[8] Hain,K. and Lüst,R.: Zur Stabilitat zylindersymmetrischer Plasmakonfigurationen mit Volumenstromen, Z. Naturforsch 13a, 936-940 (1958).

[9] Hameiri,E.: On the essential spectrum of ideal magnetohydrodynamics, Comm. Pure Appl. Math. 38, 43-66 (1985).

[10] Kako,T.: On the absolutely continuous spectrum of MHD plasma confined in the flat torus, Math. Meth. in the Appl. Sci. 7, 432-442 (1985).

[11] Kako,T.: On the essential spectrum of MHD plasma in toroidal region, Proc. Japan Acad. Ser.A. 60, 53-56 (1984).

[12] Miyamoto,K.: Plasma physics for nuclear fusion, Iwanami, Tokyo (1976) (in Japanese).

[13] Rappaz,J.: Approximation of the spectrum of a non-compact operator given by the magnetohydrodynamic stability of a plasma, Numer. Math. 28, 15-24 (1977).

[14] Ushijima,T.: On the linearized magnetohydrodynamic systems of equations for a confined plasma in a vacuum region, Computing Meth. Appl. Sci. Eng., V. Glowinski,R. and J.L.Lions (editors), 507-527 (1982).

[15] Yosida,K.: Functional analysis, Springer, Berlin-Heiderberg-New York (1968).

[16] Kako,T. and Descloux,J.: Spectral approximation for linearized MHD operators in cylindrical region, Rapport Département de Mathématiques EPFL (1986).

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