数理解析研究所講究録
第 656 巻 1988 年 81-104

Diffusion of Vortices in Planar Navier-Stokes Flow

Yoshikazu GIGA (儀我美一・北大理)
Tetsuro MIYAKAWA (宮川鉄朗・広島大理)
Hirofumi OSADA (長田博文・奈良女子大理)

1. Introduction

We consider the flow of a viscous and incompressible fluid in the whole plane R^2, assuming that the vorticity of the flow is initially concentrated in a small region. The motion of the fluid is described by the two-dimensional Navier-Stokes system :

$$(1.1)\qquad \begin{aligned} &u' - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0, \quad \nabla\cdot u = 0,\\ &u \to 0 \quad \text{as} \quad |x| \to \infty, \quad u(x,0) = a(x), \quad \nabla\cdot a = 0, \end{aligned}$$

where u and p represent unknown velocity and pressure, respectively, $\nu > 0$ is the kinematic viscosity, $(u\cdot\nabla) = \sum_i u^i\partial/\partial x_i$, $\nabla\cdot u = \sum_i \partial u^i/\partial x_i$ and $u' = \partial u/\partial t$. The density of the fluid is assumed to be one. Our assumption for the initial velocity a is formulated as follows :

(1.2) The initial vorticity : $\nabla \times a = \partial a^2/\partial x_1 - \partial a^1/\partial x_2$ is a finite Radon measure on R^2.

The velocity fields satisfying (1.2) include those with vortex sheets and vortex lines, which are both important in the vortex theory for ideal fluids. Recently, Marchioro and Pulvirenti [19], [20] and Turkington [31] have studied the relation between the classical theory of the motion of vortex lines and the solutions of the Euler equation, i.e., the system (1.1) with $\nu = 0$. For the Navier-Stokes system (1.1) Benfatto, Esposito and Pulvirenti [3] constructed a global smooth solution of (1.1), assuming that $\nabla\times a$ is a finite pure-point measure :

$$\nabla \times a = \sum_{j=1}^{m} \alpha_j\, \delta(x-z_j)$$

and ν is sufficiently large compared with $\sum_j |\alpha_j|$; here $\delta(x-z_j)$ is the Dirac measure supported at $z_j \in R^2$. This result means that the point source vorticity can be smoothed and diffuse following the Navier-Stokes flow when ν is large.

Our main purpose in this paper is to show the same existence result as above under the more general condition (1.2) without any restriction on the size and on the form of the initial measure $\nabla\times a$. We note that our result does not follow from the classical theories of the Navier-Stokes equations as developed by Leray [17], Ladyzhenskaya [16] or Temam [30]. Indeed, the initial velocity a is not always square-summable, even locally, when $\nabla\times a$ is a measure.

We prove the existence result by a standard procedure. First we regularize the initial velocity a, construct the corresponding regular solution for (1.1), and then take a subsequence converging to the desired solution of the original problem. Necessary estimates for extracting the subsequence are derived from the so-called vorticity equation for $v = \nabla\times u$:

$$v' - \nu\Delta v + (u\cdot\nabla)v = 0, \qquad u = K * v \tag{1.3}$$

for smooth initial data $v(x,0) = \nabla\times a$, where

$$K(x_1,x_2) = (-x_2,x_1)/2\pi|x|^2, \qquad x = (x_1,x_2) \tag{1.4}$$

and $*$ denotes the convolution. Note that there is no vorticity stretching term in (1.3) since the space dimension is 2. We regard (1.3) as a linear parabolic equation for v and write corresponding fundamental solution as $\Gamma_u(x,t;y,s)$. A bound for Γ_u due to Osada [25] (see also [24] and [26]) gives our key estimates:

$$C_1(t-s)^{-1}\exp -C_2|x-y|^2/(t-s) \leq \Gamma_u(x,t;y,s) \leq \leq C_3(t-s)^{-1}\exp -C_4|x-y|^2/(t-s) \tag{1.5}$$

<u>where the constants $C_j > 0$, $j=1,2,3,4$, depend only on ν and L^1-norm of $\nabla\times a$.</u>

Estimates of this type with C_j independent of the smoothness of coefficients were first shown in [1] (see also [2]) for linear parabolic equations of divergence form. The result in [25] extends estimates in [1] to a class of linear equations of nondivergence form which includes (1.3) as a typical example.

We start with presenting the existence results for (1.1) on R^n, $n \geq 2$, with initial data in L^p, $p > n$. Parts of our results are already known (see, e.g., [7] and [21]); however, we give our version here for later use. In particular, we discuss higher regularity of solutions up to $t = 0$. As a byproduct we obtain the persistent property (in the sense of Kato [15]) of solutions of the two-dimensional problem in the Sobolev spaces $W^{m,p}$, $p > 2$, $m = 0,1,2,\ldots$ More precisely, we show that if $a \in W^{m,p}(R^2)$ and $\nabla \times a \in L^q(R^2)$ with $1/q = 1/2 + 1/p$, then the corresponding (global) solution stays in $W^{m,p}(R^2)$ for all time with a bound independent of the viscosity ν. Persistent property of this sort is systematically studied in [15], [27] and [34]. However, it seems to us that our result is not included in either of them.

In Section 3 we state our key a-priori estimates for solutions of the two-dimensional problem given in Section 2, and apply them to the proof of existence (and uniqueness in some special cases) of solutions of the original problem, i.e., the problem (1.1) in R^2 under the condition (1.2). It is to be noticed that the theory of the Lorentz spaces ([4]) plays an important role in giving a precise meaning to the initial condition.

For other results on the initial value problem for nonlinear equations with measures as initial data, we refer to [5], [35], [23], [18], [22], [24], [29].

The details of the results presented in this note will be published elsewhere ([36]).

2. Existence and Uniqueness in R^n with Initial Data in L^p

As is a usual practice ([7,10,11,12,14,32,33,34]), we consider (1.1) in the form of the integral equation :

$$(2.1) \qquad u(t) = e^{\nu t\Delta} a + S[u](t), \quad t > 0,$$

where

$$(2.2) \qquad S[u](t) = S[u,u](t) \ ; \quad S[u,w](t) = -\int_0^t e^{\nu(t-s)\Delta} P(u\cdot\nabla)w(s)ds.$$

Here $e^{t\Delta}$ is the solution operator for the heat equation in R^n; P is the projection on each $L^p(R^n)^n$, $1 < p < \infty$, onto the subspace of divergence-free vector fields. Using the boundedness of the operator P and the so-called L^p-L^q estimates for solutions of the heat equation, we can easily prove

Lemma 2.1. Let $2 < n < p < \infty$, $T > 0$ and $\sigma = 1/2-n/2p$. Then :

(i) $|S[u,w]|_{p,T} \le M(\nu T)^\sigma |u|_{p,T} |w|_{p,T}/\nu$ provided that $\nabla\cdot u = 0$;

(ii) $|(\nu t)^{1/2} \nabla S[u,w]|_{p,T} \le M(\nu T)^\sigma |u|_{p,T} |(\nu t)^{1/2} \nabla w|_{p,T}/\nu$;

(iii) $|S[u,w]|_{q,T} \le M(\nu T)^\sigma |u|^{2\sigma}_{p,T} |(\nu t)^{1/2}\nabla u|^{1-2\sigma}_{p,T} |\nabla w|_{q,T}/\nu$, $q^{-1} = 1-p^{-1}$,

where $|w|_{p,T} = \sup_{0<t<T} \|w(t)\|_p$, $\|\cdot\|_p$ is the L^p-norm, and M is a positive constant depending only on n and p.

We note that the assertion (iii) is proved with the aid of the Gagliardo-Nirenberg inequality ([9, p.24, Th. 9.3]). Using Lemma 2.1, we try to solve (2.1) by the successive approximation :

$$(2.3) \qquad u_{j+1} = u_0 + S[u_j] , \quad u_0 = e^{\nu t\Delta} a, \quad j = 0,1,2,\ldots$$

Our existence result for the initial data in L^p $(p > n)$ is the following.

Proposition 2.2. (i) Let a be in $L^p(R^n)$ with $p > n$ and $\nabla\cdot a = 0$. Then there is a unique local solution u of (2.1) which, for some $T > 0$, belongs to the space $B_{p,T}$ of continuous and bounded functions on $[0,T]$ with values in $L^p(R^n)$. Further we have

(2.4) $$|u|_{p,T} \leq 2\|a\|_p .$$

(ii) The time T can be taken so that

(2.5) $$T \geq C\nu^{-1+1/\sigma} / \|a\|_p^{1/\sigma}, \quad \sigma = 1/2 - n/2p ;$$

(2.6a) $$(\nu t)^{1/2}\, \nabla u \in B_{p,T} \quad \text{with} \quad |(\nu t)^{1/2}\, \nabla u|_{p,T} < C\|a\|_p \; ; \text{ and}$$

(2.6b) If $\nabla a \in L^q(R^n)$ with $1/q = 1/p+1/n$, then $|\nabla u|_{q,T} \leq 2\|\nabla a\|_q$

where C depends only on n and p.

(iii) Let $m \geq 0$ be an integer and suppose that $\nabla^k a \in L^p(R^n)$, $k = 0,\dots, m$. Then in addition to (i) and (ii) the time T can be taken so that

(2.7a) $$\nabla^k u \in B_{p,T} \quad \text{and} \quad |\nabla^k u|_{p,T} \leq C', \quad k = 0,\dots, m ;$$

(2.7b) $$(\nu t)^{1/2}\, \nabla^{m+1} u \in B_{p,T} \quad \text{and} \quad |(\nu t)^{1/2}\, \nabla^{m+1} u|_{p,T} \leq C' ;$$

(2.7c) $$\nabla^k \partial_t^h u \in B_{p,T} \quad \text{and} \quad |\nabla^k \partial_t^h u|_{p,T} \leq C' \quad \text{for} \quad k+2h \leq m ,$$

where C' depends only on n, m, p and bounds for ν and $\|\nabla^k a\|_p$, $k = 0,\dots, m$.

Assertion (i) follows in a standard manner if we apply Lemma 2.1 to (2.3). (2.5) is due to the factor $(\nu T)^\sigma$ which appears in the estimates of Lemma 2.1. Note that $\sigma > 0$ since $p > n$. (2.6a)-(2.7b) follows in a similar way if we take ∇^k of (2.3) and apply the same argument as in (i). For the details we refer to 36 ; see also 12 . (2.7c) is immediately obtained from (2.5), (2.6a)-(2.7b) and the equation : $u' = \nu\Delta u - P(u\cdot\nabla)u$.

The next result establishes a regularizing effect for our solutions.

Proposition 2.3. (i) Let $a \in L^p(R^n)$, $p > n$, with $\nabla \cdot a = 0$, and let u be the corresponding local solution given in Proposition 2.2. Then, $\nabla^k \partial_t^h u$ is continuous from $[\varepsilon,T)$ to $L^p(R^n)$ for all $k,h > 0$ and $0 < \varepsilon < T$. Further we have

$$\sup_{[\varepsilon,T)} \|\nabla^k \partial_t^h u\|_p(t) \leq C$$

with C depending only on ε, p, n, k, h and a bound for $\|a\|_p$. In particular, u is smooth for $t > 0$ and solves the Navier-Stokes system in the classical sense for $t > 0$.

(ii) Suppose further that $\nabla^k a \in L^p(R^n)$ for all $k \geq 0$. Then all derivatives $\nabla^k \partial_t^h u$ are bounded and continuous on $R^n \times [0,T)$ so that

$$\sup_{[0,T)} \|\nabla^k \partial_t^h u\|_\infty(t) \leq C$$

with C depending only on p, n, k, h, ν and bounds for $\max_{0 \leq \ell \leq k+2h+1} \|\nabla^\ell a\|_p$.

Proof. (i) By (2.6a), $\|\nabla u\|_p(t_0) \leq C$ for $0 < t_0 < T$ with C depending only on n, p, t_0 and $\|a\|_p$. We then solve the Navier-Stokes system for $t \geq t_0$ with initial data $u(\cdot,t_0)$ and obtain, due to the uniqueness, $\|\nabla^2 u\|_p(2t_0) \leq C$. Repeating this process yields that $\|\nabla^m u\|_p(mt_0)$ is bounded by the same C so long as $mt_0 < T$. Since t_0 may be taken arbitrarily small, this shows that $\nabla^m u$ is continuous from $[\varepsilon,T)$ to $L^p(R^n)$ for all $\varepsilon > 0$ with a bound C depending only on p, n, m, ε and $\|a\|_p$. Combining this with (2.7c) gives the estimate in (i). (ii) follows from (2.7c) and the Sobolev inequality.

From now on, we restrict ourselves to the two-dimensional case. In this case, the vorticity $v = \nabla \times u = \partial u^2/\partial x_1 - \partial u^1/\partial x_2$ is a scalar which satisfies the so-called vorticity equation :

$$v' - \nu\Delta v + (u \cdot \nabla)v = 0 \; ; \qquad v(x,0) = (\nabla \times a)(x,0). \tag{1.3}$$

Suppose that a lies in $L^p(R^2)$, $p > 2$, together with all its derivatives. Then by Proposition 2.3 u is bounded on $R^2\times[0,T)$ together with all its derivatives ; thus the linear parabolic operator

$$L_u = \partial_t - \nu\Delta + (u\cdot\nabla)$$

has a unique fundamental solution

$$\Gamma_u(x,t\ ;\ y,s), \quad 0 \leq s < t < T, \quad x,y \in R^2$$

such that $L_u\Gamma_u = 0$ as a function of (x,t) and

$$\lim_{t\to s} \int_{R^2} \Gamma_u(x,t\ ;\ y,s)f(y)dy = f(x)$$

for every bounded and continuous f on R^2 ; see [8]. The fundamental solution Γ_u has the following properties :

(2.8) $$\int_{R^2} \Gamma_u(x,t\ ;\ y,s)dy = 1, \quad 0 < s < t < T.$$

(2.9) $$\int_{R^2} \Gamma_u(x,t\ ;\ y,s)dx = 1, \quad 0 < s < t < T.$$

Note that (2.9) follows from the condition : $\nabla\cdot u = 0$. The result below is immediately obtained.

Proposition 2.4. (i) Let $\nabla^k a \in L^p(R^2)$, $k = 0,1,\dots$, for some $p > 2$ and $\nabla\cdot a = 0$. Let u be the local solution given in Proposition 2.2. Then $v = \nabla\times u$ is expressed as

(2.10) $$v(x,t) = \int_{R^2} \Gamma_u(x,t\ ;\ y,0)(\nabla\times a)(y)dy, \quad 0 < t < T.$$

(ii) Suppose further that $\nabla\times a \in L^q(R^2)$ for some q with $1 < q < \infty$; then

(2.11) $$\|v\|_q(t) \leq \|\nabla\times a\|_q , \quad 0 \leq t < T.$$

We have thus established a good estimate (2.11) for the vorticity v. Our next task is to find a way in which to recover the velocity u from its vorticity v.

To state the relation between u and $v = \nabla\times u$, we introduce some function spaces. By $\mathfrak{M}$ we denote the space of all finite Radon measures on R^2 with norm given by the total variation. A measurable function f on R^2 is said to belong to $L^{p,\infty}(R^2)$, $1 < p < \infty$, iff

$$\|f\|_{p,\infty} = \sup_{\alpha>0} \alpha(\, \mathrm{meas}\{x \,;\, |f(x)| > \alpha\})^{1/p}$$

where meas denotes Lebesgue measure on R^2. Although $\|\cdot\|_{p,\infty}$ is not a norm, it is known (see [4]) that $L^{p,\infty}(R^2)$ is a Banach space with respect to a norm which is equivalent to $\|\cdot\|_{p,\infty}$. $L^{p,\infty}$ is often called a Lorentz space.
Let us now consider the function

$$K(x) = (-x_2,\, x_1)/2\pi|x|^2 \,, \quad x = (x_1,\, x_2)$$

and the corresponding convolution operator $U = K*V$. Note that K is in $L^{2,\infty}(R^2)$, but is not contained in any $L^p(R^2)$, $1 \leq p \leq \infty$.

Lemma 2.5. (i) $U = K*V$ satisfies the estimates :

(2.12a) $\|U\|_p \leq C\|K\|_{2,\infty}\|V\|_q$ if $V \in L^q(R^2)$, $1 < q < 2$ and $1/p = 1/q - 1/2$;

(2.12b) $\|U\|_{2,\infty} \leq C\|K\|_{2,\infty}\|V\|_{\mathfrak{M}}$ for $V \in \mathfrak{M}$;

(2.12c) $\|\nabla U\|_r \leq C\|V\|_r$ for $V \in L^r(R^2)$, $1 < r < \infty$,

with C independent of V, where $\|V\|_{\mathfrak{M}}$ denotes the total variation of the Radon measure V.

(ii) If $U \in L^p(R^2)$, $2 < p < \infty$, $\nabla\cdot U = 0$ and $\nabla\times U \in L^q(R^2)$ with $1/q=1/p+1/2$, then

$$U = K*(\nabla\times U).$$

(iii) If $U \in L^{2,\infty}(R^2)$, $\nabla\cdot U = 0$ and $\nabla\times U \in \mathfrak{M}$, then

$$U = K*(\nabla\times U).$$

(2.12a) is the generaliyed Young's inequality ([28, p.32]). Since ∇K is a Calderon-Zygmund kernel, (2.12c) follows from the standard theory of singular integral operators ; see [13, Chap.9]. (2.12b) is shown in the following way : Consider the linear operator $Af = f*V$ for any fixed $V \in \mathcal{M}$. It is easy to see that A defines a bounded linear operator on each $L^p(R^2)$, $1 < p < \infty$, with operator-norm $\leq \|V\|_{\mathcal{M}}$. Thus it follows from an interpolation theorem ([4, Th. 5.3.4]) that A is bounded on $L^{2,\infty}(R^2)$ with norm $\leq C\|V\|_{\mathcal{M}}$. This shows (2.12b). (ii) and (iii) are easily obtained from the Liouville theorem for harmonic functions on the whole plane R^2.

The following result is now obvious from the preceding arguments.

Proposition 2.6. Let $\nabla^k a \in L^p(R^2)$, $k = 0,1,\ldots$, for some $p > 2$, $\nabla\cdot a = 0$ and $\nabla\times a \in L^q(R^2)$ with $1/q = 1/p + 1/2$. Then the solution u given in Proposition 2.2 is expressed as

$$u(x,t) = K*(\nabla\times u) = \int_{R^2} K(x-y)(\nabla\times u)(y,t)dy, \quad 0 \leq t < T.$$

Further, we have the estimate

$$(2.13) \qquad \|u\|_p(t) \leq C\|\nabla\times u\|_q(t) \leq C\|\nabla\times a\|_q , \quad 0 \leq t < T$$

with C depending only on p.

We can now show our global existence results, using estimate (2.13).

Theorem 2.7. Under the assumption of Proposition 2.6, the solution u extends uniquely to a global (in time) solution, which is again denoted u, such that $u \in B_{p,\infty}$, $\nabla u \in B_{q,\infty}$ and

$$|u|_{p,\infty} \leq C\|\nabla\times a\|_q , \qquad |\nabla u|_{q,\infty} \leq C\|\nabla\times a\|_q$$

with C depending only on p. The derivatives $\nabla^k\partial_t^h u$ belong to $B_{p,T}$ for any finite T.

Note that the extensibility of u follows from (2.13) and the estimate (2.5) for the "life span" T of the local solution. The next result is obtained by combining Theorem 2.7 with Proposition 2.3.

Theorem 2.8. Let $a \in L^p(R^2)$ for some $p > 2$ with $\nabla\cdot a = 0$ and $\nabla\times a \in L^q(R^2)$, $1/q = 1/p + 1/2$. Then there is a unique global solution u of (2.1) such that $u \in B_{p,\infty}$, $\nabla u \in B_{q,\infty}$ and

$$|u|_{p,\infty} \leq C\|\nabla\times a\|_q, \qquad |\nabla u|_{q,\infty} \leq C\|\nabla\times a\|_q$$

with C depending only on p. Moreover, the derivatives $\nabla^k\partial_t^h u$ exist on $R^2\times[\varepsilon,\infty)$ for any $\varepsilon > 0$ and satisfy, for any $T > 0$,

$$\sup_{[\varepsilon,T]}\|\nabla^k\partial_t^h u\|_\infty(t) < C$$

with C depending only on p, T, k, h, ε, ν and a bound for $\|\nabla\times a\|_q$.

We close this section by stating our version of persistent property (in the sense of [15] and [27]) of solutions of the two-dimensional problem. In what follows $W^{m,p}(R^2)$, $m = 0,1,2,\ldots$, denotes the usual L^p Sobolev space with norm : $\|\cdot\|_{W^{m,p}}$.

Theorem 2.9. Let $a \in W^{m,p}(R^2)$ for some $p > 2$ with $\nabla\cdot a = 0$ and $\nabla\times a \in L^q(R^2)$ where $1/q = 1/p+1/2$. Then the solution u of (2.1) given in this section is continuous from $[0,T]$ to $W^{m,p}(R^2)$ for all $T > 0$ and satisfies

$$\text{(2.14)} \qquad \sup_{[0,T]}\|u\|_{W^{m,p}}(t) \leq C \quad \text{uniformly for } \nu > 0.$$

Proof. The case $m = 0$ is already shown in Theorem 2.8. For the case $m = 1$, the vorticity equation for $v = \nabla\times u$, together with (2.12c), gives $\|\nabla u\|_p(t) \leq C\|v\|_p(t) \leq C\|\nabla\times a\|_p$. Combining this with (2.14) for $m = 0$ gives (2.14) for $m = 1$. We next assume $m = 2$. We apply ∇ to the vortex equation, multiply the resulting equality by $|\nabla v|^{p-2}\nabla v$ and integrate by parts, to get

(2.15) $$(\|\nabla v\|_p^p)' \leq C\|\nabla u\|_\infty \|\nabla v\|_p^p$$

with C depending only on p. To estimate $\|\nabla u\|_\infty$ we appeal to the following result of Kato [15, Lemma A3] :

(2.16) $$\|\nabla u\|_\infty \leq C(\|v\|_\infty + \|v\|_2 + \|v\|_\infty \log(1+(\|\nabla v\|_p/\|v\|_\infty)))$$

with C depending only on p. Since $\|v\|_\infty \leq \|\nabla\times a\|_\infty \leq C\|a\|_{W^{2,p}}$ and $\|v\|_2 \leq \|v\|_q^{1-2/p} \|v\|_p^{2/p} \leq \|\nabla\times a\|_q^{1-2/p}\|\nabla\times a\|_p^{2/p}$, (2.16) gives

(2.17) $$\|\nabla u\|_\infty \leq C(1 + \log^+\|\nabla v\|_p),$$

with C depending only on p, $\|\nabla\times a\|_q$ and $\|a\|_{W2,p}$. (Note that here we have used the Sobolev inequality.) Estimates (2.15) and (2.17) together imply that

(2.18) $$\|\nabla v\|_p(t) \leq C , \quad t\in(0,T),$$

where C depends also on T. Using $\nabla^2 u = \nabla K*(\nabla v)$ and the fact that ∇K is a Calderon-Zygmund kernel, we see that $\|\nabla^2 u\|_p(t) \leq C$ on $(0,T)$. This implies (2.14) for $m = 2$. Suppose finally that $m \geq 3$. We apply ∇^k to the vorticity equation and then multiply by $|\nabla^k v|^{p-2}\nabla^k v$. After an integration by parts, we get, by the Sobolev inequality,

$$(\|v\|_{W^{m-1,p}}^p)' \leq C\|u\|_{W^{m-1,p}} \|v\|_{W^{m-1,p}}^p$$

with C depending only on m and p. Integrating this, using induction on m, together with the relation $u = K*v$, we get the desired result.

<u>Remark</u>. Kato [15] and Ponce [27] discuss persistency in the L^2 Sobolev spaces. Their results are then extended in [34] to L^p case. Namely, they prove in [34] the persistency in the spaces $H^{2,p}$, $s > 1+2/p$. However, our result above is not covered by the results in [34] when $m = 0$ or 1.

3. Existence of solutions in R^2 with measures as initial vorticity

In this section we show the existence (and uniqueness in some special cases) of solutions of the Navier-Stokes system in R^2 corresponding to the initial data a such that $a \in L^{2,\infty}(R^2)$ and $\nabla\times a \in \mathcal{M}$. The (generalized) Young's inequality implies that the function $a_\eta = e^{\nu\eta\Delta} a$, $\eta > 0$, lies in $L^p(R^2)$ for all $2 < p < \infty$ and that $\nabla\times a_\eta$ is in $L^q(R^2)$ for all $1 \leq q \leq \infty$, together with all their derivatives. The result in Section 2 therefore implies that there exists for each $\eta > 0$ a unique global solution u_η with initial value a_η. Here we wish to let $\eta \to 0$ in order to obtain a solution corresponding to the initial value a. Necessary estimates for u_η are derived from the following result, in which $\Gamma_\eta(x,t\ ;\ y,s)$ denotes the fundamental solution of the linear parabolic operator $L_\eta = \partial_t - \nu\Delta + (u_\eta\cdot\nabla)$.

<u>Proposition 3.1</u>. (i) There are constants $C_j > 0$, $j=1,2,3,4$, depending only on ν and a bound for $\|\nabla\times a\|_{\mathcal{M}}$ so that

$$(3.1)\qquad C_1(t-s)^{-1}\exp(-C_2|x-y|^2/(t-s)) \leq \Gamma_\eta(x,t\ ;\ y,s) \leq C_3(t-s)^{-1}\exp(-C_4|x-y|^2/(t-s))$$

for $x,y \in R^2$ and $0 \leq s < t$.

(ii) There is a β, $0 < \beta < 1$, depending only on ν and a bound for $\|\nabla\times a\|_{\mathcal{M}}$ so that

$$(3.2)\qquad |\Gamma_\eta(x,t\ ;\ y,s)-\Gamma_\eta(x',t'\ ;\ y',s')| \leq C_5(\ |s-s'|^{\beta/2} + |y-y'|^{\beta} + |t-t'|^{\beta/2} + |x-x'|^{\beta}\ )$$

for $\tau < t-s,\ t'-s' < \infty$ and $x, x', y, y' \in R^2$, where C_5 depends only on ν, $\tau > 0$ and a bound for $\|\nabla\times a\|_{\mathcal{M}}$.

Note that estimates (3.1) and (3.2) are uniform for the parameter $\eta > 0$.

Assuming Proposition 3.1 for a moment, we continue the discussion as to how we can obtain the desired solution. From Proposition 3.1, we obtain

Proposition 3.2. Let u_η be the unique global solution corresponding to the initial velocity a_η. Then we have the following estimates :

(3.3) $\|v_\eta\|_1(t) \le \|\nabla\times a\|_{\mathcal{M}}$, $v_\eta = \nabla\times u_\eta$; $\|u_\eta\|_{2,\infty}(t) \le C\|\nabla\times a\|_{\mathcal{M}}$ for $t > 0$

where $\|\cdot\|_{2,\infty}$ is the norm of $L^{2,\infty}(R^2)$ and C depends only on $\|K\|_{2,\infty}$.

(3.4a) $\|v_\eta\|_r(t) \le Ct^{-1+1/r}\|\nabla\times a\|_{\mathcal{M}}$ for $t > 0$ and $1 < r < \infty$;

(3.4b) $\|\nabla u_\eta\|_r(t) \le Ct^{-1+1/r}\|\nabla\times a\|_{\mathcal{M}}$ for $t > 0$ and $1 < r < \infty$;

(3.4c) $\|u_\eta\|_r(t) \le Ct^{1/r-1/2}\|\nabla\times a\|_{\mathcal{M}}$ for $t > 0$ and $2 < r < \infty$,

with C depending only on r, ν and a bound for $\|\nabla\times a\|_{\mathcal{M}}$.

(3.5) $$\sup_{[\varepsilon,T]} \|\nabla^k\partial_t^h u_\eta\|_\infty(t) \le C, \quad \varepsilon > 0$$

with C depending only on ε, h, k, ν, T and a bound for $\|\nabla\times a\|_{\mathcal{M}}$.

Proof. The first estimate in (3.3) is a direct consequence of the formula,

(3.6) $$v_\eta(x,t) = \int_{R^2} \Gamma_\eta(x,t;y,0)(\nabla\times a_\eta)(y)dy ,$$

the relation (2.9) and the estimate : $\|\nabla\times a_\eta\|_1 \le \|\nabla\times a\|_{\mathcal{M}}$. The second estimate in (3.3) follows from the first one and the generalized Young's inequality (2.12b). (3.4a) is obtained from (3.6) and (3.1), while (3.4c) is obtained from (3.4a) and (2.12a). Since $\nabla u_\eta = \nabla K * v_\eta$, (3.4b) follows from the Calderon-Zygmund theory. (3.5) is a consequence from (3.4c) and Proposition 2.3 (i).

Due to the estimate (3.5), one can apply the Ascoli-Arzela theorem to extract a subsequence of u_η, as $\eta \to 0$, which converges uniformly on any finite interval $[\varepsilon,T]$ to a function u together with all the derivatives. Obviously, the function u is a classical solution of the Navier-Stokes system for $t > 0$.

It remains to give a precise meaning to the initial condition $u(0) = a$. This is carried out with the aid of the following two results, which are also consequences of Proposition 3.1 (i).

Lemma 3.3. The functions v_η, $\eta > 0$, are uniformly bounded and equicontinuous on any finite interval $[0,T]$ with respect to the weak topology of measures. Namely, for each continuous function ϕ on R^2 vanishing at infinity, the pairings $(v_\eta(t),\phi)$, $\eta > 0$, are uniformly bounded and equicontinuous on any $[0,T]$.

Lemma 3.4. The functions u_η, $\eta > 0$, are uniformly bounded and equicontinuous on any $[0,T]$ with respect to the weak* topology of the space $L^{2,\infty}(R^2)$. (Recall that $L^{2,\infty}$ is the dual of the Lorentz space $L^{2,1}$; see [4].)

The boundedness of v_η and u_η are obvious from (3.3) ; so it suffices to show the equicontinuity. We first consider the case where ϕ is rapidly decreasing in the sense of L. Schwartz. In this case, the equicontinuity of $(v_\eta(t),\phi)$ is directly verified by calculating the difference $(v_\eta(t),\phi)-(v_\eta(s),\phi)$ with the aid of the estimates given in Proposition 3.2. Since the rapidly decreasing functions are dense in the Banach space of continuous functions vanishing at infinity, we get Lemma 3.3. Lemma 3.4 is proved by contradiction with the aid of Lemma 3.3 and the relation $u_\eta = K * v_\eta$. For another proof of Lemma 3.3, which uses a special kind of metric on the set of measures as treated in [6], we refer to [36]; see also [19], [20] and [31].

We can now mimick the proof of the Ascoli-Arzela theorem to show that the limit function u obtained above is chosen in such a way that it is weakly* continuous from $[0,\infty)$ to $L^{2,\infty}(R^2)$ and the corresponding vorticity $v = \nabla\times u$ is weakly continuous from $[0,\infty)$ to $\mathfrak{M}$.

The foregoing arguments are summarized in the following form, which is the first part of our main results.

Theorem 3.5. (Existence). Suppose that $a \in L^{2,\infty}(R^2)$, $\nabla \cdot a = 0$, and that $\nabla \times a$ is a finite measure. Then the problem (1.1) has a global solution u which is smooth for $t > 0$ such that

(i) $u : [0,\infty) \to L^{2,\infty}(R^2)$ is bounded and continuous in the weak* topology and satisfies $u(0) = a$.

(ii) $v = \nabla \times u : [0,\infty) \to \mathfrak{M}$ is bounded and continuous in the weak topology of measures and satisfies $v(0) = \nabla \times a$.

(iii) The estimates

(3.7) $$\|u\|_r(t) \leq Ct^{1/r-1/2} \quad \text{for } t > 0, \ 2 < r < \infty ;$$

(3.8) $$\|\nabla u\|_r(t) \leq Ct^{-1+1/r} \quad \text{for } t > 0, \ 1 < r < \infty$$

holds with C depending only on r, ν and a bound for $\|\nabla \times a\|_{\mathfrak{M}}$.

(iv) For $0 < \varepsilon < T$ and nonnegative integers k, h, there is a constant $C > 0$ so that

$$\sup_{[\varepsilon,T]} \|\nabla^k \partial_t^h u\|_\infty(t) \leq C$$

and the C depends also on ν and a bound for $\|\nabla \times a\|_{\mathfrak{M}}$.

(v) The function u solves the integral equation (2.1) in $L^{2,\infty}(R^2)$.

Proof. It remains only to show (v). First we note that the interpolation theory of the Lorentz spaces implies that the operators $e^{\nu t\Delta}$ and P are all bounded in $L^{2,\infty}(R^2)$ (see [4]). It is easy to see that u satisfies

$$\begin{aligned} u(t) &= e^{\nu(t-\varepsilon)\Delta} u(\varepsilon) - \int_\varepsilon^t e^{\nu(t-s)\Delta} \nabla P(u \otimes u)(s)ds \\ &= e^{\nu(t-\varepsilon)\Delta} u(\varepsilon) - \int_\varepsilon^t \nabla e^{\nu(t-s)\Delta} P(u \otimes u)(s)ds \end{aligned}$$

for all $0 < \varepsilon < t$. As $\varepsilon \to 0$, we see from Theorem 3.5 (iii) that the second term tends to $S[u](t)$ in $L^2(R^2)$, and hence in $L^{2,\infty}(R^2)$, for each fixed $t > 0$. On the other hand, an elementary calculation shows that the first term tends to $e^{\nu t\Delta} a$ in the weak* topology of $L^{2,\infty}(R^2)$; hence (v) is obtained.

We next discuss the uniqueness of our solutions. To this purpose, we need the following, which is our second main result.

Theorem 3.6. (Integral Representation for $\nabla\times u$). Under the assumption of Theorem 3.5, the vorticity $v = \nabla\times u$ is expressed as

$$v(x,t) = \int_{R^2} \Gamma(x,t\,;\,y,0)(\nabla\times a)(dy), \qquad t > 0, \tag{3.9}$$

in terms of a continuous function $\Gamma(x,t;y,s)$, $x,y \in R^2$, $t > s \geq 0$, with the following properties (3.10)-(3.12):

$$\int_{R^2} \Gamma(x,t;y,s)dy = \int_{R^2} \Gamma(x,t;y,s)dx = 1, \qquad t > s \geq 0\,; \tag{3.10}$$

$$\Gamma(x,t;y,s) = \int_{R^2} \Gamma(x,t;z,t')\Gamma(z,t';y,s)dz, \qquad t > t' > s \geq 0\,; \tag{3.11}$$

$$C_1(t-s)^{-1}\exp(-C_2|x-y|^2/(t-s)) \leq \Gamma(x,t\,;\,y,s) \leq C_3(t-s)^{-1}\exp(-C_4|x-y|^2/(t-s)), \qquad t > s \geq 0, \tag{3.12}$$

with $C_j > 0$, $j=1,2,3,4$, depending only on ν and a bound for $\|\nabla\times a\|_{\mathfrak{M}}$. Moreover, the estimate

$$\|v\|_r(t) \leq Ct^{-1+1/r}, \qquad t > 0, \quad 1 < r < \infty \tag{3.13}$$

holds with C depending only on r, ν and a bound for $\|\nabla\times a\|_{\mathfrak{M}}$.

Proof. The existence of the function Γ follows easily from the uniform estimate (3.2) and the Ascoli-Arzela theorem. The remaining assertions are immediate from the foregoing arguments.

To state our uniqueness result, let us recall the classical Lebesgue decomposition of a Radon measure μ on R^n :

$$\mu = \mu_{pp} + \mu_c$$

where μ_c is characterized by the property $\mu_c(\{x\}) = 0$ for all $x \in R^n$ and μ_{pp} is expressed as a (possibly infinite) linear combination of Dirac measures.

Lemma 3.7. For any finite Radon measure μ on R^2 we have

$$\limsup_{t\to 0} t^{1-1/r} \|e^{t\Delta}\mu\|_r \leq C\|\mu_{pp}\|_{\mathcal{M}} \quad \text{for all } 1 < r < \infty,$$

where C depends only on r.

Proof. We first recall the estimate :

$$\|e^{t\Delta}\mu\|_r \leq Ct^{-1+1/r}\|\mu\|_{\mathcal{M}}.$$

Indeed, since the linear operator $Af = f*\mu$ is bounded in both L^1 and L^∞ with operator-norm $\leq \|\mu\|_{\mathcal{M}}$, applying the Riesz-Thorin theorem ([4]) yields the estimate if we take as f the heat kernel. This estimate and the Lebesgue decomposition together show that we have only to prove that

$$\text{(3.14)} \qquad \lim_{t\to 0} t^{1-1/r}\|e^{t\Delta}\mu\|_r = 0 \qquad \text{for all } 1 < r < \infty,$$

provided $\mu = \mu_c$. With no loss of generality we may assume that $\mu \geq 0$. For any fixed $\varepsilon > 0$, we take $N > 0$ so that, denoting $B(0,N) = \{x \,;\, |x| \leq N\}$, $\mu(R^2\setminus B(0,N)) < \varepsilon$ and hence $\mu_2 = (R^2\setminus B(0,N))\lrcorner\,\mu$ satisfies

$$\text{(3.15)} \qquad t^{1-1/r}\|e^{t\Delta}\mu_2\|_r \leq C \qquad \text{for all } 1 < r < \infty.$$

The support of the measure $\mu_1 = \mu - \mu_2$ is contained in $B(0,N)$ and direct calculation gives

$$\text{(3.16)} \quad (t^{1-1/r}\|e^{t\Delta}\mu_1\|_r)^r = C't^{-1}\int_{R^2}\Big(\int_{|y|<N}\exp(-|x-y|^2/4t)\mu_1(dy)\Big)^r dx$$

$$= C't^{-1}\Big(\int_{|x|>2N} + \int_{|x|<2N}\Big)\Big(\int_{|y|<N}\exp(-|x-y|^2/4t)\mu_1(dy)\Big)^r dx$$

$$= I_1(t) + I_2(t).$$

Since $|x-y| > |x|/2$ if $|x| > 2N$ and $|y| \leq N$, we get

(3.17) $\quad I_1(t) \lesssim C'\|\mu_1\|^r_{\mathfrak{M}} \, t^{-1} \int_{|x|>2N} \exp(-r|x|^2/16t)dx \to 0, \quad \text{as} \quad t \to 0.$

For $I_2(t)$, applying the Minkowski inequality yields

$$(3.18) \quad I_2(t) \leq C't^{-1} \int_{|x|\lesssim 2N}\left(\int_{|x-y|>\delta} \exp(-|x-y|^2/4t)\mu_1(dy)\right)^r dx$$

$$+ C't^{-1} \int_{|x|\lesssim 2N}\left(\int_{|x-y|\lesssim\delta} \exp(-|x-y|^2/4t)\mu_1(dy)\right)^r dx$$

$$= I_{21}(t) + I_{22}(t),$$

where $\delta > 0$ is to be chosen later. Obviously, for any fixed $\delta > 0$,

$$(3.19) \quad I_{21}(t) \lesssim C'\text{meas}(B(0,2N))\|\mu_1\|^r_{\mathfrak{M}} \times t^{-1}\exp(-r\delta^2/4t) \to 0, \quad \text{as} \quad t \to 0$$

where meas is the Lebesgue measure on R^2. On the other hand, Hölder's inequality yields

$$(3.20) \quad I_{22}(t) \lesssim C'\int_{|x|\lesssim 2N} (\mu_1(B(x,\delta)))^{r-1} \left(\int_{|x-y|<\delta} \exp(-r|x-y|^2/4t)\mu_1(dy)\right)dx/t$$

$$\lesssim C'' \sup_{|x|\lesssim 2N} (\mu_1(B(x,\delta)))^{r-1} \|e^{r^{-1}t\Delta} \mu_1\|_1$$

$$\lesssim C'' \|\mu\|_{\mathfrak{M}} \times \sup_{|x|\lesssim 2N} (\mu_1(B(x,\delta)))^{r-1},$$

where $B(x,\delta) = \{y \; ; \; |y-x| \lesssim \delta\}$. Thus, we have only to show that

$$(3.21) \quad \mu_1(B(x,\delta)) \to 0 \quad \text{as} \quad \delta \to 0 \quad \text{uniformly for} \quad |x| \lesssim 2N.$$

This is easily obtained by contradiction if we note that $\mu(\{x\}) = 0$ for $x \in R^2$. The proof is completed.

Remark. Although Lemma 3.7 is elementary, it does not seem to be well-known ; so we have presented here its complete proof.

We are now ready to state our uniqueness result.

Theorem 3.8. (Uniqueness). Suppose that $a \in L^{2,\infty}(R^2)$, $\nabla\cdot a = 0$ and that $\nabla\times a \in \mathfrak{M}$. Take $m > 0$ so that $\|\nabla\times a\|_{\mathfrak{M}} \leq m$ and let u be the solution of (1.1) given in Theorem 3.5. Then we have the following.

(i) For all $p > 2$,

$$\limsup_{t\to 0} t^{1/2-1/p} \|u\|_p(t) \leq C\|(\nabla\times a)_{pp}\|_{\mathfrak{M}} \tag{3.22}$$

with C depending only on p, m and ν.

(ii) For each $p > 2$ there is a positive constant $\varepsilon = \varepsilon(p,\nu,m)$ such that if $\|(\nabla\times a)_{pp}\|_{\mathfrak{M}} < \varepsilon$, the solution u is unique in the class of functions w with the following properties :

(a) $w : [0,\infty) \to L^{2,\infty}(R^2)$ is weakly continuous and $w(0) = a$.

(b) $w : (0,\infty) \to L^p(R^2)$ is continuous and satisfies (3.22) for $p > 2$.

(c) w solves (2.1) in $L^{2,\infty}(R^2)$.

In particular, the solution u is unique provided that $\nabla\times a = (\nabla\times a)_c$.

Assertion (i) follows directly from the formula $u = K*v$ $(v = \nabla\times u)$, (3.9), (3.12), Lemma 3.7 and (2.12a). The proof of assertion (ii) is then rather standard, so we omit it here.

We finally discuss on the derivation of our basic tool, i.e., Proposition 3.1. Consider on R^n the linear parabolic operator of the form :

$$L_b = \partial_t - \nu\Delta + (b\cdot\nabla),$$

where $b = (b^1,\dots, b^n)$ is smooth, bounded on $R^n\times[0,T)$ together with all its derivatives and satisfies the following conditions :

(3.23) $\nabla\cdot b = 0$; and there are bounded functions c^{ij}, $i,j= 1,\dots, n$, on $R^n\times[0,T)$ so that

$$b^i = \textstyle\sum_j \partial_j c^{ij}, \quad i= 1,\dots, n, \quad \text{where } \partial_j = \partial/\partial x_j .$$

We denote by $\Gamma_b(x,t\,;\,y,s)$, $x,y \in R^n$, $0 \leq s \leq t < T$, the fundamental solution of the operator L_b.

Theorem 3.9 ([25]). (i) Let L_b satisfy (3.23). Then there are constants $C_j > 0$, $j = 1,2,3,4$, depending only on ν, and a bound for $\|c^{ij}\|_{L^\infty}$ so that

$$C_1(t-s)^{-n/2}\exp(-C_2|x-y|^2/(t-s)) \leq \Gamma_b(x,t;y,s)$$
$$\leq C_3(t-s)^{-n/2}\exp(-C_4|x-y|^2/(t-s))$$

for $x,y \in R^n$ and $0 \leq s \leq t < T$.

(ii) Under the same assumption as in (i), there is a constant β, $0 < \beta < 1$, depending only on ν and a bound for $\|c^{ij}\|_{L^\infty}$ so that

$$|\Gamma_b(x,t;y,s)-\Gamma_b(x',t';y',s')|$$
$$\leq C(|x-x'|^\beta + |y-y'|^\beta + |t-t'|^{\beta/2} + |s-s'|^{\beta/2})$$

for $x,x',y,y' \in R^n$ and $0 < \tau < t-s,\ t'-s' < \infty$, with $0 \leq t,s,t',s' < T$, where C depends only on ν, τ and a bound for $\|c^{ij}\|_{L^\infty}$.

The above type of estimates were first proved by Aronson [1] and Aronson-Serrin [2] for the operators of divergence form. If c^{ij} above satisfy the anti-symmetry condition : $c^{ij} = -c^{ji}$, then L_b is of divergence form and therefore Theorem 3.9 is regarded as a generalization of the result of [1] and [2] to the case of operators of nondivergence form. For the full version of Theorem 3.9, we refer to [25]. We can obtain Proposition 3.1 from Theorem 3.9 via the following interesting lemma.

Lemma 3.10. The function

$$K = (K^1, K^2) = (-x_2, x_1)/2\pi|x|^2, \qquad x = (x_1, x_2)$$

is expressed as

$$K^1 = \partial_1 A^3 + \partial_2 A^1 \ ; \qquad K^2 = -\partial_1 A^1 - \partial_2 A^2 \ ,$$

where

$$A^1 = -x_1^2 x_2^2/\pi|x|^4 \ , \qquad A^2 = -3x_1 x_2/2\pi|x|^2 + x_1^3 x_2/\pi|x|^4 \ ,$$

$$A^3 = -3x_1 x_2^3/2\pi|x|^2 + x_1 x_2^3/\pi|x|^4 \ .$$

The proof is done through direct calculation. Now let u_η be the solution corresponding to the initial value $a_\eta = e^{\nu\eta\Delta}a$, so that $u_\eta = K * v_\eta$, $v_\eta = \nabla\times u_\eta$. Using Lemma 3.10, together with the estimate $\|v_\eta\|_1(t) \leq \|\nabla\times a_\eta\|_1 \leq \|\nabla\times a\|_{\mathfrak{M}}$, we easily see that the operator $L_\eta = \partial_t - \nu\Delta + (u_\eta\cdot\nabla)$ satisfies condition (3.23) uniformly in $\eta > 0$.

References

1. D.G. Aronson, Bounds for the fundamental solution of a parabolic equation, Bull. Amer. Math. Soc. 73 (1968), 890-896.

2. D.G. Aronson and J. Serrin, Local behavior of solutions of quasilinear parabolic equations, Arch. Rational Mech. Anal. 25 (1967), 81-122.

3. G. Benfatto, R. Esposito and M. Pulvirenti, Planar Navier-Stokes flow for singular initial data, Nonlinear Anal. 9 (1985), 533-545.

4. J. Bergh and J. Löfström, Interpolation Spaces, An introduction, Springer-Verlag, 1976.

5. H. Brezis and A. Friedman, Nonlinear parabolic equations involving measures as initial data, J. Math. pures et appl. 62 (1983), 73-97.

6. R.L. Dobrushin, Prescribing a system of random variables by conditional distributions, Theo. Prob. Appl. 15 (1970), 458-486.

7. E.B. Fabes, B.F. Jones and N.M. Riviere, The initial value problem for the Navier-Stokes equations with data in L^p, Arch. Rational Mech. Anal. 45 (1972), 222-240.

8. A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, 1964.

9. A. Friedman, Partial Differential Equations, Holt, Rinehart and Winston, 1969.

10. H. Fujita and T. Kato, On the Navier-Stokes initial value problem I., Arch. Rational Mech. Anal. 16 (1964), 269-315.

11. Y. Giga and T. Miyakawa, Solutions in L_r of the Navier-Stokes initial value problem, Arch. Rational Mech. Anal. 89 (1985), 267-281.

12. Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, J. Diff. Eq. 62 (1986), 186-212.

13. D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd ed. Springer-Verlag, 1983.

14. T. Kato, Strong L^p-solutions of the Navier-Stokes equation in R^m, with applications to weak solutions, Math. Z. 187 (1984), 471-480.

15. T. Kato, Remarks on Euler and Navier-Stokes equations in R^2, Proc. Symp.

15. T. Kato, Remarks on Euler and Navier-Stokes equations in R^2, Proc. Symp. in Pure Math. 45 Part 2, 1-8, Amer. Math. Soc., 1986.

16. O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, 1969.

17. J. Leray, Etude de diverses équations intégrales non linéaires et de quelques problêmes que pose l'hydrodynamique, J. Math. pures et appl. Serie 9, 12 (1933), 1-82.

18. T.-P. Liu and M. Pierre, Source-solutions and asymptotic behavior in conservation laws, J. Diff. Eq. 51 (1984), 419-441.

19. C. Marchioro and M. Pulvirenti, Hydrodynamics in two dimensions and vortex theory, Commun. Math. Phys. 84 (1982), 483-503.

20. C. Marchioro and M. Pulvirenti, Euler evolution for singular initial data and vortex theory, Commun. Math. Phys. 91 (1983), 563-572.

21. F.J. McGrath, Nonstationary planar flow of viscous and ideal fluids, Arch. Rational Mech. Anal. 27 (1968), 329-348.

22. H.P. McKean, Jr., Propagation of chaos for a class of nonlinear parabolic equations, Lecture series in diff. eq., Session 7: Catholic Univ., 1967.

23. Y. Niwa, Semilinear heat equations with measures as initial data, preprint.

24. H. Osada and S. Kotani, Propagation of chaos for the Burgers equation, J. Math. Soc. Japan 37 (1985), 275-294.

25. H. Osada, Diffusion processes with generators of generalized divergence form, J. Math. Kyoto Univ., to appear.

26. H. Osada, Propagation of chaos for the two dimensional Navier-Stokes equations, preprint ; Announcement : Proc. Japan Acad. 62 (1986), 8-11.

27. G. Ponce, On two dimensional incompressible fluids, Comm. PDE 11 (1986), 483-511.

28. M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vols. I, II, Academic Press, 1972, 1975.

29. A.S. Sznitman, Propagation of chaos result for the Burgers equation, Probab. Theor. Rel. Fields, 71 (1986), 581-613.

30. R. Temam, Navier-Stokes Equations. North-Holland, 1977.

31. B. Turkington, On the evolution of a concentrated vortex in an ideal fluid, preprint.

32. W. von Wahl, The Equations of Navier-Stokes and Abstract Parabolic Equations. Vieweg Verlag, 1985.

33. F.B. Weissler, The Navier-Stokes initial value problem in L^p, Arch. Rational Mech. Anal. 74 (1980), 219-230.

34. T. Kato and G. Ponce, Well-posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces $L^p_s(R^2)$, preprint.

35. P. Baras and M. Pierre, Problêmes paraboliques semi-linêaires avec donnês mesures, Applicable Anal. 18 (1984), 111-149.

36. Y. Giga, T. Miyakawa and H. Osada, Two dimensional Navier-Stokes flow with measures as initial vorticity, preprint.