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Zero-viscosity Limit of the incompressible Navier-Stokes Equation 2

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1. Problem and Result

It has been known that the solution $u(\nu,t,x)$ of the initial value problem for the incompressible Navier-Stokes equation tends to the (unique) solution $u(0,t,x)$ of the initial value problem for the incompressible Euler equation, as the viscosity coefficient $\nu > 0$ tends to zero. Moreovre it is a smooth funcyion of $\nu \in [0,1]$ in some function spaces. For example, see [4], [3] and [1].

We consider the same problem for the initial boundary value problem (I.B.V.P.) in the half space $R^n_+ = \{ x = (x_1, \cdots, x_{n-1}, x_n) = (x', x_n) ; x_n > 0 \}$, $n \geq 2$. There has been no result for this problem, though the boundary layer originated by Prandtl in 1905 provides a good approximation method.

Let $u = {}^t(u_1, \cdots, u_n) = {}^t(u', u_n) = u(\nu,t,x)$ be the velocity of the fluid at the time $t \geq 0$ and the point $x \in R^n_+$. The I.B.V.P. for the incompressible Navier-Stokes equation is written as follows :

(1.1) (1) $\partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla p = 0$, $t > 0$, $x_n \in R^n_+$,

(2) $\nabla\cdot u = 0$,

(3) $u|_{t=0} = u_0$,

(4) $\gamma u \equiv u|_{x_n=0} = 0$.

Here $\nu\in(0,1]$ is the viscosity coefficient, $u\cdot\nabla u = u_1\partial_1 u_1+\cdots+u_n\partial_n u_n$, $\nabla\cdot u = \partial_1 u_1+\cdots+\partial_n u_n$, $\Delta u = \partial_1^2 u+\cdots+\partial_n^2 u$ and $\nabla p = {}^t(\partial_1 p, \cdots, \partial_n p)$.

Similarly the I.B.V.P. for the incompressible Euler equation is written as

(1.2) (1) $\partial_t u + u\cdot\nabla u + \nabla p = 0 , \quad t > 0,\ x \in R^n_+ ,$

(2) $\nabla\cdot u = 0 ,$

(3) $u|_{t=0} = u_0 ,$

(4) $\gamma_n u \equiv u_n|_{x_n=0} = 0 .$

As for the initial data u_0 , we assume the "compatibility" :

(1.3) (1) $\nabla\cdot u_0 = 0 ,$

(2) $\gamma u = 0 .$

We intend to get the solution of (1.1) in the following form :

(1.4)
$$u(\nu,t,x) = u^0(\nu,t,x) + \varepsilon u^1(\varepsilon,t,x) + \varepsilon^2 u(\varepsilon,t,x) + \tilde{u}^0(\varepsilon,t,x',x_n/\varepsilon) + \varepsilon\tilde{u}^1(\varepsilon,t,x',x_n/\varepsilon) + \varepsilon^2\tilde{u}^2(\varepsilon,t,x',x_n/\varepsilon),$$
$$\tilde{u}^i(\varepsilon,t,x',x_n/\varepsilon) = \begin{pmatrix} \tilde{u}^{i\prime}(\varepsilon,t,x',x_n/\varepsilon) \\ \varepsilon\tilde{u}^i_n\ (\varepsilon,t,x',x_n/\varepsilon) \end{pmatrix} ,\quad i = 0,1,$$
$$\tilde{u}^2(\varepsilon,t,x',x_n/\varepsilon) = {}^t(\ \tilde{u}^{2\prime}(\cdots,x_n/\varepsilon),\ \tilde{u}^2_n(\cdots,x_n/\varepsilon)) ,$$
$$p(\nu,t,x) = p^0(\nu,t,x) + \varepsilon p^1(\varepsilon,t,x) + \varepsilon^2 p^2(\varepsilon,t,x) + \varepsilon\tilde{p}^1(\varepsilon,t,x',x_n/\varepsilon) + \varepsilon^2\tilde{p}^2(\varepsilon,t,x',x_n/\varepsilon),$$
$$\varepsilon = \sqrt{\nu} \in (\ 0,1] .$$

Each term in the above expansion is determimed to satisfy the following "Navier-Stokes equations" (1.5)-(1.9), respectively:

(1.5)
$$\partial_t u^0 + u^0\cdot\nabla u^0 - \nu\Delta u^0 + \nabla p^0 = 0 ,\quad t > 0,\quad x \in R^n_+ ,$$
$$\nabla\cdot u^0 = 0 ,$$
$$u^0|_{t=0} = u_0 ,$$
$$\gamma_n u^0 = 0 ,$$

$$(1.6)\quad \partial_t\tilde{u}^{0\prime}+(u^0+\tilde{u}^0)\cdot\nabla'\tilde{u}^{0\prime}+(u_n^0+\varepsilon\tilde{u}_n^0-\varepsilon\gamma\tilde{u}_n^0)\partial_n\tilde{u}^{0\prime}-\nu\Delta\tilde{u}^{0\prime}+\tilde{u}^0\cdot\nabla u^{0\prime}=0,$$

$$\tilde{u}_n^0(\varepsilon,t,x) = -\partial_n^{-1}\nabla'\cdot\tilde{u}^{0\prime}(\varepsilon,t,x) \equiv \int_{x_n}^{\infty}\nabla'\cdot\tilde{u}^{0\prime}(\varepsilon,t,x',\eta_n)d\eta_n ,$$

$$\nabla'\equiv{}^t(\partial_1,\cdots,\partial_{n-1}) \equiv \partial' ,$$

$$\tilde{u}^0|_{t=0} = 0 ,$$

$$\gamma'\tilde{u}^0 \equiv \tilde{u}^{0\prime}|_{x_n=0} = -\gamma' u^0 ,$$

$$(1.7)\quad \partial_t u^1 + u^0\cdot\nabla u^1 - \nu\Delta u^1 + u^1\cdot\nabla u^0 + \nabla p^1 = 0 ,$$

$$\nabla\cdot u^1 = 0 ,$$

$$u^1|_{t=0} = 0 ,$$

$$\gamma_n u^1 = -\gamma_n\tilde{u}^0 ,$$

$$(1.8)\quad \{\partial_t+(u^0+\varepsilon u^1+\varepsilon^2u^2)\cdot\nabla -\nu\Delta\}u^2 + u^2\cdot\nabla(u^0+\varepsilon u^1) + \nabla p^2 = -u^1\nabla\cdot u^1,$$

$$u^2|_{t=0} = 0 ,$$

$$(1.9)\quad \{\partial_t+(u^0+\varepsilon u^1+\varepsilon^2u^2+\tilde{u}^0+\varepsilon\tilde{u}^1+\varepsilon^2\tilde{u}^2)\cdot\nabla -\nu\Delta\}(\tilde{u}^1+\varepsilon\tilde{u}^2) + \nabla(\tilde{p}^1+\varepsilon\tilde{p}^2)$$

$$+(\tilde{u}^1+\varepsilon\tilde{u}^2)\cdot\nabla(u^0+\varepsilon u^1+\varepsilon^2u^2+\tilde{u}^0) = -\tilde{u}^0\cdot\nabla u^1 - \tilde{h}^0 ,$$

$$\tilde{u}^i|_{t=0} = 0 , \quad i = 1,2,$$

$$\tilde{h}^0 = \begin{pmatrix} (u_n^1 - \gamma u_n^1)\partial_n\tilde{u}^{0\prime} \\ \tilde{u}^0\cdot\nabla u_n^0 + (u^0+\varepsilon u^1+\tilde{u}^0)\cdot\nabla\tilde{u}_n^0 - \partial_n^{-1}\nabla'\cdot\tilde{v}^{0\prime} \end{pmatrix}, \quad \text{(See (6.2))},$$

$$\nabla\cdot(\varepsilon u^2+\tilde{u}^1+\varepsilon\tilde{u}^2) = 0 ,$$

$$\gamma(\varepsilon u^2+\tilde{u}^1+\varepsilon\tilde{u}^2) = -{}^t(\gamma' u^1,\ 0) .$$

By using the notations described in the next section, our result is stated as follows:

Theorem. Let $u_0\in H_a^{\ell,\rho,\theta}$ with $\ell>(n-1)/2+3$, $\rho>0$, $0<\theta<\pi/4$ and $a>0$, and assume the "compatibility condition" (1.3). Then, there exists a time interval $[0,T]$, $T > 0$, independent of $\nu\in(0,1]$, such that (1.1) has a (unique) solution $u(\nu,t,x)$ of the form (1.4) and each term satisfies (1.5)–(1.9), respectively, and the follwing :

(1.10) $u^0(\nu,t,x) \in \mathring{\mathcal{X}}^{\ell,\rho,\theta}_{a,\beta_0,T}$,

$u^i(\varepsilon,t,x) \in \mathcal{X}^{\ell-i,\rho,\theta}_{a,\beta_i,T}$, $i = 1,2$, $0 < \beta_0 < \beta_1 < \beta_2$,

$\tilde{u}^0(\varepsilon,t,x) = \begin{pmatrix} \tilde{u}^{i\prime}(\varepsilon,t,x',x_n) \\ \tilde{u}^i_n(\varepsilon,t,x',x_n) \end{pmatrix} \in \mathcal{X}^{\ell-1,\rho,\theta,(\mu)}_{a/\varepsilon,\beta_1,T}$, $\mu > 0$,

$\tilde{u}^i(\varepsilon,t,x',x_n) \in \mathcal{X}^{\ell-i-1,\rho,\theta}_{a/\varepsilon,\beta_2,T}$, $i = 1,2$.

In particular, $\partial^j_n u(\nu,t,x',x_n)$, $0 \le j \le \ell$, is a continuous function of $(\nu,x_n) \in [0,1]\times\Sigma(\theta',a)\setminus\{0\}\times\{0\}$ in the strong topology of $K^{\ell-j,\rho}_{\beta',T}$ with some $\beta' > 0$ and $\theta' > 0$, and $u^0(0,t,x)$ is the unique solution of (1.2) .

2. Notations and Function spaces

First we introduce several notations :

(2.1) $I(\rho) = (-\rho,\rho)^{n-1}$ (open cube) ,

$D(\rho) = R^{n-1}+\sqrt{-1}I(\rho) = \{z' = x'+\sqrt{-1}y';\ x' \in R^{n-1},\ y' \in I(\rho)\}$,

$\Sigma(\theta,a) = \Sigma_1(\theta,a)\cup\Sigma_2(\theta,a)$, $0 < \theta < \pi/4$, $a > 0$,

$\Sigma_1(\theta,a) = \{\ z_n = x_n+\sqrt{-1}y_n\ ;\ |y_n| \le x_n\tan\theta\ ,\ 0 \le x_n \le a\ \}$,

$\Sigma_2(\theta,a) = \{\ z_n = x_n+\sqrt{-1}y_n\ ;\ |y_n| \le a\tan\theta\ ,\ x_n \ge a\ \}$,

$\Omega(\rho,\theta,a) = D(\rho)\times\Sigma(\theta,a)$,

$L(y') = R^{n-1}+\sqrt{-1}y' \subset D(\rho)$,

$L(\theta',a) = L_1(\theta',a)\cup L_2(\theta',a) \subset \Sigma(\theta,a)$, $|\theta'| \le \theta$,

$L_1(\theta',a) = \{\ z_n = x_n+\sqrt{-1}y_n;\ y_n = x_n\tan\theta',\ 0 \le x_n \le a\ \}$,

$L_2(\theta',a) = \{\ z_n = x_n+\sqrt{-1}y_n;\ y_n = a\tan\theta',\ x_n \ge a\ \}$.

Next we introduce function spaces :

(2.2) For a Banach space X with the norm $|\ |_X$, $B^k([0,T];X)$ is the set of all C^k-functions from $[0,T]$ to X with the norm

$$|f|_{X,k,T} = \sum_{j=0}^{k} \sup_{0\le t\le T} |\partial^j_t f(t)|_X < \infty .$$

With $[0,T]$ replaced by $\Delta_T=[0,1]\times[0,T]$ (resp. $\Sigma(\theta,a)$), we define $B^k(\Delta_T;X)$ (resp. $B^k(\Sigma(\theta,a);X)$) in a similar way.

(2.3) For a Banach scale $\tilde{X}_\rho=\{X_\rho;\ 0\le\rho\le\rho_0\}$ (with the norm $|\ |_\rho$ of X_ρ) we define $B^k_\beta([0,T];X_\rho)$ (resp. $B^k_\beta(\Delta_T;X_\rho)$) with the norm

$$|f|_{\rho_0,k,\beta,T}=\sum_{j=0}^{k}\sup_{0\le t\le T}|\partial_t^j f(t)|_{\rho_0-\beta t},\quad \beta\ge 0,\ \rho_0-\beta T\ge 0$$

(resp. $|f|_{\rho_0,k,\beta,T}=\sum_{i+j\le k}\sup_{\Delta_T}|\partial_\varepsilon^i\partial_t^j f(\varepsilon,t)|_{\rho_0-\beta t}$).

For further details, see 4.

(2.4) $H^{-\ell,\rho}\ni f \iff$ (1) $f(x'+\sqrt{-1}y')$ is analytic in $D(\rho)$,

(2) $\partial'^{\alpha}f(x'+\sqrt{-1}y')\in L^2(L(y'))$ for $y'\in I(\rho)$, $|\alpha|\le\ell$,

(3) $|f|_{\ell,\rho}=\sum_{|\alpha|\le\ell}\sup_{y'\in I(\rho)}|\partial'^{\alpha}f(\cdot+\sqrt{-1}y')|_{L^2(L(y'))}<\infty$.

(2.5) $H_a^{\ell,\rho,\theta}\ni f \iff$ (1) $f(z',z_n)$ is analytic inside $\Omega(\rho,\theta,a)$,

(2) $\partial^{\alpha}f(z',z_n)\in B^0(\Sigma(\theta,a);H^{-0,\rho})$ for $|\alpha|\le\ell$,

(3) $|f|_{\ell,\rho,\theta}=\sum_{|\alpha|\le\ell}\sup_{z_n\in\Sigma(\theta,a)}|\partial^{\alpha}f(\cdot,z_n)|_{0,\rho}<\infty$.

(2.6) $H_a^{\ell,\rho,\theta,(\mu)}\ni f$ $(\mu\ge 0)$ $\iff$ (1) $f\in H_a^{\ell,\rho,\theta}$,

(2) $|f|_{\ell,\rho,\theta,(\mu)}=\sum_{|\alpha|\le\ell}\sup_{z_n\in\Sigma(\theta,a)}e^{\mu x_n}|\partial^{\alpha}f(\cdot,z_n)|_{0,\rho}<\infty$.

(2.7) $K_{\beta,T}^{-\ell,\rho}=\bigcap_{j\le\ell/2}B^j_\beta([0,T];H^{-\ell-2j,\rho})$,

$$|f|_{\ell,\rho,\beta,T}=\sum_{j\le\ell/2}\sup_{0\le t\le T}|\partial_t^j f(t,\cdot)|_{\ell-2j,\rho-\beta t},$$

$\mathcal{K}_{\beta,T}^{-\ell,\rho}=\bigcap_{k\le\ell}B^k([0,1];K_{\beta,T}^{-\ell-k,\rho})$.

$$(2.8)\quad K^{\ell,\rho,\theta}_{a,\beta,T} = \bigcap_{j\le \ell/2} B^j_\beta([0,T];\ H^{\ell-2j,\rho,\theta}_a\ ),$$

$$|f|_{\ell,\rho,\theta,\beta,T} = \sum_{j\le\ell/2}\ \sup_{0\le t\le T} |\partial^j_t f(t,\cdot)|_{\ell-2j,\rho-\beta t,\theta-\beta t},$$

$$\mathcal{K}^{\ell,\rho,\theta}_{a,\beta,T} = \bigcap_{k\le\ell} B^k([0,1];\ K^{\ell-k,\rho,\theta}_{a,\beta,T}\ ),\quad (\ [0,1]\ni\varepsilon\ ).$$

$$(2.9)\quad K^{\ell,\rho,\theta,(\mu)}_{a,\beta,T} = \bigcap_{j\le \ell/2} B^j_\beta([0,T];\ H^{\ell-2j,\rho,\theta,(\mu)}_a\ ),$$

$$|f|_{\ell,\rho,\theta,(\mu),\beta,T} = \sum_{j\le\ell/2}\ \sup_{0\le t\le T} |\partial^j_t f(t,\cdot)|_{\ell-2j,\rho-\beta t,\theta-\beta t,(\mu-\beta t)},$$

$$\mathcal{K}^{\ell,\rho,\theta,(\mu)}_{a,\beta,T} = \bigcap_{k\le\ell} B^k_\beta([0,1];\ K^{\ell-k,\rho,\theta,(\mu)}_{a,\beta,T}\ ),\quad (\ [0,1]\ni\varepsilon\ ).$$

$$(2.10)\quad \mathring{\mathcal{K}}^{\ell,\rho,\theta}_{a,\beta,T} = \bigcap_{k\le\ell/2} B^k([0,1];\ K^{\ell-2k,\rho,\theta}_{a,\beta,T}\ ),\quad (\ [0,1]\ni\nu\ ).$$

$(2.5)'$ $H^{\ell,\rho,\theta}_{a,q} \ni f \iff$ (1) $f(z',z_n)$ is analytic inside $\Omega(\rho,\theta,a)$,

(2) $\partial^\alpha f(z',z_n) \in L^q(L(\theta',a);H'^{0,\rho})$, $|\theta'|<\theta$, $|\alpha|\le\ell$,

(3) $|f|_{\ell,\rho,\theta,q} = \sum_{|\alpha|\le\ell}\ \sup_{|\theta'|<\theta} \| |\partial^\alpha f(\cdot,z_n)|_{0,\rho} \|_{L^q(\theta',a)} < \infty.$

$$(2.6)'\quad \tilde{H}^{\ell,\rho,\theta}_a = H^{\ell,\rho,\theta}_a \cap H^{\ell,\rho,\theta}_{a,1},$$

$$|f|^{\sim}_{\ell,\rho,\theta} = |f|_{\ell,\rho,\theta} + |f|_{\ell,\rho,\theta,1}.$$

$$(2.7)'\quad \tilde{K}^{\ell,\rho,\theta}_{a,\beta,T} = \bigcap_{j\le \ell/2} B^j_\beta([0,T];\ \tilde{H}^{\ell,\rho,\theta}_a\ ),$$

$$|f|^{\sim}_{\ell,\rho,\theta,\beta,T} = \sum_{j\le\ell/2}\ \sup_{0\le t\le T} |\partial^j_t f(t,\cdot)|^{\sim}_{\ell-2j,\rho-\beta t,\theta-\beta t},$$

$$\tilde{\mathcal{K}}^{\ell,\rho,\theta}_{a,\beta,T} = \bigcap_{k\le\ell} B^k([0,1];\ \tilde{K}^{\ell,\rho,\theta}_{a,\beta,T}\ ).$$

3. Operators and the Stokes equations

This section consists of three parts. First we introduce Poisson operator N or D (resp. $P_1(\nu)$ or $P_2(\nu)$) of the Neumann or Dirichlet Problem of the Laplace operator Δ (resp. the heat operator $\partial_t-\nu\Delta$) and other related operators.

Second we construct "evolution operators" solving the equation (1.5)-(1.9). In particular, we construct the "Poisson operator" $\mathcal{P}(\nu)$ of the Stokes equation, combining the operators defined above.

Finally we give the estimates for these operators in the function spaces introduced in 2, and the estimates of Cauchy-Kowalewski type for the firstr-order differential operators .

1. Various operators

(3.1) $rf = f|_{R^n_+}$ (the restiction of the function f of R^n onto R^n_+).

ef = a "nice" extension of the function f of R^n_+ to R^n. Here "nice" means the regularity preservjng property (cf. [5]).

$\bar{e}f(z',z_n) = f(z',z_n)$ for $x_n = \mathrm{Re}\, z_n \geq 0$, $= 0$ for $x_n < 0$.

$\gamma f = f|_{D(\rho)} = f|_{z_n=0}$ (cf.(1.1) (4)).

(3.2) $U_0(\nu,t,x) = (4\pi\nu t)^{-n/2}\, e^{-|x|^2/(4\pi\nu t)}$ (heat kernel in R^n),

$U_0(\nu,t)f(x) = \int_{R^n} U_0(\nu,t,x-\eta)f(\eta)d\eta$,

$U_0(\nu)f(t,\cdot) = \int_0^t U_0(\nu,t-s)f(s,\cdot)ds$,

$U_0'(\nu,t,x') = (4\pi\nu t)^{-(n-1)/2}\, e^{-|x'|^2/(4\pi\nu t)}$,

$\bar{U}_0(\nu,t,x) = U_0'(\nu,t,x')(4\pi t)^{-1/2}\, e^{-x_n^2/(4\pi t)}$.

$\bar{U}_0(\nu,t)$ and $\bar{U}_0(\nu)$ are defined similarly as $U_0(\nu,t)$ and $U_0(\nu)$.

Let $u(x')$ (resp. $u(t,x')$) be a function on R^{n-1}(resp.$[0.\infty)\times R^{n-1}$). We define the Fourier transform (resp. Fourier-Laplace transform) $\hat{u}(\xi')$ (resp. $\tilde{u}(\lambda,\xi')$) of $u(x')$ (resp. $u(t,x')$) by

(3.3) $$\hat{u}(\xi') = (2\pi)^{-(n-1)/2} \int e^{-ix'\cdot\xi'} u(x')dx' , \quad i = \sqrt{-1} ,$$

$$(\text{resp. } \tilde{u}(\lambda,\xi') = \int_0^\infty e^{-\lambda t}\, \hat{u}(t,\cdot)dt) .$$

We call the multiplier $\sigma(T)(\xi')$ in ξ' (resp. $\sigma(T)(\lambda,\xi')$ in (ξ',λ)) the symbol of the operator T , if there holds

$$(3.4) \qquad (Tu)^{\wedge}(\xi') = \sigma(T)(\xi')\hat{u}(\xi')$$

$$(\text{resp. } (Tu)^{\sim}(\lambda,\xi') = \sigma(T)(\lambda,\xi')\tilde{u}(\lambda,\xi')) .$$

Poisson operators N, D, $P_1(\nu)$ and $P_2(\nu)$ are defined by the following equations, respectively :

$$(3.5) \qquad \Delta Nu = 0 , \quad \gamma\partial_n Nu = g \text{ (given boundary data)},$$

$$\Delta Du = 0 , \quad \gamma u = g ,$$

$$(3.6) \qquad (\partial_t - \nu\Delta)P_1(\nu)u = 0 , \quad P_1(\nu)u|_{t=0} = 0 , \quad \gamma\partial_n P_1(\nu)u = g ,$$

$$(\partial_t - \nu\Delta)P_2(\nu)u = 0 , \quad P_2(\nu)u|_{t=0} = 0 , \quad \gamma P_2(\nu)u = g .$$

Clearly we have

$$(3.2)' \qquad \sigma(U_0'(\nu,t))(\xi') = e^{-\nu t|\xi'|^2},$$

$$(3.5)' \qquad \sigma(N)(\xi',x_n) = -1/|\xi'| \; e^{-|\xi'|x_n} ,$$

$$\sigma(D)(\xi',x_n) = e^{-|\xi'|x_n} ,$$

$$D = \partial_n N , \quad \partial_n D = \partial_n^2 N = -\Delta' N = (\Lambda'^2 N),$$

$$(3.6)' \qquad \sigma(P_1(\nu))(\lambda,\xi') = -(\lambda/\nu+|\xi'|^2)^{-1/2} \; e^{-(\lambda/\nu+|\xi'|^2)^{1/2}x_n} ,$$

$$\sigma(P_2(\nu))(\lambda,\xi') = e^{-(\lambda/\nu+|\xi'|^2)^{1/2}x_n} , \quad P_2(\nu) = \partial_n P_1(\nu),$$

For later use we define the Poisson operators $\bar{P}_j(\nu)$, $j=1,2$, by

$$(3.7) \qquad \sigma(\bar{P}_j(\nu))(\lambda,\xi') = \{-(\lambda+\nu|\xi'|^2)^{1/2}\}^{j-2} \; e^{-(\lambda+\nu|\xi'|^2)^{1/2}x_n} ,$$

which is associated with the heat operator $(\partial_t - \nu\Delta' - \partial_n^2)$.

We introduce two kinds of singular integral operators. The first group, Q^∞, P^∞, $N' = {}^t(N_1,\cdots,N_{n-1})$ and Λ' , are of Calderon-Zygmund type and act in the function spaces on R^n and R^{n-1}, respectively :

$$(3.8) \qquad \sigma(Q^\infty)(\xi) = \begin{pmatrix} \xi'\,{}^t\xi'/|\xi|^2 & \xi'\xi_n/|\xi|^2 \\ \xi_n\,{}^t\xi'/|\xi|^2 & \xi_n^2/|\xi|^2 \end{pmatrix} , \quad P^\infty = 1 - Q^\infty = \begin{pmatrix} P^{\infty'} \\ P_n^\infty \end{pmatrix} .$$

(Only here in (3.8), we adopt the Fourier transform in R^n.)

(3.9) $\sigma(N')(\xi') = (i\xi'/|\xi'|)$, $i = \sqrt{-1}$,

(3.10) $\sigma(\Lambda')(\xi') = |\xi'|$, $\Lambda_1' = \Lambda' + 1$.

(3.11) $\sigma(\omega(\nu))(\lambda,\xi') = |\xi'|/(\lambda/\nu + |\xi'|^2)^{1/2}$,

$\Omega(\nu) = N'\omega(\nu)\,{}^tN' = \omega(\nu)N'N'$, $\omega_1(\nu) = \Lambda'^{-1}\omega(\nu)$.

(3.12) $\sigma(\tau(\nu))(\lambda,\xi') = |\xi'|/\{(\lambda/\nu + |\xi'|^2)^{1/2} + |\xi'|\}$,

$\tau_1(\nu) = \omega(\nu) - \tau(\nu) = \omega(\nu)\tau(\nu)$.

By symbol calculus, we have equalities :

(3.13) $Q^\infty = \nabla\Delta^{-1}\nabla\cdot$, $\nabla\cdot Q^\infty = \nabla\cdot$, $\nabla\cdot P^\infty = 0$,

(3.14) $\sigma(\omega(\nu,t))(\xi') = \pi^{-1/2}\nu^{1/2}t^{-1/2}|\xi'|e^{-\nu t|\xi'|^2}$,

$\sigma(\omega^2(\nu,t))(\xi') = \nu|\xi'|^2 e^{-\nu t|\xi'|^2}$, $\omega^2 = \omega(\nu)\omega(\nu)$,

$\sigma(\tau_1(\nu))(\xi') = \pi^{-1}\nu|\xi'|^2\int_0^\infty e^{-\nu t|\xi'|^2(1+s)}s^{-1/2}(1+s)^{-1}ds$.

(3.15) $P_1(\nu) = -\omega_1(\nu)P_2(\nu)$, $\bar{P}_1(\nu) = -\nu^{1/2}\omega_1(\nu)\bar{P}_2(\nu)$,

$\gamma U_0(\nu.t) = -1/(2\nu)\{P_1(\nu,t)^* + P_1(\nu,t)^{*\vee}\}$,

where T^* means the adjoint of T, and $\check{f}(x',x_n) = f(x',-x_n)$ for $x_n > 0$.

For later use we define the modified operators $\bar{Q}^\infty$ and $\bar{P}^\infty$ by

(3.8)′ $\sigma(\bar{Q}^\infty)(\xi) = \begin{pmatrix} \varepsilon^2\xi'\,{}^t\xi' & \varepsilon\xi'\xi_n \\ \varepsilon\xi_n\,{}^t\xi' & \xi_n^2 \end{pmatrix}(\varepsilon^2|\xi'|^2+\xi_n^2)^{-1}$, $\bar{P}^\infty = 1-\bar{Q}^\infty$.

We note that the identity $1 = Q^\infty + P^\infty$ gives the Helmholtz decomposition in R^n. SImilarly the following operators Q and P give the same decomposition in R^n_+ (associated with the Euler equation):

(3.16) $P = rP^\infty e - \nabla N\gamma_n P^\infty e$, $Q = 1 - P$.

2. The "Stokes equations"

Define the "evolution operator" $V(\nu,t)$ (and $V(\nu)$) by

(3.17) $V(\nu,t) = rP^\infty U_0(\nu,t)e - \nabla N\gamma_n P^\infty U_0(\nu,t)e$ (or $= PrU_0(\nu,t)e$)

($V(\nu)f(t) = \int_0^t V(\nu,t-s)f(s,\cdot)ds$).

Then $V(\nu,t)$ satisfies

$$(3.18)\qquad (\partial_t - \nu\Delta)V(\nu,t) + \nabla N\gamma_n P^\infty \partial_t U_0(\nu,t)e = 0\ ,\quad t > 0,\ x \in R^n_+\ ,$$
$$\nabla\cdot V(\nu,t) = 0\ ,$$
$$V(\nu,0) = P\ ,$$
$$\gamma_n V(\nu,t) = 0\ .$$

The evolution operator $U_1(\nu)$ (resp. $U_2(\nu)$) of the Neumann (resp. Dirichlet) problem for the heat operator $\partial_t - \nu\Delta$ is given by

$$(3.19)\qquad U_1(\nu) = rU_0(\nu)\bar{e} - P_1(\nu)\gamma\partial_n U_0(\nu)\bar{e}$$

(resp. $U_2(\nu) = rU_0(\nu)\bar{e} - P_2(\nu)\gamma U_0(\nu)\bar{e} = rU_0(\nu)\bar{e} + P_1(\nu)\gamma\partial_n U_0(\nu)\bar{e}$).

These operators satisfy

$$(3.20)\qquad (\partial_t - \nu\Delta)U_i(\nu)f = f(t,x)\ ,\quad t > 0,\ x \in R^n_+\ ,$$
$$U_i(\nu)f|_{t=0} = 0\ ,$$
$$\gamma\partial_n^{2-i}U(\nu)f = 0\ ,\quad i = 1,2\ .$$

By replacing $U_0(\nu)$ and $P_i(\nu)$ with $\bar{U}_0(\nu)$ and $\bar{P}_i(\nu)$, we also define $\bar{U}_i(\nu)$, which is associated with the heat operator $\partial_t - \nu\Delta' - \partial_n^2$. Note

$$(3.21)\qquad \partial_n^{-1}\ \bar{U}_2(\nu) = \bar{U}_1(\nu)\partial_n^{-1} = \{r\bar{U}_1(\nu)\bar{e} - \bar{P}_1(\nu)\gamma\bar{U}_0(\nu)\bar{e}\}\partial_n^{-1}\ .$$

The Poisson operator $\mathcal{P}(\nu)$ of the original Stokes equation is defined by solving

$$(3.22)\qquad (\partial_t - \nu\Delta)w + \nabla p = 0\ ,\quad t > 0,\ x \in R^n_+\ ,$$
$$\nabla\cdot w = 0\ ,$$
$$w|_{t=0} = 0\ ,$$
$$\gamma w = g(t,x') = {}^t(g', g_n)\ .$$

Let $w = w^1 + w^2 + w^0$, and put

$$(3.23)\qquad w^1 = \nabla N f_0\ ,\quad f_0|_{t=0} = 0\ ,$$
$$w^2 = P(\nu)f = {}^t(\ P_2(\nu)f',\ P_1(\nu)f_n)\ ,\quad \nabla'\cdot f' + f_n = 0\ ,$$
$$w^0 = U(\nu)\nabla q = {}^t(\ U_2(\nu)\nabla' q\ ,\ U_1(\nu)\partial_n q)\ ,\quad \Delta q = 0\ .$$

Clearly each w^i satisfies the first three conditions of (3.22) and the following "boundary conditions"

(3.24) $\quad \gamma w^1 = {}^t(\, -N'f_0,\ f_0\,)$,

$\gamma w^2 = {}^t(\, f',\ \gamma P_1(\nu) f_n)$,

$\gamma w^0 = {}^t(\, 0',\ 1/\nu\ \gamma P_1(\nu)\gamma q + 2\gamma\partial_n U_0(\nu)\bar{e}q\,)$.

We determine q by the following condition

(3.25) $\quad \Delta q = 0 \ , \quad \gamma q = -\nu f_n \ , \quad$ i.e. $\quad q = -\nu D f_n = \nu D \nabla' \cdot f'$.

Then, w satisfies (3.22), if the following equation is satisfied ;

(3.26) $\quad f' - N' f_0 = g'$,

$f_0 + P_2(\nu)^* D\Lambda' N' \cdot f' = g_n$.

Here we have used the equalities (cf. (3.15) or (3.31)) :

$$2\nu\gamma\partial_n U_0(\nu)\bar{e} = P_2(\nu)^* \quad \text{and} \quad \nabla' = \Lambda' N' .$$

Hence we obtain

(3.26)' $\quad f' + N' P_2(\nu)^* D\Lambda' N' \cdot f' = g' + N'_n g \equiv Mg$.

By symbol culculus (and the identity : $N' \cdot N' = -1$), we have

(3.27) $\quad P_2(\nu)^* D\Lambda' = \tau(\nu), \quad \{1 - \tau(\nu)\}^{-1} = \omega(\nu)$,

(3.28) $\quad \{1 + N' P_2(\nu)^* D\Lambda' N' \cdot\}^{-1} = \{1 + N'\tau(\nu)\, {}^t N'\}^{-1} = 1 + \Omega(\nu)$.

Thus (3.26) is solved by

(3.29) $\quad f' = \{1 + \Omega(\nu)\} Mg$,

$f_0 = g_n - \tau(\nu) N' \cdot \{1 + \Omega(\nu)\} Mg = g_n - \{\tau(\nu) - \tau_1(\nu)\} N' \cdot Mg$.

Substituting (3.29) into (3.22) and rearranging the expression of w , we obtain

(3.30)
$$\begin{aligned} w = \mathcal{P}(\nu) g &= \mathcal{P}_1(\nu) g + \mathcal{P}_2(\nu) g + \mathcal{P}_0(\nu) g \\ &= \nabla N g_n - \nabla N \tau(\nu) N' \cdot \{1 + \Omega(\nu)\} Mg \\ &\quad + P_2(\nu) \begin{pmatrix} E' & - \tau_1(\nu) N'^t/2 \\ \omega(\nu)/2 & - \tau_1(\nu) N' \cdot /2 \end{pmatrix} \{1 + \Omega(\nu)\} Mg \\ &\quad + \nu \nabla r \nabla' \cdot U_0(\nu) \bar{e}\ D\{1 + \Omega(\nu)\} Mg \ . \end{aligned}$$

We note that the boundary layer arises only from $\mathcal{P}_2(\nu) g$.

3. Estimates

Fix $\rho_0 > 0$ and $0 < \theta_0 < \pi/4$. Then we have the following estimates which hold uniformly in ρ and θ with $0 \le \rho \le \rho_0$ and $0 \le \theta \le \theta_0$.

Lemma 3.1. Let $\nu \in (0,1]$, $\varepsilon = \sqrt{\nu}$, $\ell \ge 0$ and $a > 0$. Then : (*1*)

(3.31) $rU_0(\nu,t)e = O(1)$, $r\bar{U}_0(\nu,t)e = O(1)$,

$(\varepsilon\Lambda')^{\kappa} rU_0(\nu,t)e = O(t^{-\kappa/2})$, $(\varepsilon\Lambda')^{\kappa} r\bar{U}_0(\nu,t)e = O(t^{-\kappa/2})$,

$(\varepsilon\Lambda')^{\kappa}\partial_n r\bar{U}_0(\nu,t)e = O(t^{-1/2\ -\kappa/2})$, $\kappa \ge 0$,

uniformly in ε, ℓ and a in the function spaces $H_a^{\ell,\rho,\theta}$, $H_{a/\varepsilon}^{\ell,\rho,\theta,(\mu)}$ and $\tilde{H}_{a/\varepsilon}^{\ell,\rho,\theta}$. The same holds if the extension e is replaced by $\bar{e}$.

(*2*) There hold the following relations :

(3.32) $\bar{P}_1(\nu) = (\varepsilon\Lambda')^{-1}\ \bar{P}_2(\nu)\omega(\nu) = \bar{P}_1(\nu,t) *_t$ (convolution in t) ,

$2\gamma\bar{U}_0(\nu,t)\bar{e} = \bar{P}_1(\nu,t)^*$, $2\gamma\partial_n\bar{U}_0(\nu,t)\bar{e} = \bar{P}_2(\nu,t)^*$.

As operators from $H^{-\ell,\rho}$ to $H_{a/\varepsilon}^{\ell,\rho.\theta,(\mu)}$ (resp. $H_{a/\varepsilon,1}^{\ell,\rho,\theta}$) ,

(3.33) $(\varepsilon\Lambda')^{\kappa}\ \bar{P}_1(\nu,t) = O(t^{-1/2\ -\kappa/2})$ (resp. $O(t^{-\kappa/2})$) , $\kappa \ge 0$,

$\bar{P}_2(\nu,t) = O(t^{-1})$ (resp. $O(t^{-1/2})$) ,

uniformly in ε ,ℓ and a . From $H_a^{\ell,\rho,\theta}$ (resp. $H_{a,1}^{\ell,\rho,\theta}$) to $H^{-\ell,\rho}$,

(3.34) $P_i(\nu,t)^* = O(\varepsilon^{3-i}\ t^{-(i-1)/2})$ (resp. $O(\varepsilon^{2-i}\ t^{-i/2})$),

$\gamma\partial_n^{i-1}\bar{U}_0(\nu,t)$, $\bar{P}_i(\nu,t)^* = O(t^{-(i-1)/2})$ (resp. $O(t^{-i/2})$) .

(*3*) As operators acting in $H^{-\ell,\rho}$,

(3.35) $(\varepsilon\Lambda')^{-\kappa}\omega(\nu,t)$, $(\varepsilon\Lambda')^{-\kappa}\tau(\nu,t)$, $(\varepsilon\Lambda')^{-\kappa}\tau_1(\nu,t) = O(t^{-1+\kappa/2})$, $0 \le \kappa \le 1$,

$N' = O(1)$

uniformly in ε .

(*4*) As operators acting in $H_a^{\ell,\rho,\theta}$ and $H_{a/\varepsilon,1}^{\ell,\rho,\theta}$,

Q^{∞}, P^{∞}, $\bar{Q}^{\infty}$, $\bar{P}^{\infty} = O(1)$.

(*5*) As the operators acting from $H^{-\ell,\rho}$ to $H_a^{\ell,\rho,\theta}$,

$D = O(1)$, $\nabla N = D\ {}^t(-N',1) = O(1)$.

Lemma 3.2. Under the corresponding conditions in Lemma 3.1, we have :

(3.36) $\bar{U}_2(\nu) = r\bar{U}_0(\nu,t)\bar{e}*_t + \bar{u}_2(\nu,t)*_t,$

$(\varepsilon\Lambda')^{\kappa}\bar{u}_2(\nu,t) = O(t^{-\kappa/2}), \quad \partial_n\bar{u}_2(\nu,t) = O(t^{-1/2})$,

(3.37) $\mathcal{P}_1(\nu)\gamma = \nabla N\gamma_n + \nabla N\hbar_1(\nu,t)*_t(\varepsilon\Lambda')$, $(\varepsilon\Lambda')^{\kappa}\hbar_1(\nu,t) = O(t^{-(1+\kappa)/2}),$

$\mathcal{P}_0(\nu)\gamma = \varepsilon\nabla\hbar_0(\nu,t)*_t(\varepsilon\Lambda')$, $(\varepsilon\Lambda')^{\kappa}\hbar_0(\nu,t) = O(t^{-\kappa/2}),$

$\varepsilon\partial_n\mathcal{P}_0(\nu)\gamma = -\mathcal{P}_0(\nu)\gamma\varepsilon\Lambda' - \nabla P_1(\nu)\varepsilon\nabla'\cdot(1+\Omega(\nu))M\gamma$,

$\bar{\mathcal{P}}_2(\nu)\gamma = \bar{P}_2(\nu)\begin{pmatrix} E' & 0 \\ 0 & 0 \end{pmatrix}M\gamma + \bar{\hbar}_2(\nu,t)*_t(\varepsilon\Lambda'),$

$(\varepsilon\Lambda')^{\kappa}\,\bar{\hbar}_2(\nu,t) = O(t^{-(1+\kappa)/2})$.

Here $*_t$ means convolution in t (on [0,t]), and $\bar{\mathcal{P}}_2(\nu)$ is defined by replacing $P_2(\nu)$ with $\bar{P}_2(\nu)$ in the definition (3.29) of $\mathcal{P}_2(\nu)$.

Lemma 3.3. Let $f(z') \in H'^{\ell,\rho}$. THen,

(3.38) $|\partial_j f|_{\ell,\rho'} \le |f|_{\ell,\rho}/(\rho-\rho'), \quad \rho > \rho' \ge 0, \ 1 \le j \le n-1.$

Lemma 3.4. Let $f(z',z_n) \in H_a^{\ell,\rho,\theta}$, and put $\chi(z_n) = \min\{1, |z_n|\}$. THen there exists $C(\theta_0) > 0$, independent of a , such that

(3.39) $|\chi(z_n)\partial_n f|_{\ell,\rho,\theta'} \le C(\theta_0)|f|_{\ell,\rho,\theta}/(\theta-\theta')$, $\theta > \theta' \ge 0,$

$|\chi(z_n)\partial_n f|_{\ell,\rho,\theta',(\mu)} \le C(\theta_0)|f|_{\ell,\rho,\theta,(\mu)}/(\theta-\theta') + \mu|f|_{\ell,\rho,\theta',(\mu)}.$

$|\frac{1}{\varepsilon}\chi(\varepsilon z_n)\partial_n f|_{\ell,\rho,\theta',(\mu')} \le C(\theta_0)|f|_{\ell,\rho,\theta,(\mu)}\{1/(\theta-\theta')+1/(\mu-\mu')\}.$

Remark. In what follows, we put $U_0(0,t) = 1$ and $U_0(0) = \delta(t)*_t$. Then, $rU_0(\nu,t)e$ (resp. $r\bar{U}_0(\nu)e$) is strongly continuous in $(\nu,t) \in [0,1]\times[0,\infty)$ in $H_a^{\ell,\rho,\theta}$ (resp. in ν in $K_a^{\ell,\rho,\theta,(\mu)}$ and $\tilde{K}_a^{\ell,\rho,\theta}$).

4. Abstract Cauchy-Kowalewski theorem

We give a survey on the abstract Cauchy-Kowalewsky theorem ([2]). Let $\tilde{X}_\rho = \{X_\rho;\ 0 \le \rho \le \rho_0\}$ be a Banach scale with the norm $|\ |_\rho$

of X_ρ, i.e. $X_{\rho'} \supset X_\rho$ and $|\ |_{\rho'} \le |\ |_\rho$ for $0 \le \rho' \le \rho \le \rho_0$. Define $X_{\rho_0,\beta,T}$ and $Y_{\rho_0,\beta,T}$ by

(4.1) $X_{\rho_0,\beta,T} = B^0_\beta([0,T];\ X_\rho) \ni f(t) \iff$

(1) $f(t) \in B^0([0,T'];\ X_\rho)$ for $\rho \le \rho_0-\beta T'$, $T' \le T$,

(2) $|f|_{\rho_0,\beta} = \sup_{0\le t\le T} |f(t)|_{\rho_0-\beta t} < \infty$,

(4.2) $Y_{\rho_0,\beta,T} \ni f(t) \iff (1)\ f(t) \in B^0_\beta([0,T'];\ X_\rho)$ for $T' < T$,

(2) $\|f\|_\beta = \sup_{0\le\rho\le\rho_0-\beta t} |f(t)|_\rho \varphi(\beta t/(\rho_0-\rho)) < \infty$, $\varphi(t)=(1-t)e^{-t}$.

We also define

(4.3) $X_{\rho,\beta,T}(R) = \{f(t) \in X_{\rho,\beta,T}\ ;\ |f|_{\rho,\beta} \le R\}$,

$\tilde{X}_{\rho,\beta,T} = B^0([0,1];\ X_{\rho,\beta,T})$, $(\ [0,1] \ni \varepsilon\ )$,

$\tilde{X}_{\rho,\beta,T}(R)$: similarly as $X_{\rho,\beta,T}(R)$.

Let $F(\varepsilon,t,u(\cdot))$ be a mapping from $[0,1]\times[0,\tau]\times X_{\rho,\beta_0,\tau}(R)$ into $\tilde{X}_{\rho',\beta_0,\tau}$ for $0 \le \rho' < \rho \le \rho_0-\beta_0\tau$ and $0 < \tau \le T_0$, and satisfy

(F.1) $|F(\varepsilon,t,u(\cdot))-F(\varepsilon,t,v(\cdot))|_{\rho'} \le \int_0^t C|u(s)-v(s)|_{\rho(s)}/(\rho(s)-\rho')ds$

for each $u,v \in X_{\rho,\beta,\tau}$, $0 \le \rho' < \rho(s) \le \rho-\beta s$, $\beta \ge \beta_0$, $0 \le t \le T_0$,

(F.2) $|F(\varepsilon,t,0)|_{\rho_0-\beta t} \le R_0 < R$, $\varepsilon \in [0,1]$, $0 \le t \le \tau \le T_0$.

Here C, R and R_0 are constants independent of ε .

Consider the following equation

(4.4) $u(t) = F(\varepsilon,t,u(\cdot))$, $0 \le t \le T\ (\le T_0)$.

Then, we have

Theorem ACK. (Abstract Cauchy-Kowalewski theorem). Assume (F.1) and (F.2). Then, there exist $\beta > \beta_0$ and $T \le T_0$, $0 < T \le \rho_0/\beta$, such that the equation (4.4) has a unique solution $u(\varepsilon,t) \in \tilde{X}_{\rho_0,\beta,T}(R)$. We can choose β such as

(4.5) $\beta = \max\ \{\ 4\beta_0/3\ ,\ 8Ce\ ,\ 16Ce^2R_0/(R-R_0)\ \}$.

Sketch of Proof. First we note

(4.6) $$\|u\|_\beta \le |u|_{\rho_0,\beta} \le (1 - \beta'/\beta)e\,\|u\|_{\beta'}\ ,\quad \beta > \beta' \ge 0\ .$$

By virtue of (F.1), we have

(4.7) $$\|F(\varepsilon,t,u) - F(\varepsilon,t,v)\|_\beta \le (2Ce/\beta)\|u-v\|_\beta\ ,\quad \beta \ge \beta_0\ ,$$

for eavh $u,v \in X_{\rho_0,\beta,T}(R)$.

Choose β satisfying (4.5), and put

(4.8) $$u_0(t) = F(\varepsilon,t,0)\ ,$$

$$u_{n+1}(t) = F(\varepsilon,t,u_n(\cdot))\ ,\quad n \ge 0\ ,$$

$$\beta_n = \beta(1 - 2^{-n-1}),\quad \text{i.e. } \beta - \beta_n = \beta 2^{-n-1}.$$

Then, $u_{n+1} \in \tilde{X}_{\rho_0,\beta_{n+1},T}$, if $u_n \in \tilde{X}_{\rho_0,\beta_n,T}(R)$ and $T \le \rho_0/\beta$. (4.7) and (4.8) imply

(4.9) $$\|u_{n+1} - u_n\|_{\beta_n} \le (2Ce)/\beta_n\ \|u_n - u_{n-1}\|_{\beta_n},$$

(4.10) $$|u_{n+1} - u_n|_{\rho_0,\beta} \le (1 - \beta_n/\beta)e\|u_{n+1} - u_n\|_{\beta_n} = 2^{n+1}e\|u_{n+1} - u_n\|_{\beta_n}$$

$$= 2^{n+1}e(2Ce/\beta_n)^n\ \|u_1 - u_0\|_{\beta_n}$$

$$\le 8/3\ e(4Ce/\beta)^n\ \|u_1 - u_0\|_{\beta_1}.$$

On the other hand (F.2) implies

$$\|u_0\|_{\beta_1} \le |u_0|_{\rho_0,\beta_0} \le R_0\ ,\quad \|u_1 - u_0\|_{\beta_1} \le (2Ce/\beta_1)\|u_0\|_{\beta_1}.$$

Hence, because of the choice of β, we have

(4.11) $$|u_{n+1} - u_n|_{\rho_0,\beta} \le 16/9\ e(4Ce/\beta)^{n+1}R_0\ ,$$

$$|u_{n+1}|_{\rho_0,\beta} \le \{1 + 4e(4Ce/\beta)\}R_0 \le R\ .$$

This shows that $\{u_n\}$ converges in $\tilde{X}_{\rho_0,\beta,T}(R)$ and the limit $u(\varepsilon,t)$ satisfies (4.4), if $T \le \min\{T_0,\rho_0/\beta\}$. Uniqueness is easily proved.

5. The first approximation $u^0(\nu,t,x)$

We solve the equation (1.5) by using the "evolution operator" $V(\nu,t)$ defined by (3.17). We consider the equation

(5.1) $$u^0 = V(\nu,t)u_0 - V(\nu)u^0\cdot\nabla u^0 \equiv F(\nu,t,u^0) .$$

The solution u^0 of (5.1) is clearly a solution of (1.5).

We note that $V(\nu,t)$ (resp. $V(\nu)$) is strongly continuous in (ν,t) in $H_a^{\ell,\rho,\theta}$ (resp. in ν in $K_a^{\ell,\rho,\theta}$), by virtue of Remark in 3 .

Fix ρ_0 and θ_0 so that $\rho_0 > 0$ and $0 < \theta_0 < \pi/4$, and put

(5.2) $$G(u) = u\cdot\nabla u , \quad u \in \mathring{H}_a^{\ell,\rho,\theta} = \{ u \in H_a^{\ell,\rho,\theta};\ \gamma_n u = 0 \},$$
$$\ell \geq (n-1)/2 +1 , \ 0 < \rho \leq \rho_0 , \ 0 < \theta \leq \theta_0 , \ a > 0 .$$

By virtue of Sobolev embedding theorem, Lemma3.3 and 3.4, we obtain the uniform estimates (in ρ, θ and $\nu \in (0,1]$):

(5.3) $$|G(u)|_{\ell,\rho',\theta'} \leq C|u|_{\ell,\rho',\theta'}\{|u|_{\ell,\rho,\theta'}/(\rho-\rho')+|u|_{\ell,\rho',\theta}/(\theta-\theta')\},$$
$$|G(u)-G(v)|_{\ell,\rho',\theta'} \leq C(|u|_{\ell,\rho',\theta'}+|v|_{\ell,\rho',\theta'})\times$$
$$\times\{|u-v|_{\ell,\rho,\theta'}/(\rho-\rho') + |u-v|_{\ell,\rho',\theta}/(\theta-\theta')\},$$
$$u,v \in \mathring{H}_a^{\ell,\rho,\theta}, \quad 0 \leq \rho' < \rho \leq \rho_0,\ 0 \leq \theta' < \theta \leq \theta_0,\ 0 \leq \nu \leq 1.$$

The constant C is independent of ρ', ρ, θ', θ, $a > 0$ and ν . Thus the mapping $F(\nu,t,u(\cdot)) = V(\nu,t)u_0 - V(\nu)G(u)$ appearing in (5.1) satisfies the conditions (F.1) and (F.2) in $\mathring{H}_a^{\ell,\rho,\theta}$ with arbitrary $T_0 > 0$. Hence, applying Theorem ACK , we have

Theorem 5.1. Let $\ell \geq (n-1)/2 +3$, $0 < \rho \leq \rho_0$, $0 < \theta \leq \theta_0 < \pi/4$ and $a > 0$. Assume $u_0 \in H_a^{\ell,\rho,\theta}$ and the condition (1.3). Then, there exist $T > 0$ and $\beta_0 > 0$ such that (5.1) has a unique solution $u^0(\nu,t)$ $\in \mathring{\mathcal{X}}^{\ell,\rho,\theta}_{a,\beta_0,T}$, which is defined from $\mathring{H}_a^{\ell,\rho,\theta}$ in the same way as in 2 .

6. The first boundary layer $\tilde{u}^0(\varepsilon,t,x',x_n/\varepsilon)$

Let $\varepsilon = \sqrt{\nu} \in [0,1]$. In order to solve (1.6), we change variables as follows :

(6.1)
$$x_n \to \varepsilon x_n\ ,\ \partial_n \to \partial_n/\varepsilon\ ,$$
$$\tilde{u}^{0'}(\varepsilon,t,x',x_n/\varepsilon) \to \tilde{u}^{0'}(\varepsilon,t,x',x_n),\quad \tilde{u}^0_n(\cdot\cdot,x_n/\varepsilon) \to \tilde{u}^0_n(\cdot\cdot,x_n)/\varepsilon\ ,$$
$$u^0(\nu,t,x) \to \begin{pmatrix} u^{0'}(\nu,t,x',\varepsilon x_n) \\ u^0_n(\nu,t,x',\varepsilon x_n)/\varepsilon \end{pmatrix} \equiv \bar{u}^0(\varepsilon,t,x',x_n)\ .$$

Then, the equation (1.6) is rewriten as

(6.2)
$$\partial_t\tilde{u}^{0'} + (\bar{u}^0+\tilde{u}^0)'\cdot\nabla'\tilde{u}^{0'} + (\bar{u}^0_n+\tilde{u}^0_n-\gamma\tilde{u}^0_n)\partial_n\tilde{u}^{0'} - \nu\Delta'\tilde{u}^{0'} - \partial_n^2\tilde{u}^{0'} + \tilde{u}^0\cdot\nabla\bar{u}^{0'} = 0,$$
$$\tilde{u}^0_n(\varepsilon,t,x) = -\partial_n^{-1}\nabla'\cdot\tilde{u}^{0'}(\varepsilon,t,x) \equiv \int_{x_n}^{\infty}\nabla'\cdot\tilde{u}^{0'}(\varepsilon,t,x',\eta_n)d\eta_n\ ,$$
$$\tilde{u}^0|_{t=0} = 0\ ,$$
$$\gamma'\tilde{u}^0 \equiv \tilde{u}^{0'}|_{x_n=0} = -\gamma'\bar{u}^0 = -\gamma'u^0\ .$$

We put

(6.3)
$$\tilde{u}^{0'} = \bar{U}_2(\nu)\tilde{v}^{0'} - \bar{P}_2(\nu)\gamma'u^0\ ,$$
$$\tilde{u}^0_n = -\partial_n^{-1}\nabla'\cdot\tilde{u}^{0'} \equiv -\partial_n^{-1}\bar{U}_2(\nu)\nabla'\cdot\tilde{v}^{0'} + \bar{P}_1(\nu)\gamma\nabla'\cdot u^{0'}$$
$$= -\{r\bar{U}_0(\nu)\bar{e} - \bar{P}_1(\nu)\gamma\bar{U}_0(\nu)\bar{e}\}\partial_n^{-1}\nabla'\cdot\tilde{v}^{0'} + \bar{P}_1(\nu)\gamma u^{0'}\quad \text{(See (3.21))}\ .$$

Then, the last three conditions of (6.2) are automatically satisfied.

Substituting (6.3) into (6.2), we have an equation for $\tilde{v}^{0'}$:

(6.4)
$$\tilde{v}^{0'} + (\bar{u}^{0'}+\tilde{u}^{0'})\cdot\underline{\nabla'\bar{U}_2(\nu)\tilde{v}^{0'}} + (\bar{u}^0_n + \underline{\tilde{u}^0_n - \gamma\tilde{u}^0_n})\partial_n\bar{U}_2(\nu)\tilde{v}^{0'}$$
$$+ \{\tilde{u}^{0'}\cdot\nabla' + (\underline{\tilde{u}^0_n - \gamma\tilde{u}^0_n})\partial_n\}\bar{u}^{0'}$$
$$- \{(\bar{u}^{0'}+\tilde{u}^{0'})\cdot\nabla' + (\bar{u}^0_n + \underline{\tilde{u}^0_n - \gamma\tilde{u}^0_n})\partial_n\}\bar{P}_2(\nu)\gamma'u^0 = 0\ .$$

Note (See (3.15).)

(6.5)
$$\tilde{u}^0_n - \gamma\tilde{u}^0_n = -\int_0^{x_n}\nabla'\cdot\tilde{u}^{0'}(\varepsilon,t,x',\xi_n)d\xi_n\ ,$$

(6.6)
$$\bar{P}_2(\nu)\gamma'u^0 = \bar{P}_2(\nu)\gamma'\{V(\nu,t)u_0 - V(\nu)u^0\cdot\nabla u^0\}\ ,$$
$$\bar{P}_2(\nu)\gamma'V(\nu,t) = \bar{P}_2(\nu)\gamma U_0(\nu,t)\{P^{\infty\prime} + N'P^{\infty}_n\}e$$
$$= -\bar{P}_1(\nu)\nu^{-1/2}\{P_2(\nu,t)^* + P_2(\nu,t)^{*V}\}\{P^{\infty\prime} + N'P^{\infty}_n\}e\ .$$

From (3.33) and (3.34), it follows $\bar{P}_1(\nu)\nu^{-1/2}P_2(\nu,t) = O(1)$, and $\bar{P}_2(\nu)\gamma' u^0 \in \mathcal{K}_\infty^{\ell,\rho,\theta}$. Hence, only the underlined terms contain the first derivatives of $\tilde{v}^{0\prime}$ in linear order, and other terms are continuous in $\tilde{v}^{0\prime}$ in $K_{a/\varepsilon}^{\ell-1,\rho,\theta}$. Thus, by virtue of Lemma 3.3 and 3.4, we can apply Theorem ACK inorder to solve (6.4) . Since (6.4) has a unique solution $\tilde{v}^{0\prime} \in \mathcal{K}_{a/\varepsilon,\beta_1,T_1}^{\ell-1,\rho,\theta,(\mu)}$ with some $\beta_1 > \beta_0$ and $0 < T_1 \le T$, we have

Theorem 6.1. Under the assumptions of Theorem 5.1, the "Navier-Stokes equation" (6.2) has a solution $\tilde{u}^0(\varepsilon,t,x',x_n) \in \mathcal{K}_{a/\varepsilon.\beta_1,T_1}^{\ell-1,\rho,\theta,(\mu)}$, where $\beta_0 < \beta_1$ and $0 < T_1 \le T$.

7. The second approximation $u^1(\varepsilon,t,x)$

We solve the equation (1.7) in three steps. (*1*) First we solve

(7.1) $$\partial_t u + u^0\cdot\nabla u - \nu\Delta u + \nabla p = 0 , \quad t > 0 , \ x \in R_+^n ,$$
$$\nabla\cdot u = 0 ,$$
$$u|_{t=0} = v_0 ,$$
$$\gamma_n u = 0 .$$

We write the solution u of (7.1) as $u = V(\nu,t;u^0)v_0$, which is the definition of the "evolution operator" $V(\nu,t;u^0)$. Since (7.1) is linear, it is easy to solve it in a framework of Theorem ACK . However, we sketch briefly how to construct $V(\nu,t;u^0)$ in order to get better estimates (cf. [1]).

First we consider the transport equation

(7.2) $$\partial_t v + w\cdot\nabla v = 0 \quad (w = eu^0) , \ t > s , \ x \in R^n , \quad v|_{t=s} = v_0 .$$

We assume $w(s,x)$ and $v_0(x) \in H_a^{k,\rho-\beta s,\theta-\beta s}(R^n)$ $(k \ge \ell-1)$, which is defined in a similar way as in (2.5) with $\Omega(\rho,\theta,a)$ replaced by

$$\Omega(\rho,\theta,a)\cup\check{\Omega}(\rho,\theta,a/2) , \quad \check{\Omega}(\rho,\theta,a) = \{(x',x_n);(x',-x_n)\in\Omega(\rho,\theta,a)\} .$$

If $w = w(x)$ does not depend on t, $A = -w\cdot\nabla$ generates a (time-local) semigroup e^{tA} in $K^{k,\rho,\theta}_{a,\beta,T}(R^n)$, since $e^{tA} = 1 + tA + (tA)^2/2! + \cdots$ converges on $[0,T]$ strongly in $K^{k,\rho,\theta}_{a,\beta,T}$. Because, with the terminologies of 4 , the estimate : $|Af|_{\rho'} \le C|f|_\rho/(\rho-\rho')$, implies

$$|(tA)^j f|_{\rho-\beta t} \le t^j C^j |f|_\rho (\beta t/j)^{-j} \le (C/\beta)^j j^j |f|_\rho \ .$$

Hence, Stirling's formula : $j! = (2\pi)^{-1/2}\ e^{-j} j^{j+1/2}\{1 + o(1)\}$, gives our conclusion with $\beta \ge 2Ce$. If $w = w(\varepsilon,t,x) \in \mathcal{K}^{k,\rho,\theta}_{a,\beta_0,T}$, Cauchy's connected segment method can be applied to prove that $-w\cdot\nabla$ generates evolution operator $T(t,s;w)$ such that $v = T(t,s;w)v_0$ is the unique solution of (7.2) . Second we put

$$(7.3)\qquad V_0(\nu,t,s;w) = P^\infty U_0(\nu,t-s)T(t,s;w)\ ,$$

$$V_1(\nu,t,s;w) = V_0(\nu,t,s;w) + \int_s^t V_0(\nu,t,r;w)R_1(\nu,r,s;w)dr\ .$$

If we choose $R_1(t,s) \equiv R_1(\nu,t,s;w)$ satisfying

$$(7.4)\qquad R_1(t,s) - \int_s^t R_0(t,r)R_1(r,s)dr = R_0(t,s)\ ,$$

$$R_0(t,s) \equiv R_0(\nu,t,s;w) \equiv [P^\infty U_0(\nu,t-s),\ w(\varepsilon,t,\cdot)\cdot\nabla]\ ,$$

then, we obtain the evolution operator of the linear N-S equation :

$$(7.5)\qquad \partial_t V_1 + w\cdot\nabla V_1 - \nu\Delta V_1 + Q^\infty R_1 = 0\ ,\ t > s\ ,$$

$$\nabla\cdot V_1 = 0\ ,$$

$$V_1|_{t=s} = P^\infty\ .$$

Since $R_0(\nu,t,s;w) = O(1)$ in $\mathcal{K}^{k,\rho,\theta}_{a,\beta,T}\ (R^n)$, $\beta \ge \beta_0$, the Volterra equation (7.4) is solved by succesive approximation .

Next we put

$$(7.6)\qquad V(\nu,t,s;u^0) = rV_1(\nu,t,s;eu^0)e - \nabla N\gamma_n V_1(\nu,t,s;eu^0)$$

$$+ \int_s^t \{rV_1(\nu,t,r;u^0)e - \nabla N\gamma_n V_1(\nu,t,r;eu^0)\}S(\nu,r,s;u^0)dr\ ,$$

$$(7.7)\qquad S(t,s) - \int_s^t S_1(t,r)S(r,s)ds = S_1(t,s)\ ,$$

$$S_1(t,s) \equiv S_1(\nu,t,s;u^0) \equiv \{[u^{0\prime}(t)\cdot\nabla',\nabla]N + u^0_n(t)\partial_n\nabla N\}\gamma_n V_1(t,s)\ .$$

In the same way as above, (7.7) has a unique solution $S(\nu,t,s;u^0)$, which is O(1) in $\mathcal{X}^{k,\rho,\theta}_{a,\beta,T}$. Thus, we have the evolution operator $V(\nu,t,s;u^0)$ of (7.1) .

(ℓ) The Poisson operator $\mathcal{Q} = \mathcal{Q}(\nu;u^0)$ of the equation (7.1) is given by solving

$$(7.8)\qquad \begin{aligned} &\partial_t v + u^0\cdot\nabla v - \nu\Delta v + \nabla p = 0 \ , \ t > 0 \ , \ x \in R^n_+ \ , \\ &\nabla\cdot v = 0 \ , \\ &v|_{t=0} = 0 \ , \\ &\gamma_n v = g \quad (g|_{t=0} = 0) \ . \end{aligned}$$

We put

$$(7.9)\qquad v(t) = \nabla Ng + \int_0^t V(\nu,t,s;u^0)f(s,\cdot)ds \ .$$

then, v(t) satisfies the last three conditions of (7.9) . The first equation will be satisfied, if f is determined by

$$(7.10)\qquad f(t) + [u^0(t)\cdot\nabla,\nabla]Ng = 0 \ , \quad [u^0\cdot\nabla,\nabla]N = O(1) \ .$$

$\mathcal{Q}(\nu;u^0)$ is defined by $v(t) = \mathcal{Q}(\nu;u^0)g$, which is O(1) from $\mathcal{X}^{-k,\rho}_{\beta,T}$ to $\mathcal{X}^{k,\rho,\theta}_{a,\beta,T}$, if $k \le \ell$ and $\beta \ge \beta_0$.

($\mathcal{S}$) The solution u^1 of the equation (1.7) is described as

$$(7.11)\qquad u^1(t) = -\int_0^t V(\nu,t,s;u^0)u^1(s)\cdot\nabla u^0(s)ds - \mathcal{Q}(\nu;u^0)\gamma_n\tilde{u}^{0\prime} .$$

Since thjs is a linear Voltrra equation in $\mathcal{X}^{\ell-1,\rho,\theta}_{a,\beta_1,T_1}$, we have a unique solution $u^1(\varepsilon,t,x)$ in the same space . thus, we have

Theorem 7.1. Under the assumptions of Theorem 5.1 the "N-S equation" (1.7) has a solution $u^1(\varepsilon,t,x) \in \mathcal{X}^{\ell-1,\rho,\theta}_{a,\beta_1,T_1} \cap \mathcal{X}^{\ell-1,\rho,\theta}_{a,2,\beta_1,T_1}$.

8. The complemntary terms

We are at the final stage, though we can continue our procedure which was used to get $\tilde{u}^0$ and u^1. It provides an asymptotic solution of (1.1). In order to solve (1.8) and (1.9), we put

(8.1) (1) $u^2 = rP^\infty U_0(\nu)ev^2 - \{\mathcal{P}_1(\nu)+\mathcal{P}_0(\nu)\}\gamma P^\infty U_0(\nu)e(v^2+\tilde{v}^1/\varepsilon) + \{\mathcal{P}_1(\nu)+\mathcal{P}_0(\nu)\}g/\varepsilon$,

(2) $\tilde{u}^1+\varepsilon\tilde{u}^2 = rP^\infty U_0(\nu)e(\tilde{v}^1+\varepsilon\Lambda_1'\tilde{v}^2) - \mathcal{P}_2(\nu)\gamma P^\infty U_0(\nu)e(\varepsilon v^2+\tilde{v}^1) - \mathcal{P}(\nu)\gamma P^\infty U_0(\nu)e\varepsilon\Lambda_1'\tilde{v}^2 + \mathcal{P}_2(\nu)g$,

(3) $g = -{}^t(\gamma' u^1, 0) \in \mathcal{X}^{\prime\ell-1,\rho}_{\beta_1,T_1}$, (and $\varepsilon g \in \mathcal{X}^{\prime\ell,\rho,\theta}_{\beta_1,T_1}$) .

Then, all conditions of (1.8)-(1.9) are satisfied except for the first two equations . Later we will show that we may assume

(8.2)
$$u^2(\varepsilon,t,x) = {}^t(u^{2\prime}(\varepsilon,t,x),\ u^2_n(\varepsilon,t,x)) ,$$
$$\tilde{u}^1(\varepsilon,t,x',x_n/\varepsilon) = {}^t(\tilde{u}^{1\prime}(\varepsilon,t,x',x_n/\varepsilon),\ \varepsilon\tilde{u}^1_n(\varepsilon,t,x',x_n/\varepsilon)) ,$$
$$\tilde{u}^2(\varepsilon,t,x',x_n/\varepsilon) = {}^t(\tilde{u}^{2\prime}(\varepsilon,t,x',x_n/\varepsilon),\ \tilde{u}^2_n(\varepsilon,t,x',x_n/\varepsilon)) .$$

We make the change of the variables :

(8.3)
$$(x',x_n) \longrightarrow (x',\varepsilon x_n) , \quad \nabla = {}^t(\nabla',\partial_n) \longrightarrow \bar{\nabla} = {}^t(\nabla',\partial_n/\varepsilon) ,$$
$$Q^\infty \longrightarrow \bar{Q}^\infty = \bar{\nabla}(\nabla'^2+\partial_n^2/\varepsilon^2)^{-1}\bar{\nabla}\cdot , \quad P^\infty \longrightarrow \bar{P}^\infty = 1-\bar{Q}^\infty ,$$
$$u^i(x',x_n) \longrightarrow u^i(x',\varepsilon x_n) \equiv \bar{u}^i(x',x_n) ,$$
$$\tilde{u}^i(x',x_n/\varepsilon) \longrightarrow \tilde{u}^i(x',x_n) \equiv \tilde{u}^i .$$

Then, we have (See (3.8)')

(8.4)
$$\bar{Q}^\infty = \begin{pmatrix} \bar{Q}^{\infty\prime\prime} & \bar{R}^\infty \\ {}^t\bar{R}^\infty & \bar{S}^\infty \end{pmatrix} , \quad \bar{P}^\infty = \begin{pmatrix} \bar{P}^{\infty\prime\prime} & -\bar{R}^\infty \\ -{}^t\bar{R}^\infty & \bar{T}^\infty \end{pmatrix} = \begin{pmatrix} E' & 0 \\ 0 & 0 \end{pmatrix} + \bar{W}^\infty ,$$
$$\sigma(\bar{R}^\infty)(\xi) = (\ \varepsilon\xi'\xi_n/(\varepsilon^2|\xi'|^2+\xi_n^2)\ ) = \varepsilon\sigma(\bar{S}^\infty)(\xi)\xi'/\xi_n ,$$
$$\sigma(\bar{S}^\infty)(\xi) = \xi_n^2/(\varepsilon^2|\xi'|^2+\xi_n^2) ,$$
$$\sigma(\bar{T}^\infty)(\xi) = \varepsilon^2|\xi'|^2/(\varepsilon^2|\xi'|^2+\xi_n^2) = \varepsilon\sigma({}^tN'\bar{R}^\infty)(\xi)|\xi'|/\xi_n^{-1} ,$$
$$\sigma(\bar{Q}^{\infty\prime\prime})(\xi) = \varepsilon\sigma(\bar{R}^{\infty\,t}N')(\xi)|\xi'|/\xi_n^{-1} ,$$
i.e. $\bar{R}^\infty = \varepsilon\Lambda'\bar{S}^\infty N'\partial_n^{-1}$, $\bar{T}^\infty = \varepsilon\Lambda' N'\cdot\bar{R}^\infty\partial_n^{-1}$, $\bar{Q}^{\infty\prime\prime} = \varepsilon\Lambda'\bar{R}^{\infty\,t}N'\partial_n^{-1}$,
$\bar{R}^\infty,\ \bar{S}^\infty,\ \bar{T}^\infty,\ \bar{W}^\infty (= {}^t(\bar{W}^{\infty\prime},\ \bar{W}^\infty_n) = \varepsilon\Lambda'\bar{w}^\infty\partial_n^{-1}) = O(1)$, $\bar{w}^\infty = O(1)$.

Substitute (8.1) into (1.8), and drop the boundary layers and potential parts. Then, we obtain an equation for v^2 :

$$(8.5)\quad v^2(t)+\underline{\{u^0+\varepsilon u^1+\varepsilon^2u^2\}\cdot\nabla rP^\infty U_0(\nu)ev^2}$$
$$-[(u^0+\varepsilon u^1)\cdot\nabla,\nabla]\{N\gamma_n+N\hbar_1(\nu)\varepsilon\Lambda'+\varepsilon\hbar_0(\nu)\varepsilon\Lambda'\}P^\infty U_0(\nu)ev^2$$
$$-[(u^0+\varepsilon u^1)\cdot\nabla,\nabla]N\gamma_n\bar U_0(\nu)\bar w^\infty e\underline{\Lambda'\partial_n^{-1}\tilde v^1}$$
$$-[(u^0+\varepsilon u^1)\cdot\nabla,\nabla]\{N\hbar_1(\nu)+\varepsilon\hbar_0(\nu)\}\bar U_0(\nu)\underline{\{e\Lambda'^t(\tilde v^1,0)+\bar w^\infty e\Lambda'\partial_n^{-1}\tilde v^1\}}$$
$$-\varepsilon u^2\underline{\cdot\nabla\ \nabla\{N\gamma_n+N\hbar_1(\nu)\varepsilon\Lambda'+\varepsilon\hbar_0(\nu)\varepsilon\Lambda'\}\bar P^\infty\bar U_0(\nu)e(\varepsilon\bar v^2+\tilde v^1)}$$
$$-[(u^0+\varepsilon u^1)\cdot\nabla,\nabla]\{N\hbar_1(\nu)+\varepsilon\hbar_0(\nu)\}\Lambda' g\ -\varepsilon u^2\cdot\nabla\{\nabla N\hbar_1(\nu)+\varepsilon\nabla\hbar_0(\nu)\}\varepsilon\Lambda' g$$
$$+\ u^2\cdot\nabla(u^0+\varepsilon u^1)\ =\ f^1\ \equiv\ -u^1\cdot\nabla u^1\ ,$$

where only the underlined terms contain the first derivatives of v^2 and $\tilde v^1$ in linear order. Note

$$(8.6)\quad g'=-\gamma' u^1=-\gamma' V(\nu,t,s;u^0)*_s u^1\cdot\nabla u^0+(-N')\gamma\tilde u_n^0\ ,$$
$$\varepsilon\Lambda'\gamma' V(\nu,t,s;u^0)=O((t-s)^{-1/2})\ ,$$
$$\gamma\tilde u_n^0=\gamma\{\bar U_0(\nu)\bar e-\bar P_1(\nu)\gamma\bar U_0(\nu)\bar e\}\partial_n^{-1}\nabla'\tilde v^{0\prime}-\bar P_1(\nu)\gamma' u^0\quad((6.3)).$$

This and (3.36) of Lemma 3.2 imply that the last term containing g on the left hand side of (8.5) is in $\mathcal{X}^{\ell-2.\rho,\theta}_{a,\beta_1,T_1}$.

Taking (8.1)-(ℓ) and (8.4) into account, we set for $\tilde u^1$ and $\tilde u^2$

$$(8.\bar 1)\ (\ell)\quad \tilde u^{1\prime}=r\bar U_0(\nu)e\tilde v^{1\prime}-\bar{\mathcal P}_2(\nu)'\gamma\bar P^\infty\bar U_0(\nu)e(\varepsilon\bar v^2+\tilde v^1)+\bar{\mathcal P}_2(\nu)' g\ ,$$
$$\varepsilon\tilde u_n^1=\varepsilon r\bar U_0(\nu)\bar w_n^\infty e\underline{\Lambda'\partial_n^{-1}\tilde v^1}\quad(=r\bar U_0(\nu)\bar W_n^\infty e\tilde v^1\ )$$
$$-\ \varepsilon\bar\hbar_2(\nu)_n\underline{\Lambda'\gamma\bar P^\infty\bar U_0(\nu)e}(\varepsilon\bar v^2+\underline{\tilde v^1})-\varepsilon\bar\hbar_2(\nu)_n\Lambda' g\ ,$$
$$\tilde u^2=r\bar U_0(\nu)\{\left(\begin{matrix}\bar w^{\infty\prime}\\0\end{matrix}\ \underline{e\Lambda'\partial_n^{-1}\tilde v^1}\right)+\bar P^\infty e\underline{\Lambda_1'\tilde v^2}\}\ -\bar{\mathcal P}(\nu)\gamma\bar P^\infty\bar U_0(\nu)e\underline{\Lambda_1'\tilde v^2}.$$

Substitute (8.1) into (1.9) and change the variables by (8.3) (and by $(8.\bar 1)$ (ℓ)) . Then, we obtain the equations for $\tilde v^1$ and $\tilde v^2$:

$$(8.7)(\prime)\quad \tilde v^{1\prime}+\{\underline{\bar u^0\cdot\bar\nabla+\tilde u^{0\prime}\cdot\nabla'}+\tilde u_n^0\partial_n+\varepsilon(\bar u^1+\varepsilon\bar u^2+\tilde u^1+\varepsilon\tilde u^2)\cdot\bar\nabla\}r\underline{\bar U_0(\nu)e\tilde v^{1\prime}}$$
$$-\ \{(\underline{\bar u^0+\tilde u^0)'\cdot\nabla'}+\varepsilon(\bar u^1+\varepsilon\bar u^2+\tilde u^1+\varepsilon\tilde u^2)'\cdot\nabla'\}\bar{\mathcal P}_2(\nu)'\gamma\bar P^\infty\bar U_0(\nu)e(\varepsilon\bar v^2+\underline{\tilde v^1})$$
$$-\ \{(\bar u_n^0/\varepsilon+\bar u_n^1+\tilde u_n^0)\partial_n+(\varepsilon\bar u_n^2+\varepsilon\tilde u_n^1+\underline{\varepsilon\tilde u_n^2)\partial_n}\}\bar{\mathcal P}_2(\nu)'\gamma\bar P^\infty\bar U_0(\nu)e\underline{(\varepsilon\bar v^2+\tilde v^1)}$$
$$+\ (\bar u^0+\varepsilon\bar u^1+\varepsilon\bar u^2+\tilde u^0+\varepsilon\tilde u^1+\varepsilon^2\tilde u^2)\cdot\bar\nabla\ \bar{\mathcal P}_2(\nu)' g$$
$$+\ \tilde u^{1\prime}\cdot\nabla'(\bar u^0+\varepsilon\bar u^1+\varepsilon^2\bar u^2+\tilde u^0)'+\varepsilon\tilde u^{2\prime}\cdot\nabla\tilde u^{0\prime}+(\bar u_n^2+\underline{\tilde u_n^1+\tilde u_n^2})\partial_n\tilde u^{0\prime}$$
$$=\ \tilde h^{0\prime}-(\tilde u^{0\prime}\cdot\nabla'+\tilde u_n^0\partial_n)\tilde u^{1\prime}\ ,$$

$$(\mathcal{2})\quad \tilde{v}^1_n + \{u^0:\underline{\bar{\nabla}}+\tilde{u}^0:\underline{\nabla'}+\tilde{u}^0_n\partial_n+\varepsilon(\bar{u}^1+\varepsilon\bar{u}^2+\tilde{u}^1+\varepsilon\tilde{u}^2):\bar{\nabla}\}r\bar{U}_0(\nu)\underline{\bar{W}^\infty_n e\tilde{v}^1}$$
$$-(u^0+\varepsilon\bar{u}^1+\varepsilon^2\bar{u}^2+\tilde{u}^0+\varepsilon\tilde{u}^1+\varepsilon^2\tilde{u}^2)\cdot\underline{\bar{\nabla}}.\bar{\hbar}_2(\nu)_n\varepsilon\Lambda'.\gamma\bar{P}^\infty.\bar{U}_0(\nu)\underline{(\varepsilon v^2+\tilde{v}^1)}$$
$$+(\bar{u}^0+\varepsilon\bar{u}^1+\varepsilon^2\bar{u}^2+\tilde{u}^0+\varepsilon\tilde{u}^1+\varepsilon^2\tilde{u}^2)\cdot\bar{\nabla}\bar{\hbar}_2(\nu)_n\varepsilon\Lambda' g$$
$$+(\tilde{u}^1+\varepsilon\tilde{u}^2)\cdot\bar{\nabla}(\bar{u}^0_n+\varepsilon\bar{u}^1_n+\varepsilon^2\bar{u}^2_n+\varepsilon\tilde{u}^0_n) = \tilde{h}^0_n - \tilde{u}^0\cdot\bar{\nabla}\bar{u}^1_n\ ,$$
$$(\mathcal{3})\quad \Lambda_1'\tilde{v}^2 + \{\underline{\bar{u}^0\cdot\bar{\nabla}+\tilde{u}^0\cdot\nabla'}+\tilde{u}^0_n\partial_n+\varepsilon(\bar{u}^1+\varepsilon\bar{u}^2+\tilde{u}^1+\varepsilon\tilde{u}^2):\bar{\nabla}\}r\bar{P}^\infty.\bar{U}_0(\nu)e\underline{\Lambda_1'\tilde{v}^2}$$
$$-(\bar{u}^0+\varepsilon\bar{u}^1+\varepsilon^2\bar{u}^2+\tilde{u}^0+\varepsilon\tilde{u}^1+\varepsilon^2\tilde{u}^2)\cdot\underline{\bar{\nabla}}\ \bar{\hbar}_2(\nu)\varepsilon\Lambda'.\gamma\bar{P}^\infty.\bar{U}_0(\nu)e\underline{\Lambda_1'\tilde{v}^2}$$
$$+\{(\underline{\tilde{u}^1_n+\tilde{u}^2_n})\partial_n/\varepsilon+\tilde{u}^2\cdot\nabla'\}\ {}^t((\bar{u}^0+\varepsilon\bar{u}^1+\varepsilon^2\bar{u}^2,\ 0\ )\ .$$

Here the underlined terms contain the first derivatives of $\tilde{v}^1$ and $\tilde{v}^2$ in linear order. We note that the (8.8) results from (8.9) :

$$(8.8)\quad (u^1,\tilde{u}^1{}',\tilde{u}^1_n,\tilde{u}^2) \in K^{\ell-1,\rho,\theta}_{a,\beta,T}\times K^{\ell-2,\rho,\theta,(\mu)}_{a/\varepsilon,\beta,T}\times K^{\ell-2,\rho,\theta}_{a/\varepsilon,\beta,T}\times K^{\ell-3,\rho,\theta}_{a/\varepsilon,\beta,T}\ ,$$
$$(\varepsilon u^1,\varepsilon\tilde{u}^1{}',\varepsilon\tilde{u}^1_n,\varepsilon\tilde{u}^2) \in K^{\ell,\rho,\theta}_{a,\beta,T}\times K^{\ell-1,\rho,\theta,(\mu)}_{a/\varepsilon,\beta,T}\times K^{\ell-1,\rho,\theta}_{a/\varepsilon,\beta,T}\times K^{\ell-2,\rho,\theta}_{a/\varepsilon,\beta,T}\ ,$$
$$(8.8)\quad (v^1,\tilde{v}^1{}',\tilde{v}^1_n,\tilde{v}^2) \in K^{\ell-1,\rho,\theta}_{a,\beta,T}\times K^{\ell-2,\rho,\theta,(\mu)}_{a/\varepsilon,\beta,T}\times \tilde{K}^{\ell-2,\rho,\theta}_{a/\varepsilon,\beta,T}\times K^{\ell-2,\rho,\theta}_{a/\varepsilon,\beta,T}$$
$$\equiv L^{\ell-1,\rho,\theta}_{a,\beta,T}\ ,$$

The same holds if the symbol letter K (or L) is replaced by $\mathcal{K}$ ($\mathcal{L}$). ($\mathcal{L}$ is defind from L as in 2). We put

$$(8.9)\qquad w = {}^t(v^1,\tilde{v}^1{}',\tilde{v}^1_n,\tilde{v}^2)\ .$$

Then, the equations (8.5) and (8.7) ($\mathcal{1}$),($\mathcal{2}$),($\mathcal{3}$) are written as

$$(8.10)\qquad w = G(\varepsilon,t,w(\cdot))\ ,$$

in the function space $L^{\ell-1,\rho,\theta}_{a,\beta,T}$ (or in $\mathcal{L}^{\ell-1,\rho,\theta}_{a,\beta,T}$) , with $\beta > \beta_1$ and $0 < T < T_1$. Thus, we can apply Theorem ACK in order to solve (8.9), and we obtain

Theorem 8.1. Under the assumptions of Theorem 5.1 the "Navier-Stokes equation" (1.8)-(1.9) has a (unique) solution $(u^1,\ \tilde{u}^1+\varepsilon\tilde{u}^2)$ with $\tilde{u}^1 = {}^t(\tilde{u}^1{}',\varepsilon\tilde{u}^1_n)$ such that $(u^2,\tilde{u}^1{}',\tilde{u}^1_n,\tilde{u}^2)\in \mathcal{K}^{\ell-2,\rho,\theta}_{a,\beta_2,T_2}\times\mathcal{K}^{\ell-2,\rho,\theta,(\mu)}_{a/\varepsilon,\beta_2,T_2}\times\tilde{\mathcal{K}}^{\ell-2,\rho,\theta}_{a/\varepsilon,\beta_2,T_2}\times\mathcal{K}^{\ell-3,\rho,\theta}_{a/\varepsilon,\beta_2,T_2}$ with $\beta_1 < \beta_2$ and $0 < T_2 < T_1$.

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