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THE DISCREPANCY OF SEQUENCES AND APPLICATIONS

(Abstract of a survey lecture)

Robert F. Tichy
Technical University Vienna

Let $(x_n)_{n=1}^{\infty}$ denote a sequence of points in the s-dimensional unit cube $U_s=[0,1)^s$. We are interested in the distribution behaviour of this sequence in U_s. (x_n) is called uniformly distributed (u.d.) if the number $A(x_n,I,N)$ of points x_n $(1 \leqq n \leqq N)$ contained in an arbitrary s-dimensional interval $I \subseteqq U_s$ is asymptotically equal N times the volume $\lambda(I)$ of I $(N \to \infty)$. As a quantitative measure for the distribution behaviour we may introduce the discrepancy

$$(1) \qquad D_N(x_n) = \sup_I \left| \frac{A(x_n,I,N)}{N} - \lambda(I) \right| .$$

For an arbitrary sequence (y_n) of points in $\mathbb{R}^s$ we define $D_N(y_n)$ to be the discrepancy of the modulo 1 reduced sequence $(\{y_n\})$. One of the starting points of the theory of u.d. sequences is the investigation of the distribution behaviour of $(n\alpha)$, α irrational (Kronecker's approximation theorem). A systematic treatment of the subject was initiated by H. Weyl (1916). Today there exist two classical monographs: one by L. Kuipers and H. Niederreiter (1974) and one by E. Hlawka (1979).

1. SPECIAL SEQUENCES

One basic problem in the theory of u.d. sequences is to give estimates for the discrepancy of special sequences. For $(n\alpha)$ such estimates are well-known and classical, e.g.

$$D_N(n\alpha) = O\left(\frac{\log N}{N}\right) , \tag{3}$$

if α has a bounded continued fraction expansion, and the estimate is essentially best possible. In some papers (cf. a joint paper with G. Turnwald in Journal Number Th. 1987) I have been interested in sequences related to the sum $s(q;n)$ of q-ary digits of the positive integer n. We consider the sequence $(x\,s(q;n))_{n=0}^{\infty}$, where x is of finite approximation type η i.e. for every $\varepsilon>0$ there exists a constant $c>0$ such that

$$\| hx \| \geq \frac{c}{h^{\eta+\varepsilon}} \tag{4}$$

for all positive integers h($\|.\|$ denotes the distance to the nearest integer).

THEOREM: Let x be of finite approximation type η. Then for every $\varepsilon>0$.

$$D_N(xs\ (q;n)) \leq \frac{c(q,x,\varepsilon)}{(\log N)^{1/2\eta-\varepsilon}}$$

for all integers $N>1$. If x is not of approximation type η' for any $\eta'<\eta$ then for every $\varepsilon>0$ and infinitely many N

$$D_N(xs(q;n)) \geq \frac{1}{(\log N)^{1/2\eta+\varepsilon}} .$$

Furthermore for every irrational x and infinitely many N

$$D_N(xs(q;n)) \geq \frac{c'(d,x)}{(\log N)^{1/2}}$$

The maine tool for the proof are two general inequalities, namely the inequality of Erdös-Turan and Koksma's inequality.

2. IRREQULARITIES OF DISTRIBUTION.

The theory of irregularities of distributions answers the question how bad the distribution of N points may be in a certain region in the best possible case, i.e. we ask for lower bounds for the discrepancy. Starting point of this theory was a problem of Van der Corput (1935) who conjectured

$$(5) \qquad \limsup_{N\to\infty} \; ND_N(x_n)=\infty \quad .$$

This conjecture was proved by Van Aardenne - Ehrenfest. K. F. Roth (1956) improved the result to the estimate

$$(6) \qquad D_N(x_n) \geqq c\sqrt{\log N}$$

for infinitely many N and W. Schmidt (1972) gave a final solution with the best possible estimate

$$(7) \qquad ND_N(x_n) \geqq c \log N$$

for infinitely many N.

In 1987 there appeared an excellent monograph by J. Beck and W. Chen on the theory of irregularities of distribution (Cambridge University Press). Both authors have heavily influenced this theory and they are mainly interested in the distribution behaviour of points in special given regions (eg. circular discs, on spheres etc.) As an example we consider a set $\omega=\{u_1,...,u_N\}$ of N points in the circular disc D of unit area assigned with the weights α_i with

$$\alpha_i \geq 0 \ , \ \alpha_1+\ldots+\alpha_N=1.$$

For each disc - segment S(i.e. an intersection of D with a half-plane) we define

(8) $$D_N(\omega;S)=\left|\sum_{u_i\in S}\alpha_i-\mu(S)\right|$$

with the usual area $\mu(S)$ of the segment S. We set

(9) $$\Delta(N)=\inf_{\omega,\alpha_i}\ \sup_S D_N(\omega,S).$$

Then my Ph.D. student R. Winkler obtained in his thesis for sufficiently large N and arbitrary $\varepsilon>0$:

(10) $$\Delta(N)> N^{-\frac{3}{4}-\varepsilon}.$$

This generalizes J. Beck's result for the weights $\alpha_1=\ldots=\alpha_N=\frac{1}{N}$; and the estimate (10) is essentially best possible.

3. WEIGHTED MEANS.

It is an easy exercise to prove that the sequence $((\log n))$, $n>1$ is u.d. modulo 1, but $(\log n)$ is not u.d. mod 1. In 1953 the Japanese mathematician Tsuji generalized the definition of u.d. sequences such that also sequences of very slow growth can be treated. The idea is to consider a general weight sequence $P=(p_n)$, $p_n>0$ and the discrepancy

(11) $$D_N(P,x_n)=\sup_I\left|\frac{1}{P(N)}\sum_{n=1}^N p_n 1_I(\{x_n\})-\lambda(I)\right|,$$

where $P(N)=\sum_{n=1}^N p$ and 1_I denotes the characteristic function of the interval I. The sequence (x_n) is called P-u.d. mod 1 if $D_N(P,x_n)\to 0$ for $N\to\infty$. Tsuji proved that $(\log n)$ is P-u.d. for $p_n=\frac{1}{n}$. In several papers (some of them jointly with H.

Niederreiter) I investigated the weighted discrepancy (11). In the following we consider the very important case of the harmonic mean $p=(\frac{1}{n})$ and give an application to the theory of power series. The question is to construct a power series which is uniformly convergent on the circle of convergence but not absolutely convergent at any point of this circle. Thus the circle of convergence is the natural border: an analytic continuation beyond this circle is not possible. We consider the power series

$$f(z)=\sum_{n=2}^{\infty}\frac{z^n}{n\log n}\,e^{2\pi i\log n} \tag{12}$$

and set $z=e^{2\pi i\Theta}$ ($\Theta\in\mathbb{R}$), since the circle of convergence is $|z|=1$. Obviously $f(z)$ is not absolutely convergent for any z with $|z|=1$. Furthermore we have by Abel's summation formula ($x_k=\Theta k+\log k$)

$$\left|\sum_{n=M}^{M+L}\frac{e^{2\pi i(\Theta n+\log n)}}{n\log n}\right| \leq \left|\sum_{n}\left(\frac{1}{\log n}-\frac{1}{\log(n+1)}\right)\sum_{k=1}^{n}\frac{1}{k}e^{2\pi i x_k}\right| \tag{13}$$

Now it can be shown that the sequence (x_k) is u.d. mod 1 with respect to the harmonic mean. The proof of this result immediately yields the estimate

$$\left|\sum_{k=1}^{n}\frac{1}{k}\,e^{2\pi i x_k}\right| = O(\log^{1-\alpha}n) \tag{14}$$

with $0<\alpha<1$. From this it follows that the sum (13) can be estimated by

$$\sum_{n}\frac{1}{n(\log n)^{1+\alpha}}<\infty .$$

Hence $f(z)$ is uniformly convergent on $|z|=1$.

Further applications of u.d. sequences with respect to weighted means are due to P. Schatte and K. Nagasaka: they investigated in detail Benford's law - a statistical law for the distribution of digits of random numbers.

4. APPLICATIONS TO NUMERICAL ANALYSIS.

The basic tool for these applications is the inequality of Koksma-Hlawka:

$$(15)\qquad \left|\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int\limits_{U_s}\!\dots\!\int f(x)\,d\alpha\right| \leq V(f)D_N(x_n)$$

for every function f of bounded variation $V(f)$ in the sense of Hardy - Krause and an arbitrary sequence (x_n) in U_s. The s-dimensional integral is approximated by a sum over a sequence of N points with small discrepancy. By this method one obtains the same order of convergence as by summing up over all lattice points with mesh-width $\frac{1}{N}$, but one needs not so many points. Of course, it is very important for this method to determine sequences with small discrepancy and which can be easily computed. There are two general methods: The first one is to use linear recurrences mod m, i.e.

$$y_{n+s} = a_1 y_{n+s-1} + a_2 y_{n+s-2} \dots + a_s y_n$$

over $\mathbb{Z}_m$. For special parameters $a_1,\dots,a_s$, $m\in\mathbb{Z}$ the sequence $\left(\frac{y_n}{m},\frac{y_{n+1}}{m},\dots,\frac{y_{n+s-1}}{m}\right)_{n=0}^{m-1}$ has a discrepancy $\ll \frac{(\log m)^s}{m}$.

This method was studied in several papers by Niederreiter. The second method is the method of good lattice points which has been developed independently by Hlawka and Korobov; see the monographs by Hua-Wang (Springer 1981) and Korobov. This method makes use of a lattice point $g\in\mathbb{Z}^s$ such that the sequence

$$\left(n\frac{g}{m}\right)_{n=0}^{m-1}$$

is of small discrepancy, i.e. its discrepancy is $\frac{(\log m)^s}{m}$. In several papers I have given applications of this method to the numerical evaluation of Fourier coefficients and to the numerical solution of partial differential equations.