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Yang-Mills connections and the index bundles

筑波大学 数学系 伊藤光弘 (Mitsuhiro Itoh)

1. Let P be a C^∞ G-principal bundle over a compact connected oriented Riemannian 4-manifold M (G is compact and semisimple). The moduli space of Yang-Mills connections on P which are anti-self-dual (or self-dual) carries a finite dimensional space structure. As is known, it proposes an effective machinery in studying the low dimensional topology and complex manifold theory ([4] [5]).

We shall investigate in this paper certainly defined finite dimensional vector bundles, namely index bundles, over the moduli space and then develop geometry of them from a viewpoint of metric connection and curvature.

The motivation of this paper is to make a sufficient study of the following conjecture: if the base manifold M is a complex Kahler surface with an ample line bundle, then the moduli space admits reasonably a holomorphic line bundle of positive Chern class.

A Yang-Mills connection is to be defined by a connection which is stationary to the variation with respect to the Yang-Mills functional. However any connection is in an original meaning a first order differential operator ---- covariant differentiation.

Indeed, every connection A on P gives a covariant derivative $\nabla_A^{\mathbb{E}}$ on any vector bundle $\mathbb{E}$ associated to P. We suppose that there exist another real (or complex) vector bundle $V \longrightarrow M$ and also an elliptic operator $\mathcal{D}$; $\Gamma^1(V) \longrightarrow \Gamma^2(V)$ associating with V. V is for example a holomorphic vector bundle and $\mathcal{D}$ is the operator associated to the twisted Dolbeault complex. Then on the tensor product $V \otimes \mathbb{E}$ a family of elliptic operators $\mathcal{D}_A$; $\Gamma^1(V \otimes \mathbb{E}) \longrightarrow \Gamma^2(V \otimes \mathbb{E})$ is defined by coupling $\mathcal{D}$ to A.

Obviously, subspaces $\operatorname{Ker} \mathcal{D}_A$, $\operatorname{Coker} \mathcal{D}_A$ are of finite dimension and from the Atiyah-Singer index theorem their difference, the numerical index, is independent of a choice of connection. If we move a connection in the space $\mathcal{A}$ of connections on P, we get a family of formal differences $\{\operatorname{Ind} \mathcal{D}_A\}$, $\operatorname{Ind} \mathcal{D}_A = \operatorname{Ker} \mathcal{D}_A - \operatorname{Coker} \mathcal{D}_A$. As the group $\mathcal{G}$ of gauge transformations of P acts equivariantly on $\mathcal{D}_A$, $\operatorname{Ind} \mathcal{D} = \{\operatorname{Ind} \mathcal{D}_A ;\ A \in \mathcal{A}\}$ can be regarded as an element of $K(\mathcal{B})$, $\mathcal{B} = \mathcal{A}/\mathcal{G}$. Thus we can argue such virtual (or probably proper) vector bundles over $\mathcal{B}$ and also over the moduli space of anti-self-dual connections $\mathcal{M}$, the subspace of $\mathcal{B}$.

So we pose a question: how extent we are able to get knowledge of the index bundle $\operatorname{Ind} \mathcal{D}$.

The following is indeed known with respect to this question. The index formula on a family $\operatorname{Ind} \not{D} = \{ \operatorname{Ind} \not{D}_A \}$ associated to Dirac operators coupled to connections gives the Chern character formula in an integral form as $\operatorname{ch}(\operatorname{Ind} \not{D}) = \int_M \hat{a}(M) \operatorname{ch}(\mathcal{E})$, where $\hat{a}(M)$ is the characteristic form of the spinor bundles so that $\int_M \hat{a}(M)$ is the $\hat{A}$-genus and Ⓔ is the vector bundle associated to the Poincaré bundle $\mathbb{P} \longrightarrow M \times$ Ⓑ ([2]). Therefore the Chern forms are computable in principle. On the othe hand the determinant line bundle det Ind Ⓓ = (Λ^aKer $Ⓓ_A$) ⊗ (Λ^bCoker $Ⓓ_A)^{-1}$, a = dim Ker $Ⓓ_A$, b = dim Ker Coker $Ⓓ_A$, defines a proper line bundle ([3], [12]). Bismut and Freed apply superconnection formalism and also a heat equation method to get a Hermitian connection and the curvature of det Ind Ⓓ ([3], see also [6], [12]). Their method is rather analytical. However, the method which we adopt here is fully simple and accessible in a geometrical sense. Namely, we stay the standpoint that the scheme of projection Ⓐ ⟶ Ⓑ presents a principal bundle structure with infinite dimensional group Ⓖ and it inherits a natural connection. Regarding Ind Ⓓ over Ⓑ as a subbundle of certain Hilbert space vector bundle associated to Ⓐ ⟶ Ⓑ, we restrict the connection to Ind Ⓓ to define not only a connection, but second fundamental form. So the vector bundle version of Gauss equation on curvature is available.

Although the conjecture is half solved as we shall see, the main results which we obtain by the Gauss equation are the following.

The index bundle over the moduli space has the curvature of type (1,1) provided that V is a complex vector bundle over a Kähler surface M with an Einstein Hermitian connection (Theorem 5.3). Hence the complexified index bundle inherits a holomorphic structure. Further the Ricci form Φ can be expressed in terms of the second fundamental form and the ambient space curvature term $\bar{\Phi}$ (Theorem 6.1) so that the difference

$\Phi - \bar{\Phi}$ has an estimation even if Φ is not estimated by itself.

This note will appear elsewhere.

2. Principal bundle with group Ⓖ

We denote by Ⓐ the space of all irreducible connections on P. The action of Ⓖ on Ⓐ, $(g, A) \longrightarrow g(A) = g^{-1}dg + g^{-1}.A.g$, defines the space of orbits of irreducible connections on P, Ⓐ $\longrightarrow$ Ⓑ which actually admits a principal bundle structure. Since the stabilizer of each connection is the center Z_G of G, we should take the quotient group Ⓖ$/Z_G$ instead of Ⓖ in order that the action is free.

At each $A \in$ Ⓐ the tangent space T_AⒶ splits into the sum of vertical and horizontal subspaces;

$$T_A\textcircled{A} = \textcircled{V}_A \oplus \textcircled{H}_A .$$

This splitting is Ⓖ-equivariant. The vertical space is isomorphic to $\Omega^0(\text{ad } P)$, the space of adjoint bundle valued 0-forms, which is the Lie algebra of Ⓖ. So we have the distribution of horizontal spaces defining a connection on the bundle. Its connection form $\omega : T\textcircled{A} \longrightarrow \Omega^0(\text{ad } P)$ has the following expression.

PROPOSITION 2.1. (i) The principal bundle Ⓐ $\longrightarrow$ Ⓑ with group Ⓖ admits a natural connection whose connection form ω is

$$\omega(\alpha) = G_A(D_A{}^*\alpha), \qquad \alpha \in T_A\textcircled{A} \cong \Omega^1(\text{ad } P), \quad (2.1)$$

where G_A is $(D_A{}^*D_A)^{-1}$.

(ii) The curvature form $\Omega^\omega = d\omega + 1/_2[\omega \wedge \omega]$, the $\Omega^0(\text{ad } P)$-valued 2-form on A, is represented by

$$\Omega^\omega(X,Y) = -2\ G_A(\{X,Y\}), \qquad (2.2)$$

$X,\ Y \in T_{[A]}$Ⓑ ($[A] \in$ Ⓑ), where we restrict it to the origin of a slice S in Ⓐ at A

REMARKS. (i) The bilinear operation $\{.,.\}$ $\Omega^1(\mathrm{ad}\ P)\times\Omega^1(\mathrm{ad}\ P) \longrightarrow \Omega^0(\mathrm{ad}\ P)$ is defined by

$$\{X, Y\} = \sum_{i,j} h^{ij}\,[X_i, Y_j], \quad X = \sum X_i dx^i,$$

$Y = \sum Y_i dx^i$ ($(h^{ij}) = (h_{ij})^{-1}$, here h is the Riemannian structure)

(ii) For each A in Ⓐ a slice S is defined to be transversal to gauge orbits and hence gives a neighborhood centered at $[A]$ in Ⓑ. Therefore each tangent vector to the Hilbert manifold Ⓑ is identified with vector tangential to S at a corresponding connection([7],§2).

Proof. (i) Because of the covariance of the gauge action, for any $g \in$ Ⓖ $(R_g{}^*\omega)_A(\alpha) = \omega_{g(A)}(R_g{}^*\alpha) = g(G_A D_A{}^*\alpha)$ so that ω is the connection form, since $\omega(D_A\psi) = \psi$ for each vertical vector $D_A\psi$.

(ii) Extending X and Y to fields $\widetilde{X}$ and $\widetilde{Y}$ over S, we have $\Omega^\omega(X,Y) = X\omega(\widetilde{Y}) - Y\omega(\widetilde{X})$, because $X,\ Y \in \mathrm{Ker}\ D_A{}^*$. Since $X\omega(\widetilde{Y})$ is the derivative at $t=0$ of $\omega(\widetilde{Y})$ along the line $A_t = A + tX$, it reduces to $-G_A(\{X,Y\})$.

With respect to Proposition 2.1 we should mention about the universal connection on the Poincaré bundle. In [2] Atiyah and Singer defined the Poincaré bundle $\mathbb{P}$ with a connection $\mathbb{A}$.

Actually, the action of gauge group $\textcircled{G}$ on the product $P \times \textcircled{A}$ yields a bundle over $M \times \textcircled{B}$, $\mathbb{P} = (P \times \textcircled{A})/_G \longrightarrow M \times \textcircled{B} = G \backslash \mathbb{P}$ with structure group G by taking further a quotient of the G-action on $P \times \textcircled{A}$. By making use of the natural connection ω on $\textcircled{A} \longrightarrow \textcircled{B}$, we can reformulate the connection $\mathbb{A}$ on $\mathbb{P}$ as $\mathbf{A}$ equals A when restricted to $M \times [A]$ and also equals $ev_x(\omega)$ to $\{x\} \times \textcircled{B}$, here ev_x is the evaluation map at x: $\Omega^0(\mathrm{ad}\ P) \longrightarrow (\mathrm{ad}\ P)_x$ and further $(\mathrm{ad}\ P)_x$ is identified with the algebra $\textcircled{g}$.

So, we obtain readily with respect to the product structure of $\times \textcircled{B}$ the curvature $\mathbb{F}$ of $\mathbb{A}$; $\mathbb{F} = \mathbb{F}^{2,0} + \mathbb{F}^{1,1} + \mathbb{F}^{0,2}$, where $\mathbb{F}^{2,0} = F(A)$, the curvature of A on P, $\mathbb{F}^{0,2} = ev_x(\Omega^{\omega}) = -2\, ev_x(G_A\{.,.\})$ and $\mathbb{F}^{1,1}$ is represented by $\mathbb{F}^{1,1}(Y,\alpha) = -\alpha(Y)$ for $(Y, \alpha) \in T_{(x,[A])}(M \times \textcircled{B})$, $(Y \in T_xM,\quad \alpha \in \mathrm{Ker}\ D_A^*)$

3. The index bundle.

Let V be a vector bundle over M with a fibre metric. We suppose that V enjoys an elliptic operator Ⓓ: $\Gamma^1(V) \longrightarrow \Gamma^2(V)$.

For convenience' sake we assume that Ⓓ is the Atiyah-Hitchin-Singer operator;

$$Ⓓ = (\nabla^*, d^{\nabla+}) \ : \quad \Omega^1(V) \longrightarrow (\Omega^0 \oplus \Omega^2_+)(V),$$

(∇^* is the adjoint of a metric connection ∇ on V and Ω^2_+ is the space of self-dual 2-forms). We can of course consider the case of Dirac operator on the spinor bundles.

Let E be a vector space on which G acts through ρ and $\mathbb{E}$ the associated vector bundle $P \times_\rho E$. Then tensoring V with $\mathbb{E}$ we get a new vector bundle $V \otimes \mathbb{E}$. It is equipped with a family of connections ∇_A, the connection ∇ coupled to connections $\nabla_A^{\mathbb{E}}$ on $\mathbb{E}$ as A moves on P.
The representation ρ of Ⓖ on sections of $V \otimes \mathbb{E}$ induces the action on coupled connections;

$$\nabla_{g(A)} = \rho(g^{-1}) \circ \nabla_A \circ \rho(g), \quad g \in Ⓖ.$$

Consider the first order elliptic operators

$$Ⓓ_A = (\nabla_A^*, d_A^+) \ : \quad \Omega^1(V \otimes \mathbb{E}) \longrightarrow (\Omega^0 \oplus \Omega^2_+)(V \otimes \mathbb{E})$$

parametrized by connections on P. Here d_A^+ is $\frac{1}{2}(d_A + *d_A)$.

Since $\mathcal{D}_A$ is $\mathcal{G}$-equivariant, the index bundle $\mathrm{Ind}\,\mathcal{D}$ $= \{ \mathrm{Ind}\,\mathcal{D}_A \}$, $\mathrm{Ind}\,\mathcal{D}_A = \mathrm{Ker}\,\mathcal{D}_A - \mathrm{Coker}\,\mathcal{D}_A$ makes a suitable sense as an element of $K(\mathcal{B})$.

From now on we assume $\mathrm{Coker}\,\mathcal{D}_A = 0$ for each connection A. If M is a complex Kähler surface, for example, and V is a Hermitian-Einstein vector bundle with positive Ricci field, then from the Weitzenböck formula $\mathrm{Coker}\,\mathcal{D}_A$ must vanish for each anti-self-dual connection A.

The index bundle is then from that assumption a finite dimensional subbundle of an infinite dimensional Hilbert space vector bundle $\mathcal{Q} = A \times_G \Omega^1(V \otimes \mathbb{E})$. So the Hilbert space bundle is decomposed into $\mathcal{Q} = \mathrm{Ind}\,\mathcal{D} \oplus \mathrm{Ind}\,\mathcal{D}^{\perp}$, $\mathrm{Ind}\,\mathcal{D}^{\perp}$ is the orthogonal complement and is spanned by the eigenspaces of positive eigenvalues with respect to the operator $\mathcal{D}^*\mathcal{D}$.

Letting $\bar{\nabla}$ be the connection on $\mathcal{Q}$ induced from the bundle $\mathcal{A} \longrightarrow \mathcal{B}$, we have

$$\bar{\nabla}_X \xi = \nabla_X \xi + \sigma_X \xi, \tag{3.1}$$

where ξ is a section of $\mathrm{Ind}\,\mathcal{D}$ and $X \in T_{[A]}\mathcal{B}$.

PROPOSITION 3.1(Gauss equation). Denote by $\bar{\Omega}$ and Ω the curvature of $\bar{\nabla}$ and ∇, respectively and by $<\ ,\ >$ the L_2-inner product on $\Omega^1(V \otimes \mathbb{E})$. Then

$$\langle \bar{\Omega}(X,Y)\xi, \eta \rangle = \langle \Omega(X,Y)\xi, \eta \rangle + \langle \sigma_X \xi, \sigma_Y \eta \rangle - \langle \sigma_Y \xi, \sigma_X \eta \rangle \tag{3.2}$$

$\xi, \eta \in \mathrm{Ind}\textcircled{D}_A, \qquad X, Y \in T_{[A]}\textcircled{B}.$

We find this proposition for submanifolds in any ordinary book on differential geometry (see for example [8]).

4. The second fundamental form and the curvature formula.

We restrict the bundle $\mathcal{A} \longrightarrow \mathcal{B}$ to the moduli space $\mathcal{M}$ of anti-self-dual connections on P.

Before getting the formula on the second fundamental form σ, we need the following $C^\infty(X)$-bilinear mapping

$$\Omega^1(\mathrm{ad}\, P) \times \Omega^1(V \otimes \mathbb{E}) \longrightarrow (\Omega^0 \oplus \Omega^2_+)(V \otimes \mathbb{E}) \tag{4.1}$$

$$(X, \phi) \longrightarrow \rho(X)\circ\phi = (\rho(X)L\phi, -(\rho(X)\Lambda\phi)^+)$$

where we regard $\rho(X)$ as an element of $\Omega^1(\mathrm{End}(V \otimes \mathbb{E}))$.

Then we obtain

PROPOSITION 4.1.

$$\sigma_X \xi = \mathbb{G}_A D_A{}^*(\rho(X)\circ\xi), \tag{4.2}$$

$X \in T_{[A]}\mathcal{M}$ $\xi \in \mathrm{Ker}\,\mathcal{D}_A$. Here $\mathbb{G}_A$ denotes the Green operator of the operator $\mathcal{D}_A{}^*\mathcal{D}_A$.

Proof. Let A_t be a curve of anti-self-dual connections with initial velocity vector X, $A_0 = A$. We extend $\xi \in \mathrm{Ker}\,\mathcal{D}_A$ to $\xi_t \in \mathrm{Ker}\,\mathcal{D}_{A_t}$ along A_t. Since $\dim \mathrm{Ker}\,\mathcal{D}_{A_t}$ is independent of t, ξ_t is written as $\xi_t = \sum_{i=1}^k \xi^i(t)\beta_i(t)$ with respect to an orthonormal basis $\{\beta_i(t),\ 1 \leq i \leq k\}$ of $\mathrm{Ker}\,\mathcal{D}_{A_t}$ each of which depends smoothly on t.

By its definition

$$\sigma_X \xi = (\overline{\nabla}_X \xi)^{\perp} = (d/_{dt}\, \xi_{t|t=0} + \rho(\omega(X))\xi)^{\perp} =$$

$(d/_{dt}\, \xi_{t|t=0})^{\perp}$, since $X \in \mathrm{Ker}\, \mathbb{D}_A^*$ and hence $\omega(X)$ vanishes. So, it reduces to $\sum \xi^i(0)(\dot{\beta}_i(0))^{\perp} = \sum \xi^i(0)\, \mathbb{G}_A(\mathbb{D}_A^* \mathbb{D}_A)(\dot{\beta}_i(0))$ by observing that $\mathbb{G}_A(\mathbb{D}_A^* \mathbb{D}_A)$ gives just the orthogonal projection to the orthogonal complement ($\cdot$ means the differentiation with respect to t). On the other hand we differentiate $(\mathbb{D}^*\mathbb{D})_{A_t} \beta_i(t) = 0$ and set $t = 0$ to get

$$(\mathbb{D}^* \mathbb{D})_A(\dot{\beta}_i(0)) + (d/_{dt|t=0}(\mathbb{D}^* \mathbb{D})_{A_t})(\beta_i(0)) = 0, \qquad (4.3)$$

$1 \leq i \leq k$. As $\beta = \beta_i(0) \in \mathrm{Ker}\, \mathbb{D}_A$ is in the ambient space $\Omega^1(V \otimes \mathbb{E})$, the second term in RHS of (4.3) is reduced to $\mathbb{D}_A^*(-\rho(X)L\beta, (\rho(X)\wedge\beta)^+)$ from which (4.2) follows.

Since we have from Prop.3.2 $\overline{\Omega}(X,Y)\xi = -2\, \rho((D_A^*D_A)^{-1}\{X,Y\})\xi$, we derive the curvature formula from the Gauss equation.

PROPOSITION 4.2. Let ∇ the connection on Ind D naturally defined. Then its curvature is represented as

$$\langle\Omega(X,Y)\xi, \eta\rangle = -2 \langle \rho((D_A^*D_A)^{-1}\{X,Y\})\xi, \eta\rangle$$
$$- \langle\mathbb{G}_A\mathbb{D}_A^*\rho(X)\circ\xi,\ \mathbb{G}_A\mathbb{D}_A^*\rho(Y)\circ\eta\rangle + \langle\mathbb{G}_A\mathbb{D}_A^*\rho(Y)\circ\xi,\ \mathbb{G}_A\mathbb{D}_A^*\rho(X)\circ\eta\rangle \qquad (4.4)$$

REMARK. We should mention on the canonical Riemannian structure on the moduli space Ⓜ defined in [7]. The Riemannian structure was defined by the crucial aid of the Kuranishi map, a map linearizing the moduli space and also the Hodge theory relating the deformation complex. Also there are diffrent definitions of it and different ways to define. Indeed S. Kobayashi applied in [9] the method of submersion due to O'Neill to discuss it. Although the curvature formula was obtained in [7] by tedious calculation, we can apply in a direct way thus described curvature formula of the index bundle to the trivial situation that V is the trivial bundle $M \times \mathbb{R}$ and Ⓓ $= (d^*, d^+) : \Omega^1 \longrightarrow \Omega^0 \oplus \Omega^2_+$, and $\mathbb{E}$ is ad P with the adjoint representation as ρ. Indeed, since $\rho(X) \llcorner Y = \{X,Y\}$, $(\rho(X) \wedge Y)^+ = [X \wedge Y]^+$, we get immediately the formula (Theorem 5.1 in [7]).

5. Holomorphic structure on the index bundle.

We assume that the base 4-manifold M is complex Kähler. Then the moduli space of anti-self-dual connections over M carries in a natural sense a complex manifold structure and a Kähler structure([7]). In fact, the almost complex structure at each tangent space of Ⓜ defining the complex structure is given by the bundle endomorphism $I : \Omega^1(\mathrm{ad}\, P) \longrightarrow \Omega^1(\mathrm{ad}\, P)$ which is induced from the base space almost complex structure $I : \Omega^1 \longrightarrow \Omega^1$.

The first observation is the following

PROPOSITION 5.1. The curvature form Ω^ω of the natural connection ω on the bundle Ⓐ $\longrightarrow$ Ⓑ restricting to Ⓜ is of type (1.1). Hence any finite dimensional complex vector bundle associated to it is endowed with a holomorphic structure compatible with the connection.

REMARK. Choose a point x in M and consider the $\mathbb{E}$-framed moduli space $\tilde{Ⓜ}_x$, i.e., the G-quotient of the set $\{(A,\phi);\ A$ is anti-self-dual and $\phi \in \mathbb{E}_x\}$, which is a vector bundle over Ⓜ with fibre E. Since $\tilde{Ⓜ}_x$ associates to the bundle Ⓐ $\to$ Ⓑ and the curvature of the induced connection is seen to be $\rho(ev_x(\Omega^\omega))$, thecomplexified framed moduli space becomes from the proposition a holomorphic vector bundle.

Proof. At $[A] \in$ Ⓜ, $\Omega^\omega(X,Y) = -2\, G_A\{X,Y\}$, $X, Y \in T_{[A]}$Ⓜ which is identified with the first cohomology $H^1(\mathrm{ad}\, P; A)$.

It suffices to show that $\{Z,W\} = 0$ for any pair of complex vectors Z, W of type $(1,0)$ or of type $(1,0)$. Since each vector of type $(1,0)$ (or of type $(0,1)$) is a $\pm\sqrt{-1}$ eigenvector of I, respectively. and also $\{X,IY\}$ is given by the inner product of $[X\wedge Y]^{+}$ with the Kahler form and hence is symmetric with respect to real X, Y, $\{Z, W\}$ obviously vanishes. Since the curvature condition on type is just the integrability of holomorphic structure ([1]), the proposition holds.

PROPOSITION 5.2. Let $\mathbb{P} \longrightarrow M \times Ⓜ$ be the Poincaré bundle related with the bundle P over M. Then the connection $\mathbb{A}$ has curvature whose type is $(1,1)$ and hence any complex vector bundle associated to $\mathbb{P}$ admits a holomorphic structure compatible with the induced connection.

This follows proposition 5.1 by observing the expression of the curvature $\mathbb{F}$.

The Poincaré bundle with the connection corrsponds to the algebro-geometrical notion of universal bundle which might impose interesting problems on us([10],[11]).

The following asserts the integrable condition on holomorphic structure of the index bundle.

THEOREM 5.3. Assume that the vector bundle V carries an Einstein-Hermitian connection ∇ with positive Ricci form. Then, (i) the complexified index bundle $\mathrm{Ind}Ⓓ^{\mathbb{C}}$ decomposes into subbundles $\mathrm{Ind}Ⓓ^{1,0}$ and $\mathrm{Ind}Ⓓ^{0,1}$ with respect to the almost complex structure I defined on Ⓜ, (ii) the curvature form Ω

restricting to Ind$\textcircled{D}^{1,0}$ (or Ind$\textcircled{D}^{0,1}$) is a (1,1)-form and hence (iii) Ind$\textcircled{D}^{1,0}$ (or Ind$\textcircled{D}^{0,1}$) is equipped with a holomorphic structure which is consistent with the connection ∇.

REMARKS.(i) Since the Einstein-Hermitian connection has curvature of type (1,1), the operator $\textcircled{D}$: $\Omega^1(V) \longrightarrow (\Omega^0 \oplus \Omega^2_+)(V)$ naturally associates to the twisted Dolbeault complex

$\Omega^0(V) \xrightarrow{\nabla''} \Omega^{0,1}(V) \xrightarrow{d^{\nabla''}} \Omega^{0,2}(V)$ and consequently by coupling anti-self-dual connections on P so does the coupled operator $\textcircled{D}_A$ to

$$\Omega^0(V \otimes \mathbb{E}) \xrightarrow{\nabla''_A} \Omega^{0,1}(V \otimes \mathbb{E}) \xrightarrow{d_A''} \Omega^{0,2}(V \otimes \mathbb{E}).$$

Therefore the subbundle Ind$\textcircled{D}^{0,1}$ is regarded as the bundle over $\textcircled{M}$ with fibre $H^1(V \otimes \mathbb{E}, d_A'')$, the first cohomology of the above coupled Dolbeault complex.

(ii) Quillen examines a new metric and its curvature on the determinant line bundle of the index bundle over a Riemann surface associated with operators $\textcircled{D}_A = \nabla_A'' : \Omega^0(\mathbb{E}) \longrightarrow \Omega^{0,1}(\mathbb{E})$ ([3],[6],[12]). The metric defined by the Ray-Singer analytic torsion is of interest. However, he argues it only over $\textcircled{A}$.

Proof. From Proposition 4.2 and Proposition 5.1 it suffices to prove that

$$\langle \sigma_Z(\xi - \sqrt{-1}I\xi), \sigma_W(\eta + \sqrt{-1}I\eta) \rangle = 0 \quad \text{for} \quad Z,\ W \in T^{1,0}\textcircled{M}$$

and $\xi, \eta \in$ Ind$\textcircled{D}_A$.

We assert first the formulas:

$$\sigma_X \xi = \mathbb{G}_A \nabla_A(\rho(X)L\xi) + I\, \mathbb{G}_A \nabla_A(\rho(IX)L\xi) - \mathbb{G}_A d_A^{+*}([\rho(X)\wedge\xi]^2) \tag{5.1}$$

$$\sigma_X I\xi = -\mathbb{G}_A \nabla_A(\rho(IX)L\xi) + I\, \mathbb{G}_A \nabla_A(\rho(X)L\xi) - \mathbb{G}_A d_A^{+*}([\rho(IX)\wedge\xi]^2) \tag{5.2}$$

$$\sigma_{IX}\xi = G_A \nabla_A(\rho(IX)L\xi) - I\, \mathbb{G}_A \nabla_A(\rho(X)L\xi) - \mathbb{G}_A d_A^{+*}([\rho(IX)\wedge\xi]^2) \tag{5.3}$$

where we set $[\rho(X)\wedge\xi]^2 = [\rho(X)\wedge\xi]^{2,0} + [\rho(X)\wedge\xi]^{0,2}$.

Indeed $[\rho(X)\wedge\xi]^+ = [\rho(X)\wedge\xi]^2 + [\rho(X)\wedge\xi]^0$ with $[\rho(X)\wedge\xi]^0 = \frac{1}{2}(\rho(X)LI\xi)\otimes\omega_X$. So $\mathbb{G}_A d_A^{+*}([\rho(X)\wedge\xi]^0) = \frac{1}{2} d_A^{+*}\mathbb{G}_A((\rho(X)LI\xi)\otimes\omega_X)$ reduces to $I\, \mathbb{G}_A\nabla_A(\rho(X)LI\xi)$ (ω_X is the Kähler from). By the aid of the formula $\rho(IX)LI\xi = \rho(X)L\xi$ which is caused by the naturality of I, we obtain (5.1). (5.2) and (5.3) also follow from the formula $[\rho(X)\wedge I\xi]^2 = [\rho(IX)\wedge\xi]^2$.

By making use of these formulas, we have

$$< \sigma_X I\xi,\ \sigma_Y I\eta > \ = \ < \sigma_{IX}\xi \ \ \sigma_{IY}\eta > \tag{5.4}$$

and hence

$$< \sigma_Z \xi,\ \sigma_W \eta > + < \sigma_Z I\xi,\ \sigma_W I\eta > \ = 0 \tag{5.5}$$

for $Z = X - \sqrt{-1}IX$, $W = Y - \sqrt{-1}IY \in T^{1,0}M$.

Therefore

$$< \sigma_Z(\xi+\sqrt{-1}I\xi),\ \sigma_W(\eta-\sqrt{-1}I\eta) > \ = \ < \sigma_Z\xi,\ \sigma_W\eta > + < \sigma_Z I\xi,\ \sigma_W I\eta >$$

$$- \sqrt{-1}\ \{ < \sigma_Z\xi,\ \sigma_W I\eta > \ - < \sigma_Z I\xi,\ \sigma_W\ \eta > \} \ = \ 0.$$

6. The Ricci form.

To get an expression of the Ricci form on the index bundle, we recall the definition of the Ricci form of a holomorphic bundle. Let F be a holomorphic vector bundle with a Hermitian fibre metric over a complex manifold M. Then the Ricci form Φ of F which is a $(1,1)$-form over M is defined by trace of the endomorphism $\Omega(X,Y)_x : F_x \to F_x$, $x \in M$, where Ω is the curvature form of the fibre metric. The 2-form $1/_{2\pi\sqrt{-1}}\ \Phi$ represents the first Chern class of F.

We consider now the index bundle $\mathrm{Ind}\textcircled{D}$ over the moduli space $\textcircled{M}$ of anti-self-dual connections. We already observed that $\mathrm{Ind}\textcircled{D}^{1,0}$ carries a holomorphic structure with a Hermitian fibre metric. Thus, if we let $\{\xi_i\}_{1\leq i\leq k}$ be an orthonormal basis of $\mathrm{Ind}\textcircled{D}_A$, then the Ricci form Φ of $\mathrm{Ind}\textcircled{D}^{1,0}$ is by the definition written as $\Phi(X,Y) = \sum_{i=1}^k < \Omega(X,Y)\phi_i, \bar\phi_i >$, $\phi_i = (\xi_i - \sqrt{-1}I\xi_i)/_{\sqrt{2}}$, $X, Y \in T_{[A]}\textcircled{M}$.

We denote by $\bar\Phi$ the 2-form on B and hence on $\textcircled{M}$ defined by $\sum_{i=1}^k < \bar\Omega(X,Y)\phi_i, \bar\phi_i >$. The 2-form $\bar\Phi$ is just the trace of the curvature endomorphism $\bar\Omega(X,Y)$ restricted to $\mathrm{Ind}\textcircled{D}$.

THEOREM 6.1. $\bar\Phi(Z,\bar Z) - \Phi(Z,\bar Z) = 4\sum_{i=1}^k\{\| \mathbb{G}_A\nabla_A(\rho(X)LI\xi_i) +$

$$I\mathbb{G}_A\nabla_A(\rho(X)L\xi_i)\|^2 - \|\mathbb{G}_A d_A^{+*}([\rho(X)\wedge\xi_i]^2)\|^2\}. \tag{6.1}$$

Proof. By the Gauss equation we have $< \Omega(X,Y)\xi, \xi >$ $=$ $< \bar{\Omega}(X,Y)\xi, \xi >$. On the other hand the pair (X,Y) generates a gauge transformation g_t which preserves the L_2-inner product $< , >$ on $\Omega^1(V \otimes \mathbb{E})$ so that $< \bar{\Omega}(X,Y)\xi, \xi >$ $=$ $\frac{1}{2}\, d/_{dt|t=0} < \rho(g_t)\xi, \rho(g_t)\xi >$ vanishes. Hence

$$< \Omega(X,Y)\xi, \xi > \; = \; 0, \qquad \xi \in \mathrm{Ind}(\mathbb{D})_A \tag{6.2}$$

Therefore from this

$$< \Omega(X,Y)\phi_i, \bar{\phi}_i > = \tfrac{1}{2}\sqrt{-1}\{ <\Omega(X,Y)\xi_i, I\xi_i> - <\Omega(X,Y)I\xi_i,\xi_i> \}$$

$$= \sqrt{-1}/_2 \; \{ <\bar{\Omega}(X,Y)\xi_i, I\xi_i> - <\bar{\Omega}(X,Y)I\xi_i,\xi_i> -2 <\sigma_X I\xi_i, \sigma_Y\xi_i>$$

$$+ 2 \; <\sigma_Y I\xi_i, \sigma_X\xi_i> \}.$$

Here we used the fact that $\bar{\Omega}(X,Y)I\xi = I(\bar{\Omega}(X,Y)\xi)$.

Thus the Ricci form is represented as

$$\Phi(X,Y) = \sqrt{-1}\, \textstyle\sum_i <\bar{\Omega}(X,Y)\xi_i, I\xi_i> - \sqrt{-1}\sum_i \{ <\sigma_X I\xi_i, \sigma_Y\xi_i> -$$

$$<\sigma_Y I\xi_i, \sigma_X\xi_i> \} \tag{6.3}$$

Therefore we see for $Z = X - \sqrt{-1}\, IX \in T^{1,0}_{[A]}(\mathbb{M})$ that $(Z,\bar{Z}) = 2\sqrt{-1}\,\Phi(X,IX)$ is given by

$$\Phi(Z,\bar{Z}) = -\, 2\textstyle\sum_i <\Omega(X,IX)\xi_i, I\xi_i> +2 \sum_i\{<\sigma_X I\xi_i, \sigma_{IX}\xi_i>$$

$$- <\sigma_{IX} I\xi_i, \sigma_X\xi_i> \} \tag{6.4}$$

By making use of formulas (5.1), (5.2), (5.3), we can reduce the terms $\langle \sigma_X I\xi_i, \sigma_{IX}\xi_i \rangle - \langle \sigma_{IX} I\xi_i, \sigma_X \xi_i \rangle$ to

$$- 2 \{ \langle \mathbb{G}_A \nabla_A (\rho(X) LI\xi_i), \mathbb{G}_A \nabla_A (\rho(X) LI\xi_i) \rangle$$

$$+ \langle \mathbb{G}_A \nabla_A (\rho(X) L\xi_i), \mathbb{G}_A \nabla_A (\rho(X) L\xi_i) \rangle$$

$$+ 2 \langle \mathbb{G}_A \nabla_A (\rho(X) LI\xi_i), I(\mathbb{G}_A \nabla_A (\rho(X) L\xi_i) \rangle \} + 2 \| \mathbb{G}_A d_A^{+*} ([\rho(X) \wedge \xi]^2) \|^2$$

$$= - 2 \| \mathbb{G}_A \nabla_A (\rho(X) LI\xi_i) + I(G_A \nabla_A (\rho(X) L\xi_i)) \|^2 + 2 \| \mathbb{G}_A d_A^{+*} ([\rho(X) \wedge \xi]^2) \|^2 .$$

Thus, (6.1) is shown.

References.

1. M.F.Atiyah, N.Hitchin, I.M.Singer, Self-duality in 4-dimensional Riemannian geometry, Proc.R.S.London 362(1978)425-461.
2. M.F.Atiyah, I.M.Singer, Dirac operators coupled to vector potentials, Proc.Natl.Acad.Sci.U.S.A.,81(1984)2597-2600.
3. J.-M.Bismut, D.S.Freed, The analysis of elliptic families I, Com. Math.Phys.,106(1986)159-176.
4. S.K.Donaldson, An application of gauge theory to the topology of 4-manifolds,J.Diff.Geo.,18(1983)269-316.
5. S.K.Donaldson,Irrationality and the h-cobordism conjecture, J.Diff.Geo.,26(1987)141-168.
6. S.K.Donaldson, Infinite determinants, stable bundles and curvature, Duke Math.J.,54(1987)231-247.
7. M.Itoh,Geometry of anti-self-dual connections and Kuranishi map,J.Math.Soc.Japan,40(1988)9-32.
8. S.Kobayashi, K.Nomizu, Foundations of differential geometry II, 1969, Interscience Publ.
9. S.Kobayashi,Submersions of CR submanifolds, Tohoku Math.J., 39(1987)
10. J.Le Potier, Fibrés stables de rang 2 sur $P_2(\mathbb{C})$, Math.Ann. 241(1979)217-256.
11. D.Mumford, P.Newstead, Periods of a moduli space of bundles on curves, Amer.J.Math.,90(1968)1200-1208.
12. D.Quillen, Determinants of Cauchy-Riemann operators over a Riemann surface, Funct.Anal.Appl.,19(1985)31-34.