数理解析研究所講究録
第 663 巻 1988 年 47-61

The Chu-Vandermonde convolution generates transformation formulas for hypergeometric series*

by Kazuo UENO
College of Liberal Arts, University of Saga
1 Honjo, Saga 840, Japan
佐賀大学教養部　数学　上野一男

(*This note is an abbreviation of an article submitted for publication in a journal.)

Abstract: Several well-known hypergeometric series identities are proved using only powerseries computation and the Chu-Vandermonde convolution for binomial coefficients.

1. Introduction.

The close relationship between binomial coefficient identities and hypergeometric series has been noticed by several authors; see for example [1, 6]. In this note we will show how some of the well-known hypergeometric series identities can be derived by using only powerseries computation and the Chu-Vandermonde convolution (the CV for short) for binomial coefficients (2.1).

In Section 2 we review the powerseries computation proof of the Kummer transformations for ${}_1F_1$ and ${}_2F_1$ (2.2)-(2.3). Some authors write this proof in their textbooks ([4; p.76], [7; p.31]); however, since such a proof does not seem to be widespread and the Kummer transformations are referred to in later sections, we repeat the proof here.

In Section 3 the transformation formulas for the Lauricella series F_A and F_D (3.1)-(3.4) are proved using only powerseries computation and the CV. The proof of (3.4) goes along a line slightly different from those for (3.1)-(3.3), which appeals to something of powerseries operator calculus. In fact we can also give purely in an operator calculus manner a proof of (3.4) as well as those of other identities for hypergeometric series of one or several indeterminates; see [8].

In Section 4 we first review the interrelationship between the CV, the Kummer transformation for ${}_2F_1$, the Saalschütz formula for ${}_3F_2$, and a quadratic transformation for ${}_2F_1$ (4.2). We show three more examples of quadratic transformations for ${}_2F_1$ (4.3)-(4.5) which can be proved using the CV through powerseries computation. The proof of (4.5) would perhaps be more interesting than those of (4.3) and (4.4), as it makes use of the Lagrange inversion formula.

2. Kummer Transformations.

Throughout the present article we fix a base field K of characteristic zero and the letters $a, b, c, \ldots, x_1, x_2, \ldots$ etc. denote <u>indeterminates</u>; thus the hypergeometric series

$$ {}_2F_1(a, b; c; x) := \sum_{i\in\mathbb{N}} \frac{a^{(i)}b^{(i)}}{c^{(i)}i!}x^i $$

is considered to be an element of $K(a, b, c)[[x]]$ and the Lauricella series

$$ F_D(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n) $$

$$:= \sum_{i_1,\ldots,i_n\in\mathbb{N}} \frac{a^{(i_1+\cdots+i_n)}b_1^{(i_1)}\cdots b_n^{(i_n)}}{c^{(i_1+\cdots+i_n)}i_1!\cdots i_n!}x_1^{i_1}\cdots x_n^{i_n} $$

is considered to be an element of $K(a, b_1, \ldots, b_n, c)[[x_1, \ldots,$

$x_n]]$, where $a^{(i)} := a(a + 1)\cdots(a + i - 1)$ denotes the rising factorial. Similar consideration applies to all the powerseries appearing in the sequel.

We have

$$\frac{a^{(k)}}{k!} = \binom{a + k - 1}{k} = \sum_{i+j=k} \binom{c + k - 1}{i} \binom{a - c}{j} \quad (k \in \mathbb{N}) \tag{2.1}$$

by the Chu-Vandermonde convolution which we call the CV throughout this paper.

Theorem 2.1. The Kummer transformations

$$_1F_1(a;\ c;\ x) = e^x\,{}_1F_1(c - a;\ c;\ -x), \tag{2.2}$$

$$_2F_1(a,\ b;\ c;\ x) = (1 - x)^{-b}\,{}_2F_1(c - a,\ b;\ c;\ \frac{x}{x - 1}), \tag{2.3}$$

and the formula (2.1) are equivalent to each other, where $_1F_1$ denotes the confluent hypergeometric series.

Proof. The formula (2.1) is equivalent to

$$\frac{a^{(k)}}{c^{(k)}k!} = \sum_{i+j=k} \frac{(c - a)^{(j)}(-1)^j}{i!c^{(j)}j!}. \tag{2.4}$$

Multiplying both sides of (2.4) by x^k and summing these terms over $k \in \mathbb{N}$, we obtain (2.2). To show (2.3) we multiply both sides of (2.4) by $b^{(k)}x^k$ and sum these terms over $k \in \mathbb{N}$; the left-hand side equals $_2F_1(a,\ b;\ c;\ x)$. Noting that $b^{(k)} = b^{(j)}(b + j)^{(i)}$ $(i + j = k)$, we see that

the right-hand side

$$= \sum_{j\in\mathbb{N}} \frac{b^{(j)}(c - a)^{(j)}}{c^{(j)}j!}(-x)^j \sum_{i\in\mathbb{N}} \frac{(b+j)^{(i)}}{i!}x^i$$

$$= (1 - x)^{-b}{}_2F_1(c - a, b; c; \frac{x}{x - 1}).$$

Conversely, we can reverse the above reasoning by taking the coefficients of x^k $(k \in \mathbb{N})$.

<u>Remark</u> 2.2. Using the Hadamard product of powerseries $(f*g := \sum_{i\in\mathbb{N}} f_i g_i x^i$ for $f = \sum_{i\in\mathbb{N}} f_i x^i$ and $g = \sum_{i\in\mathbb{N}} g_i x^i)$, we can summarize the proof of Theorem 2.1 as

$${}_1F_1(a; c; x) = (1 - x)^{-a} * \sum_{i\in\mathbb{N}} \frac{1}{c^{(i)}} x^i$$

$$= ((1 - x)^{-c}(1 - x)^{c-a}) * \sum_{i\in\mathbb{N}} \frac{1}{c^{(i)}} x^i$$

$$= e^x {}_1F_1(c - a; c; - x)$$

and

$${}_2F_1(a, b; c; x) = (1 - x)^{-a} * \sum_{i\in\mathbb{N}} \frac{b^{(i)}}{c^{(i)}} x^i$$

$$= ((1 - x)^{-c}(1 - x)^{c-a}) * \sum_{i\in\mathbb{N}} \frac{b^{(i)}}{c^{(i)}} x^i$$

$$= (1 - x)^{-b}{}_2F_1(c - a, b; c; \frac{x}{x - 1}).$$

3. Transformation formulas for the Lauricella series.

We give the proofs, making use of only powerseries computation and the CV, of the transformation formulas for the Lauricella series F_A and F_D:

(i) $F_A(a; b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n)$

$$= (1 - x_1 - \cdots - x_n)^{-a} F_A(a; c_1 - b_1, \ldots, c_n - b_n; c_1, \ldots,$$

$$c_n; \frac{x_1}{x_1 + \cdots + x_n - 1}, \ldots, \frac{x_n}{x_1 + \cdots + x_n - 1}), \tag{3.1}$$

where the left-hand side denotes

$$\sum_{i_1,\ldots,i_n \in \mathbb{N}} \frac{a^{(i_1+\cdots+i_n)} b_1^{(i_1)} \cdots b_n^{(i_n)}}{c_1^{(i_1)} \cdots c_n^{(i_n)} i_1! \cdots i_n!} x_1^{i_1} \cdots x_n^{i_n},$$

(ii) $F_A(a; b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n)$

$$= (1 - x_1)^{-a} F_A(a; c_1 - b_1, b_2, \ldots, b_n; c_1, \ldots, c_n; \frac{x_1}{x_1 - 1},$$

$$\frac{x_2}{1 - x_1}, \ldots, \frac{x_n}{1 - x_1}), \tag{3.2}$$

(iii) $F_D(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n)$

$$= (1 - x_1)^{-a} F_D(a; c - B, b_2, \ldots, b_n; c; \frac{x_1}{x_1 - 1}, \frac{x_1 - x_2}{x_1 - 1}, \ldots,$$

$$\frac{x_1 - x_n}{x_1 - 1}), \tag{3.3}$$

(iv) $F_D(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n)$

$$= (1 - x_1)^{-b_1} \cdots (1 - x_n)^{-b_n} F_D(c - a; b_1, \ldots, b_n; c; \frac{x_1}{x_1 - 1},$$

$$\ldots, \frac{x_n}{x_n - 1}). \tag{3.4}$$

Proof of (i). We compute:

the right-hand side of (3.1)

$$= \sum_{i \in \mathbb{N}} a^{(i)} (1 - (x_1 + \cdots + x_n))^{-a-i} (-1)^i \times$$

$$\times \sum_{i_1 + \cdots + i_n = i} \frac{(c_1 - b_1)^{(i_1)} \cdots (c_n - b_n)^{(i_n)}}{c_1^{(i_1)} \cdots c_n^{(i_n)} i_1! \cdots i_n!} x_1^{i_1} \cdots x_n^{i_n}. \tag{3.5}$$

Substituting $(1 - (x_1 + \cdots + x_n))^{-a-i} = \sum_{j\in\mathbb{N}} \frac{(a+i)^{(j)}}{j!} \times$

$\times \sum_{j_1+\cdots+j_n=j} \frac{j!}{j_1!\cdots j_n!} x_1^{j_1}\cdots x_n^{j_n}$ into (3.5) and noting that $a^{(i)}(a + i)^{(j)} = a^{(k)}$ with $k = i + j$, we have

$$(3.5) = \sum_{k\in\mathbb{N}} a^{(k)} \times$$

$$\times \sum_{i_1+\cdots+i_n+j_1+\cdots+j_n=k} \frac{(c_1 - b_1)^{(i_1)}\cdots(c_n - b_n)^{(i_n)}}{c_1^{(i_1)}\cdots c_n^{(i_n)} i_1!\cdots i_n! j_1!\cdots j_n!} \times$$

$$\times(-1)^{i_1+\cdots+i_n} x_1^{i_1+j_1}\cdots x_n^{i_n+j_n}$$

$$= \sum_{k\in\mathbb{N}} a^{(k)} \sum_{k_1+\cdots+k_n=k} x_1^{k_1}\cdots x_n^{k_n} \times$$

$$\times \prod_{m=1}^{n} \sum_{i_m+j_m=k_m} \frac{(-1)^{i_m}(c_m - b_m)^{(i_m)}}{i_m! j_m! c_m^{(i_m)}}. \tag{3.6}$$

The last summation factor is transformed into

$$\frac{1}{c_m^{(k_m)}} \sum_{i_m+j_m=k_m} \binom{b_m - c_m}{i_m} \binom{c_m + k_m - 1}{j_m}$$

$$= \frac{b_m^{(k_m)}}{c_m^{(k_m)} k_m!} \tag{3.7}$$

by the CV. Substitution of (3.7) into (3.6) yields the left-hand side of (3.1).

<u>Proof of (ii)</u>. We compute:

the right-hand side of (3.2)

$$= \sum_{i\in\mathbb{N}} a^{(i)}(1 - x_1)^{-a-i} \sum_{i_1+\cdots+i_n=i} \frac{(c_1 - b_1)^{(i_1)} b_2^{(i_2)} \cdots b_n^{(i_n)}}{c_1^{(i_1)} \cdots c_n^{(i_n)}} \times$$

$$\times \frac{(-1)^{i_1} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}}{i_1! i_2! \cdots i_n!}. \tag{3.8}$$

Substituting $(1 - x_1)^{-a-i} = \sum_{j\in\mathbb{N}} \frac{(a + i)^{(j)}}{j!} x_1^j$ into (3.8) and noting that $a^{(i)}(a + i)^{(j)} = a^{(i_1+\cdots+i_n+j)}$, we have

$$(3.8) = \sum_{i_1,\ldots,i_n,j\in\mathbb{N}} \frac{a^{(i_1+\cdots+i_n+j)} (-1)^{i_1} (c_1 - b_1)^{(i_1)}}{c_1^{(i_1)} \cdots c_n^{(i_n)} j! i_1!} \times$$

$$\times \frac{b_2^{(i_2)} \cdots b_n^{(i_n)}}{i_2! \cdots i_n!} x_1^{i_1+j} x_2^{i_2} \cdots x_n^{i_n}$$

$$= \sum_{k,i_2,\ldots,i_n\in\mathbb{N}} \frac{a^{(k+i_2+\cdots+i_n)} b_2^{(i_2)} \cdots b_n^{(i_n)}}{c_2^{(i_2)} \cdots c_n^{(i_n)} i_2! \cdots i_n!} x_1^k x_2^{i_2} \cdots x_n^{i_n} \times$$

$$\times \sum_{i_1+j=k} \frac{(-1)^{i_1} (c_1 - b_1)^{(i_1)}}{j! i_1! c_1^{(i_1)}}. \tag{3.9}$$

The last summation factor is transformed into

$$\frac{1}{c_1^{(k)}} \sum_{i_1+j=k} \binom{b_1 - c_1}{i_1} \binom{c_1 + k - 1}{j} = \frac{b_1^{(k)}}{c_1^{(k)} k!} \tag{3.10}$$

by the CV. Substitution of (3.10) into (3.9) yields the left-hand side of (3.2).

Proof of (iii). We compute:

the right-hand side of (3.3)

$$= \sum_{i\in\mathbb{N}} \frac{a^{(i)}(-1)^i}{c^{(i)}}(1 - x_1)^{-a-i} \times$$

$$\times \sum_{i_1+\cdots+i_n=i} \frac{(c - B)^{(i_1)} b_2^{(i_2)} \cdots b_n^{(i_n)}}{i_1! i_2! \cdots i_n!} x_1^{i_1} \times$$

$$\times \sum_{j_2+k_2=i_2} \frac{i_2!}{j_2! k_2!} x_1^{j_2}(-x_2)^{k_2} \cdots \sum_{j_n+k_n=i_n} \frac{i_n!}{j_n! k_n!} x_1^{j_n}(-x_n)^{k_n}$$

$$= \sum_{i\in\mathbb{N}} \frac{a^{(i)}(-1)^i}{c^{(i)}} \sum_{m\in\mathbb{N}} \frac{(a + i)^{(m)}}{m!} x_1^m \times$$

$$\times \sum_{i_1+j_2+k_2+\cdots+j_n+k_n=i} \frac{(c - B)^{(i_1)} b_2^{(j_2+k_2)} \cdots b_n^{(j_n+k_n)}}{i_1! j_2! k_2! \cdots j_n! k_n!} \times$$

$$\times x_1^{i_1+j_2+\cdots+j_n}(-1)^{k_2+\cdots+k_n} x_2^{k_2} \cdots x_n^{k_n}$$

$$= \sum_{i,m\in\mathbb{N}} \frac{a^{(i+m)}(-1)^i}{c^{(i)} m!} x_1^m \sum_{j+k=i} x_1^j (-1)^{k+j} \times$$

$$\times \sum_{k_2+\cdots+k_n=k} \frac{b_2^{(k_2)} \cdots b_n^{(k_n)}}{k_2! \cdots k_n!} x_2^{k_2} \cdots x_n^{k_n} \times$$

$$\times \sum_{i_1+j_2+\cdots+j_n=j} \binom{-c+B}{i_1} \binom{-b_2-k_2}{j_2} \cdots \binom{-b_n-k_n}{j_n}. \tag{3.11}$$

The last summation factor is equal to $\binom{-c+b_1-k}{j}$ by the multivariate CV. Thus, putting

$$S_k := \sum_{k_2+\cdots+k_n=k} \frac{b_2^{(k_2)} \cdots b_n^{(k_n)}}{k_2! \cdots k_n!} x_2^{k_2} \cdots x_n^{k_n}, \tag{3.12}$$

we have

$$(3.11) = \sum_{j,k,m\in\mathbb{N}} \frac{a^{(j+k+m)}}{c^{(j+k)} m!} \binom{-c+b_1-k}{j} S_k x_1^{j+m}$$

$$= \sum_{i,k\in\mathbb{N}} S_k \sum_{j+m=i} \frac{a^{(i+k)}(c + j + k)^{(m)}}{c^{(i+k)}m!}\binom{- c + b_1 - k}{j} x_1^{\,i}$$

$$= \sum_{i,k\in\mathbb{N}} S_k \frac{a^{(i+k)}}{c^{(i+k)}} x_1^{\,i} \sum_{j+m=i} \binom{c + i + k - 1}{m}\binom{- c + b_1 - k}{j}. \qquad (3.13)$$

The last summation factor is transformed into $\binom{b_1 + i - 1}{i} = b_1^{(i)}/i!$ by the CV. Substitution of (3.12) into (3.13) yields the left-hand side of (3.3).

Proof of (iv). (See also [8].) We make use of the Kummer transformation (2.2). Writing $L := K(a, b_1, \ldots, b_n, c)$, we define the L-linear endomorphism σ of $L[[x_1, \ldots, x_n]]$ by

$$\sigma(x_1^{\,i_1}\cdots x_n^{\,i_n}) := \sum_{j=1}^{n}(b_j + i_j)x_j\cdot x_1^{\,i_1}\cdots x_n^{\,i_n}$$

$$(i_1, \ldots, i_n \in \mathbb{N}).$$

By induction we have

$$\sigma^k(1) = \sum_{k_1+\cdots+k_n=k} \frac{k!}{k_1!\cdots k_n!} b_1^{(k_1)}\cdots b_n^{(k_n)} x_1^{\,k_1}\cdots x_n^{\,k_n}$$

$$(k \in \mathbb{N}). \qquad (3.14)$$

With M being the maximal ideal we have $\sigma(M^i) \subset M^{i+1}$ $(i \in \mathbb{N})$; hence any $f(\sigma) \in L[[\sigma]]$ acts as an L-linear endomorphism of $L[[x_1, \ldots, x_n]]$. Thus we have from (2.2) that

$${}_1F_1(a; c; \sigma)(1) = e^{\sigma}\,{}_1F_1(c - a; c; - \sigma)(1). \qquad (3.15)$$

The left-hand side of (3.15) is equal to that of (3.4) by virtue of (3.14). The right-hand side of (3.15) is

$$\sum_{i\in\mathbb{N}} \frac{(c - a)^{(i)}}{c^{(i)}i!}(- \sigma)^i e^{\sigma}(1);$$

hence it suffices to show that

$$(-\sigma)^{i}e^{\sigma}(1) = (1 - x_1)^{-b_1}\cdots(1 - x_n)^{-b_n}\times$$

$$\times \sum_{i_1+\cdots+i_n=i} \frac{i!}{i_1!\cdots i_n!} b_1^{(i_1)}\cdots b_n^{(i_n)}\times$$

$$\times(x_1/(x_1 - 1))^{i_1}\cdots(x_n/(x_n - 1))^{i_n} \quad (i \in \mathbb{N}) \tag{3.16}$$

to obtain (3.4). We show an equivalent form of (3.16):

$$\sigma^{i}e^{\sigma}(1) = \sum_{i_1+\cdots+i_n=i} \frac{i!}{i_1!\cdots i_n!} b_1^{(i_1)}\cdots b_n^{(i_n)} x_1^{i_1}\cdots x_n^{i_n}\times$$

$$\times(1 - x_1)^{-b_1-i_1}\cdots(1 - x_n)^{-b_n-i_n} \quad (i \in \mathbb{N}). \tag{3.17}$$

We compute:

the right-hand side of (3.17)

$$= \sum_{i_1+\cdots+i_n=i} \frac{i!}{i_1!\cdots i_n!} \sum_{j_1\in\mathbb{N}} \frac{b_1^{(i_1+j_1)}}{j_1!} x_1^{i_1+j_1}\cdots\times$$

$$\times \sum_{j_n\in\mathbb{N}} \frac{b_n^{(i_n+j_n)}}{j_n!} x_n^{i_n+j_n}$$

$$= \sum_{j\in\mathbb{N}} \frac{1}{j!} \sum_{j_1+\cdots+j_n=j} \frac{j!}{j_1!\cdots j_n!} \sum_{i_1+\cdots+i_n=i} \frac{i!}{i_1!\cdots i_n!}\times$$

$$\times b_1^{(i_1+j_1)}\cdots b_n^{(i_n+j_n)} x_1^{i_1+j_1}\cdots x_n^{i_n+j_n}$$

$$= \sum_{j\in\mathbb{N}} \frac{1}{j!} \sum_{k_1+\cdots+k_n=i+j} b_1^{(k_1)}\cdots b_n^{(k_n)} x_1^{k_1}\cdots x_n^{k_n}\times$$

$$\times\sum_A \frac{i!\qquad j!}{i_1!\cdots i_n!j_1!\cdots j_n!}, \tag{3.18}$$

where the summation $\sum_A$ ranges over the set $\{ (i_1, \ldots, i_n; j_1,$

$\ldots, j_n) \in \mathbb{N}^{2n} \mid i_1 + j_1 = k_1, \ldots, i_n + j_n = k_n, i_1 + \cdots + i_n = i,$
$j_1 + \cdots + j_n = j \}$. Eliminating $j_1, \ldots, j_n$, we have

$$\Sigma_A(\cdots) = \sum_{i_1+\cdots+i_n=i} \binom{k_1}{i_1} \cdots \binom{k_n}{i_n} \frac{i!j!}{k_1!\cdots k_n!}$$

$$= \frac{(i + j)!}{k_1!\cdots k_n!} \tag{3.19}$$

by the multivariate CV. Substitution of (3.19) into (3.18) yields the left-hand side of (3.17) by virtue of (3.14), completing the proof.

We note that, since (3.15) follows from (2.2) which is a consequence of the CV, the proof of (3.4) is in fact performed using only powerseries computation and the CV.

4. Quadratic transformations.

As we have seen in Section 2, the Kummer transformation for $F := {}_2F_1$ (2.3) follows from the CV. Applying (2.3) twice, we have

$$F(a, b; c; x) = (1 - x)^{c-a-b}F(c - a, c - b; c; x). \tag{4.1}$$

As shown in [3; pp.65-66] and [7; pp.48-49], (4.1) is equivalent to the Saalschütz formula for terminating ${}_3F_2$. Thus we see:

Proposition 4.1. <u>The Saalschütz formula is a consequence of the CV.</u>

Using the Saalschütz formula, we can directly prove one of the quadratic transformations for $F = {}_2F_1$

$$F(a, b; 1 + a - b; x)$$

$$= (1 - x)^{-a}F(a/2, (a + 1 - 2b)/2; 1 + a - b; - 4x/(1 - x)^2);$$

(4.2)

see [3; p66] or [7; pp.49-50]. By Proposition 4.1 we can say that (4.2) is also a consequence of the CV. We will further give the proofs of some other quadratic transformations for $F = {}_2F_1$ using only powerseries computation and the CV:

(i) ([3; p.65, (26)]) $F(a, a + 1/2; b; x)$

$$= (2/(1 + (1 - x)^{1/2}))^{2a} F(2a, 2a - b + 1; b; (1 - (1 - x)^{1/2})/(1 + (1 - x)^{1/2})), \qquad (4.3)$$

(ii) ([3; p.65, (27)]) $F(a, b; a + b + 1/2; 4x(1 - x))$

$$= F(2a, 2b; a + b + 1/2; x), \qquad (4.4)$$

(iii) ([3; p.66, (33)]) $F(a, b; 2b; x)$

$$= (1 - x/2)^{-a} F(a/2, a/2 + 1/2; b + 1/2; (x/(2 - x))^2). \qquad (4.5)$$

Proof of (i). It is sufficient to show that

$$(1 + t)^{-2a} F(a, a + 1/2; b; 4t/(1 + t)^2)$$
$$= F(2a, 2a - b + 1; b; t); \qquad (4.6)$$

replacing t by $(1 - (1 - x)^{1/2})/(1 + (1 - x)^{1/2})$ gives (4.3). Note that $2/(1 + (1 - x)^{1/2})$ is a powerseries with constant term unity so that (4.3) has no ambiguity as a powerseries identity.

Comparing the coefficients of x^k of both sides of (4.6), we have to show

$$\sum_{i+j=k} \frac{a^{(i)}(a + 1/2)^{(i)} 2^{2i} (2i + 2a)^{(j)} (-1)^j}{b^{(i)} i! j!}$$

$$= \frac{(2a)^{(k)} (2a - b + 1)^{(k)}}{b^{(k)} k!} \quad (k \in \mathbb{N}). \qquad (4.7)$$

We compute:

the left-hand side of (4.7)

$$= \sum_{i+j=k} \frac{(2a)^{(2i+j)}(-1)^j}{b^{(i)}i!j!}$$

$$= \frac{(2a)^{(k)}}{b^{(k)}} \sum_{i+j=k} \frac{(2a+k)^{(i)}(-1)^j(b+i)^{(j)}}{i!\qquad j!}$$

$$= \frac{(2a)^{(k)}(-1)^k}{b^{(k)}} \sum_{i+j=k} \binom{-2a-k}{i}\binom{b+k-1}{j},$$

which is equal to the right-hand side of (4.7) by the CV.

<u>Proof of (ii).</u> Replacing a by $2a$, b by $a - b + 1/2$, and $-4x/(1-x)^2$ by t in (4.2), we have

$$F(a, b; a+b+1/2; t) = (2/((1-t)^{1/2}+1))^{2a}F(2a, a-b+1/2; a+b+1/2; ((1-t)^{1/2}-1)/((1-t)^{1/2}+1)).$$

Applying (2.3) to the right-hand side of the above identity, we see that

$$F(a, b; a+b+1/2; t)$$

$$= F(2a, 2b; a+b+1/2; (1-(1-t)^{1/2})/2),$$

which is equivalent to (4.4) through the substitution $x = (1-(1-t)^{1/2})/2$.

<u>Proof of (iii).</u> We compute:

the right-hand side of (4.5)

$$= \sum_{i\in\mathbb{N}} \frac{a^{(i)}x^i}{i!2^i} \sum_{j\in\mathbb{N}} \frac{(a/2)^{(j)}(a/2+1/2)^{(j)}x^{2j}}{(b+1/2)^{(j)}j!}(2-x)^{-2j}$$

$$= \sum_{i,j,k\in\mathbb{N}} \frac{x^{i+2j+k}a^{(i)}(a/2)^{(j)}(a/2+1/2)^{(j)}(2j)^{(k)}}{2^{i+2j+k}i!(b+1/2)^{(j)}j!k!}$$

$$= \sum_{m \in \mathbb{N}} \frac{x^m}{2^m} \sum_{i+2j+k=m} \frac{a^{(i)} a^{(2j)} b^{(j)} (2j)^{(k)}}{i!(2b)^{(2j)} j! k!}; \tag{4.8}$$

The last summation factor is transformed as

$$\sum_{2j \leqq m} \frac{a^{(2j)} b^{(j)}}{(2b)^{(2j)} j!} \sum_{i+k=m-2j} \frac{a^{(i)} (2j)^{(k)}}{i!k!}$$

$$= \sum_{2j \leqq m} \frac{a^{(2j)} b^{(j)}}{(2b)^{(2j)} j!} (-1)^{m-2j} \binom{-a-2j}{m-2j} \qquad \text{(by the CV)}$$

$$= \sum_{2j \leqq m} \frac{a^{(m)} b^{(j)}}{(2b)^{(2j)} j!(m-2j)!}$$

$$= \frac{a^{(m)}}{(2b)^{(m)}} \sum_{2j \leqq m} \frac{b^{(j)} (2b+2j)^{(m-2j)}}{j!(m-2j)!}$$

$$= \frac{a^{(m)}}{(2b)^{(m)}} \sum_{2j \leqq m} (-1)^j \binom{-b}{j} \binom{2b+m-1}{m-2j}$$

$= \dfrac{a^{(m)}}{(2b)^{(m)}} \times$(the coefficient of x^m in the powerseries expansion

of $(1-x^2)^{-b}(1+x)^{2b+m-1} = (1-x)^{-b}(1+x)^{b+m-1}$). (4.9)

We show that the coefficient in the parentheses is equal to $2^m \binom{b+m-1}{m}$, which is the coefficient of x^m in the expansion of $(1+2x)^{b+m-1}$; substitution of this equality into (4.9) will yield that (4.8) is equal to the left-hand side of (4.5).

To show that the above two coefficients are mutually equal we use a form of the Lagrange inversion formula [2; p.150, Theorem D]: Let K be a field of characteristic zero. For $f(x) \in K[[x]]$ invertible with respect to powerseries composition, $g(x) \in K[[x]]$ its compositional inverse, and $H(x) \in K[[x]]$ any powerseries, the coefficients of x^m in the expansions of $xH(g(x))/g(x)f'(g(x))$ and $H(x)(f(x)/x)^{-m}$ are mutually equal, where $f'(x)$ denotes the derivative of $f(x)$ with respect to x. Take $f(x) := x/(1 - x)$, $g(x) := x/(1 + x)$, and $H(x) := (1 - x)^{-b-m}(1 + x)^{b+m-1}$ to obtain the desired result.

Remark 4.2. As shown in [3; p.66], the formula

$$F(a, b; 2b; 4x/(1 + x)^2) = (1 + x)^{2a} \times$$

$$\times F(a, a + 1/2 - b; b + 1/2; x^2) \tag{4.10}$$

follows from (4.3) and (4.5); hence (4.10) is also a consequence of the CV through powerseries computation.

References.

1. G. E. ANDREWS, Applications of basic hypergeometric functions, SIAM Review 16 (1974), 441-484.

2. L. COMTET, "Advanced Combinatorics," Reidel, Boston, 1974.

3. A. ERDELYI et al., "Higher Transcendental Functions," McGraw-Hill, New York, 1953.

4. Y. KOMATU, "Special Functions," Asakura, Tokyo, 1967 (in Japanese).

5. W. MILLER, Jr., "Symmetry and Separation of Variables," Addison-Wesley, Reading, Mass., 1977.

6. R. ROY, Binomial identities and hypergeometric series, Amer. Math. Monthly, Jan. 1987, 36-46.

7. L. C. SLATER, "Generalized Hypergeometric Functions," (reprint) Univ. Microfilms International, Ann Arbor, Mich., 1980.

8. K. UENO, Hypergeometric series formulas through operator calculus, in preparation.