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A Note on the Calculation of Monodromy Groups of Hypergeometric Systems

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1. Introduction

Consider monodromy groups of the hypergeometric system of differential equations

$$(1.1)\qquad (t-B)\frac{dX}{dt} = AX \qquad (t\in\mathbb{C}),$$

where B is a diagonal matrix of the form

$$(1.2)\qquad B = \mathrm{diag}(\overbrace{\lambda_1,\ldots,\lambda_1}^{n_1}, \overbrace{\lambda_2,\ldots,\lambda_2}^{n_2}, \ldots, \overbrace{\lambda_p,\ldots,\lambda_p}^{n_p})$$

$$(\ \lambda_i\neq\lambda_j\ (i\neq j),\quad n_i\geq 1,\quad n_1+n_2+\cdots+n_p=n\ )$$

and $A\in M_n(\mathbb{C})$. This system has only regular singularities at $t=\lambda_j$ $(j=1,2,\ldots,p)$ and $t=\infty$ in the whole complex t-plane, and hence is Fuchsian. Since the form of (1.1) is invariant under the linear transformation X=DY, where D is a block-diagonal constant matrix of the form

$$(1.3)\qquad D = \mathrm{diag}\ (D_1 \oplus D_2 \oplus \cdots \oplus D_p),$$

the D_j being n_j by n_j matrices, it may be assumed in (1.1) that , denoting $A=(A_{ij};\ i,j= 1,2,\ldots,p)$, where the A_{ij} are n_i by n_j matrices, we take the diagonal blocks A_{ii} $(i=1,2,\ldots,p)$ as Jordan canonical forms.

In the paper [1; see also 2], K.Okubo established an ingenious method of calculating monodromy groups of (1.1) in a generic case, i.e., in a case when there appear no logarithmic solutions.

We shall here give an outline of the method. Assume that A is similar to a diagonal matrix, i.e.,

(1.4) $$A \sim \mathrm{diag}\,(\nu_1, \nu_2, \ldots, \nu_n)$$

together with the condition:

$$\nu_i \not\equiv 0, \quad \nu_i \not\equiv \nu_j \pmod{\mathbb{Z}} \qquad (i \neq j;\ i, j = 1, 2, \ldots, n),$$

and all A_{ii} $(i=1,2,\ldots,p)$ are *diagonal* matrices whose diagonal elements ρ_{ij} $(j=1,2,\ldots,n_i)$ are not negative integers and not congruent to each other modulo $\mathbb{Z}$. Then we can choose an appropriate n by n_i matrix solution $X_i(t)$ near each singularity $t=\lambda_i$, which is constructed of n_i non-holomorphic (column vectorial) solutions, and can verify the so-called extended Gauß' formula

(1.5) $$\det(X_1(t), X_2(t), \ldots, X_p(t))$$

$$= \prod_{i=1}^{p} \prod_{j=1}^{n_i} (t-\lambda_i)^{\rho_{ij}}\, \Gamma(\rho_{ij}+1) \Big/ \prod_{k=1}^{n} \Gamma(\nu_k+1),$$

which implies that the matrix $X(t)=(X_1(t), X_2(t), \ldots, X_p(t))$ forms a fundamental matrix solution of (1.1). By means of the fundamental matrix solution X(t), one can derive generators M_i $(i=i,2,\ldots,p)$ of the monodromy group:

$$(1.6) \qquad X(t) \xrightarrow{\ \circlearrowleft \lambda_i\ } X(t)M_i, \qquad M_i \in GL(n,\mathbb{C})$$

and analyze their properties in detail. Then, taking account of the relation

$$(1.7) \qquad M_1 \, M_2 \cdots M_p = M_\infty,$$

where M_∞ is a representation of the monodromy group corresponding to a negative circuit around $t=\infty$, K.Okubo concludes that if the hypergeometric system (1.1) has no accessory parameters, then one can always calculate explicit values of the generators $M_i\,(i=1,2,\ldots,p)$ only by the relation (1.7).

In this note we shall explain the calculation of the monodromy group of (1.1) in a *non-generic* case. For simplicity, we here assume that

$$(1.8) \qquad A_{ii} = \rho_i + J_i^*, \qquad J_i = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \qquad (i=1,2,\ldots,p),$$

where the asterisk denotes the transposition of a matrix, and the ρ_i are not negative integers and not congruent to each other modulo $\mathbb{Z}$. Other excluded cases can be dealt with by a slight modification of the consideration stated below.

2. *Gauß equation*

As an example, we first treat of the Gauß equation

$$t(t-1)y''+\{(\alpha+\beta+1)t-\gamma\}y'+\alpha\beta y = 0$$

in a case where $\gamma=p+1$, p being a positive integer. This single differential equation can be reduced to the hypergeometric system

$$\left(t - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \frac{dX}{dt} = \begin{pmatrix} -p & 1 \\ a & \rho \end{pmatrix} X, \tag{2.1}$$

where $\rho=p-\alpha-\beta$ and $a=-(p-\alpha)(p-\beta)$. Here it is assumed that $\rho\not\equiv 0$ (mod $\mathbb{Z}$) and $\alpha\not\equiv\beta$ (mod $\mathbb{Z}$), $\alpha\not\equiv 0$, $\beta\not\equiv 0$ (mod $\mathbb{Z}$).

Now it is easy to see that a non-holomorphic solution near $t=0$ corresponding to the characteristic exponent $-p$ involves a logarithmic term. In fact, let $y_0(t)$ be a holomorphic solution of the form

$$y_0(t) = \sum_{m=0}^{\infty} g(m)t^m \qquad (|t|<1). \tag{2.2}$$

Then, according to the Frobenius method, one can obtain a logarithmic solution $x_0(t)$ associated with $y_0(t)$ as follows:

$$x_0(t) = y_0(t)\log t + \hat{x}_0(t) \qquad (|t|<1),$$

where the convergent power series $\hat{x}_0(t)$ is expressed as

$$\hat{x}_0(t) = t^{-p}\sum_{m=0}^{p-1} \hat{g}(m)t^m + \sum_{m=0}^{\infty} \partial[g(m)]t^m.$$

In the above, the coefficient $\hat{g}(m)$ can be determined by the recurrence equation

$$B(m-p)\hat{g}(m) = (m-1-p-A)\hat{g}(m-1) \qquad (0 \le m \le p-1)$$

subject to the condition

$$-(1+A)\hat{g}(p-1) = Bg(0),$$

and ∂ denotes the differentiation with respect to m, i.e.,

$$\partial[g(m)] = dg(m)/dm.$$

By such a construction, we obtain a fundamental matrix solution of the form

(2.3) $$X_0(t) \equiv (y_0(t),\ x_0(t)) = (y_0(t),\ \hat{x}_0(t))t^J \qquad (|t|<1),$$

where J is a shifting matrix, i.e.,

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

As for other solutions, one can easily see that near t=1 there exist a non-holomorphic solution of the form

(2.4) $$x_1(t) = (t-1)^{\rho} \sum_{m=0}^{\infty} g_1(m)(t-1)^m \qquad (|t-1|<1)$$

and a holomorphic solution $y_1(t)$, which corresponds to the characteristic exponent $\rho=0$. Near $t=\infty$, there exist two non-holomorphic solutions of the form

(2.5) $$x^k(t) = t^{\nu_k} \sum_{s=0}^{\infty} h_k(s)t^{-s} \qquad (k=1,2),$$

where the characteristic exponents ν_k (k=1,2) are eigenvalues of A, i.e., $\nu_1=-\alpha$ and $\nu_2=-\beta$.

We here remark that $X(t)=(y_0(t),\ x_1(t))$ forms a fundamental set of solutions of (2.1). In fact, by taking

$$g(0) = (1,1)^*, \qquad g_1(0) = (0,1)^*,$$

one can prove that there holds

$$\det X(t) = (t-1)^{\rho}[\ \Gamma(\rho+1)/(\ \Gamma(1-\alpha)\Gamma(1-\beta))] \tag{2.6}$$

in a simply connected domain including the intersection $(|t|<1)\cap(|t-1|<1)$. In the above and hereafter, we denote analytic continuations of solutions by the same notation.

We always have linear combinations between solutions. We put

$$x_1(t) = \delta y_0(t) + c_0x_0(t) \tag{2.7}$$

and then we immediately see from (2.6) that the constant c_0 is not equal to zero. As a holomorphic solution near t=1, we define $y_1(t)$ by the relation

$$y_0(t) = y_1(t) + c_1x_1(t), \tag{2.8}$$

and then we can see that $X_1(t)=(y_1(t),x_1(t))$ forms a fundamental matrix solution near t=1, because of

$$\det X_1(t) = \det X(t).$$

All preparations having done, we are now in a position to calculate generators of the monodromy group of (2.1). Since the circuit matrices are given as follows:

$$X_0(t) \xrightarrow{\;(0)\;} X_0(t)\exp(2\pi iJ),$$

$$X_1(t) \xrightarrow{\;(1)\;} X_1(t)\begin{pmatrix} 1 & 0 \\ 0 & e_1 \end{pmatrix},$$

where $e_1=\exp(2\pi i\rho)$, from (2.7) and (2.8) we immediately obtain generators $M_i\,(i=0,1)$ of the monodromy group with respect to $X(t)$

(2.9)
$$M_0 = \begin{pmatrix} 1 & \delta \\ 0 & c_0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \delta \\ 0 & c_0 \end{pmatrix} = \begin{pmatrix} 1 & 2\pi ic_0 \\ 0 & 1 \end{pmatrix},$$
$$M_1 = \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c_1(e_1-1) & e_1 \end{pmatrix}.$$

However, by a diagonal transformation $X\,\mathrm{diag}(d_0,d_1)$, where $d_0d_1\neq 0$, that is, by changing a fundamental matrix solution only up to constant factors, we can assign any value to one of constants c_0, c_1 not yet determined in (2.9) without the change of the form of generators.

We here put $2\pi ic_0=1$ and leave c_1 as a constant undetermined in (2.9). Then we have

$$M_0M_1 = \begin{pmatrix} d+1 & e_1 \\ d & e_1 \end{pmatrix} \qquad (d=c_1(e_1-1)),$$

which is equal to the circuit matrix with respect to $X(t)$ along a negative circuit around $t=\infty$, and is therefore similar to $\mathrm{diag}(f_1,f_2)$ ($f_1=\exp(-2\pi i\alpha)$, $f_2=\exp(-2\pi i\beta)$) which is the circuit matrix with respect to a fundamental set of solutions (2.5).

Hence we have

$$\det(M_0M_1-f) = f^2-(d+1+e_1)f+e_1$$
$$= (f-f_1)(f-f_2)$$
$$(e_1= f_1f_2\text{: Fuchs' relation}).$$

From this, we obtain

$$d = f_1+f_2-e_1-1 = -(f_1-1)(f_2-1).$$

We have thus determined the required generators of the monodromy group of (2.1):

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ -(f_1-1)(f_2-1) & e_1 \end{pmatrix}, \tag{2.10}$$

where $f_1=\exp(-2\pi i\alpha)$, $f_2=\exp(-2\pi i\beta)$ and $e_1=\exp(2\pi i\rho)$.

3. *Monodromy group in a non-generic case*

We shall now explain a general treatment of obtaining the monodromy group of (1.1) under the assumption (1.8). In this case, following the Frobenius method, we can again derive the logarithmic matrix solutions $X_i(t)$ near $t=\lambda_i$ of n by n_i matrix form:

$$(3.1)\quad \begin{cases} X_i(t) = \hat{X}_i(t)(t-\lambda_i)^{J_i}, \\ \hat{X}_i(t) = (t-\lambda_i)^{\rho_i} \sum_{m=0}^{\infty} G_i(m)(t-\lambda_i)^m \qquad (i=1,2,\dots,p) \end{cases}$$

where the factor $(t-\lambda_i)^{J_i}$ denotes n_i by n_i matrix of logarithmic polynominals. The coefficient matrix

$$G_i(m)=(g_i(m),g_i^{[1]}(m),\dots,g^{[n_i-1]}(m)),$$

each element $g(m)$ of which is an n-dimensional column vector, is determined as follows: The first element $g_i(m)$ is given as a particular solution of the system of linear difference equations

$$(B-\lambda_i)(m+\rho_i)g_i(m) = (m-1+\rho_i-A)g_i(m-1)$$

subject to the initial condition

$$g_i(0) = (\overset{n_1}{\wedge}{0},\dots,\overset{n_{i-1}}{\wedge}{0},\ \overbrace{(0,\dots,0,1)}^{n_i},\ \overset{n_{i+1}}{\wedge}{0},\dots,\overset{n_p}{\wedge}{0})^*,$$

and then other elements are defined as

$$g_i^{[k]}(m) = \partial^k[g_i(m)] = \frac{1}{k!}\frac{d^k}{dm^k}(g_i(m)) \qquad (k=1,2,\dots,n_i-1).$$

In the paper [3], we showed that the matrix

$$X(t) = (X_1(t),X_2(t),\dots,X_p(t))$$

forms a fundamental matrix solution of (1.1), verifying the extended Gauß-Kummer's formula

$$\det X(t) = \prod_{i=1}^{p} (t-\lambda_i)^{n_i \rho_i} (-1)^{n_i(n_i-1)/2} (\Gamma(\rho_i+1))^{n_i} \Big/ \prod_{k=1}^{n} \Gamma(\nu_k+1).$$

Using this fact, we shall calculate generators of the monodromy group. One can express each $X_i(t)$ near other singularities $t=\lambda_k$ $(k\neq i)$ as follows:

$$X_i(t) = X_k(t)C_{ki} + Y_{ki}(t) \qquad (k\neq i;\ i,k=1,2,\ldots,p),$$

where the $Y_{ki}(t)$ are n by n_i matrices of holomorphic solutions at $t=\lambda_k$, and the C_{ki} are n_k by n_i constant matrices. Then it is easy to see that

$$X(t;\lambda_k) = (Y_{k1}(t),\ Y_{k2}(t),\ldots,X_k(t),\ldots,Y_{kp}(t))$$

forms a fundamental matrix solution near $t=\lambda_k$, because of the relation

$$\det X(t) = \det X(t;\lambda_k).$$

In fact, we have the connection formulas

(3.2) $$X(t) = X(t;\lambda_k)L_k \qquad (k=1,2,\ldots,p),$$

where

(3.3) $$L_k = \begin{pmatrix} I_{n_1} & & & & & & 0 \\ & I_{n_2} & & & & & \\ & & \ddots & & & & \\ C_{k1} & C_{k2} & \cdots & I_{n_k} & \cdots & & C_{kp} \\ & & & & \ddots & & \\ & 0 & & & & & I_{n_p} \end{pmatrix}.$$

Since the circuit matrix with respect to $X_k(t)$ around $t=\lambda_k$ is given as

$$X_k(t) \xrightarrow{\circlearrowleft(\lambda_k)} X_k(t)\exp(2\pi i(\rho_k+J_k)),$$

we immediately obtain the circuit matrix with respect to $X(t;\lambda_k)$ in the form

$$X(t;\lambda_k) \xrightarrow{\circlearrowleft(\lambda_k)} X(t;\lambda_k)E_k ,$$

where

$$E_k = \mathrm{diag}(I_{n_1} \oplus I_{n_2} \oplus \cdots \oplus \exp(2\pi i(\rho_k+J_k)) \oplus \cdots \oplus I_{n_p}) \qquad (k=1,2,\ldots,p). \tag{3.4}$$

Therefore the generators M_k $(k=1,2,\ldots,p)$ of the monodromy group of (1.1) with respect to $X(t)$ are expressed as

$$M_k = L_k^{-1}E_kL_k \qquad (k=1,2,\ldots,p). \tag{3.5}$$

In the above, the number of constants to be determined is equal to

$$\sum_{k=1}^{p} (n-n_k)n_k .$$

However, we see:

(i) By a non-singular diagonal transformation, i.e., by the change of a fundamental matrix solution up to constant factors, one can assign any values to (n-1) constants among them.

(ii) By means of the relation (1.7), one can determine some number of constants in (3.5). For example, suppose that A is similar to a diagonal matrix of the form

$$(3.6)\qquad A \sim \mathrm{diag}(\overbrace{\nu_1,\ldots,\nu_1}^{\theta_1}, \overbrace{\nu_2,\ldots,\nu_2}^{\theta_2}, \ldots, \overbrace{\nu_q,\ldots,\nu_q}^{\theta_q})$$

$$\theta_1 + \theta_2 + \cdots + \theta_q = n \qquad (1 \le q \le n).$$

Then the eigenvalues of $M_1 M_2 \cdots M_p$ are $f_j = \exp(2\pi i \nu_j)\,(j=1,2,\ldots,q)$, and

$$\mathrm{rank}(M_1 M_2 \cdots M_p - f_j) = n - \theta_j \qquad (j=1,2,\ldots,q).$$

In this case, one can easily see that $\sum_{j=1}^{q} \theta_j^2$ constants in (3.5) are determined by the remaining. Here we have only to pay attention to the identity of the Fuchs relation

$$\det(M_1 M_2 \cdots M_p) = \prod_{k=1}^{p} \det[\exp(2\pi i(\rho_k + J_k))]$$

$$= \exp(2\pi i \sum_{k=1}^{p} n_k \rho_k)$$

$$= \prod_{j=1}^{q} f_j^{\theta_j} = \exp(2\pi i \sum_{j=1}^{q} \theta_j \nu_j).$$

From (i) and (ii), if (3.6) is assumed, the number of undetermined constants in (3.5) becomes

$$(3.7) \qquad \mathcal{N} = \sum_{k=1}^{p} (n-n_k)n_k - (n-1)-(\sum_{j=1}^{q} \theta_j^2 - 1)$$

$$= n^2-n+2- \sum_{k=1}^{p} n_k^2 - \sum_{j=1}^{q} \theta_j^2 ,$$

which is exactly equal to the number of accessory parameters in A of (1.1). (See [2]).
Moreover, if for some k and i, $\nu_k=\nu_i$ (mod $\mathbb{Z}$) in (3.6), then $f_k=f_i$, and

$$\text{rank}(M_1M_2\cdots M_p-f_k) = n-\theta_k-\theta_i .$$

Hence in this case the number of undetermined constants is more diminished.

Anyhow, if $\mathcal{N}\leq 0$, then we can determine explicitly all generators (3.5) only by algebraic calculations.

As an example of illustrating the above fact, consider a case when $n_1=n_2=\cdots=n_{p-1}=1$ and hence $n_p=n-(p-1)$. Assume that eigenvalues of A are mutually distinct modulo $\mathbb{Z}$. Then the number of accessory parameters is equal to

$$\mathcal{N} = n^2-n+2-(p-1+(n+1-p)^2) - (n)$$

$$= (p-2)(2n-1-p),$$

whence the accessory parameter free case occurs only for p=2. Such a differential equation is just the generalized hypergeometric equation of Fuchsian type.

4. *Generalized hypergeometric equation*

Following the method described in the preceding section, we shall now calculate the monodromy group of the generalized hypergeometric equation

$$(4.1)\qquad (t-B)\frac{dX}{dt} = AX,$$

where $B=\mathrm{diag}(0,\ldots,0,1)$, and A is of the form

$$A = \begin{pmatrix} \rho & & & & 0 & \beta_1 \\ 1 & \rho & & & & \vdots \\ & 1 & \ddots & & & \\ 0 & & \ddots & \ddots & & \\ & & & 1 & \rho & \beta_{n-1} \\ \alpha_1 & \alpha_2 & \cdots & \cdots & \alpha_{n-1} & \rho_1 \end{pmatrix}, \quad \rho\not\equiv 0 \pmod{\mathbb{Z}},$$

which is assumed to be similar to a diagonal matrix, i.e.,

$$A \sim \mathrm{diag}\,(\nu_1, \nu_2, \ldots, \nu_n)$$

$$\nu_i\not\equiv 0,\ \nu_i\not\equiv\nu_j \pmod{\mathbb{Z}} \quad (i\neq j;\ i,j=1,2,\ldots,n).$$

This hypergeometric system, of course, corresponds to the well-known single generalized hypergeometric equation of Fuchsian type

$$t^{n-1}(t-1)y^{(n)} = \sum_{k=0}^{n-1}(a_k+b_k t)t^{k-1}y^{(k)} \qquad (a_0=0).$$

with $(n-1)$ multiple (modulo $\mathbb{Z}$) characteristic exponents at the origin.

In this case, one can find the logarithmic solutions of n by $(n-1)$ matrix form

$$\begin{cases} X_0(t) = \hat{X}_0(t)t^J, \\ \hat{X}_0(t) = t^\rho \sum_{m=0}^{\infty} G(m)t^m, \end{cases}$$

J again being an (n-1) by (n-1) shifting matrix, near t=0, and a non-holomorphic solution of a column vectorial form

$$X_1(t) = (t-1)^{\rho_1} \sum_{m=0}^{\infty} g_1(m)(t-1)^m,$$

near t=1. Then we have

$$(4.2)\quad \det(X_0(t),\ X_1(t))$$
$$= t^{(n-1)\rho}(t-1)^{\rho_1}(-1)^{(n-1)(n-2)/2}(\Gamma(\rho+1))^{n-1}\Gamma(\rho_1+1)/\prod_{k=1}^{n}\Gamma(\nu_{k+1}),$$

and hence, we can take $X(t)=(X_0(t),\ X_1(t))$ as a fundamental matrix solution of (4.1) to calculate generators of the monodromy group.

We next define holomorphic solutions $Y_0(t)$ and $Y_1(t)$ by the following connection formulas

$$(4.3)\quad \begin{cases} X_0(t) = X_1(t)(c_1, c_2, \ldots, c_{n-1}) + Y_1(t), \\ X_1(t) = X_0(t)(c_1', c_2', \ldots, c_{n-1}')^* + Y_0(t), \end{cases}$$

where $Y_0(t)$ and $Y_1(t)$ are a column vector and an n by (n-1) matrix, respectively. Then we see immediately from (4.2) that

$$\begin{cases} X(t;\ 0) = (X_0(t),\ Y_0(t)), \\ X(t;\ 1) = (Y_1(t),\ X_1(t)), \end{cases}$$

form fundamental sets of solutions near t=0 and t=1, respectively, and obtain circuit matrices with respect to them as follows:

$$X(t;0) \xrightarrow{\ \circlearrowleft\, 0\ } X(t;0)E_0$$

$$E_0 = \begin{pmatrix} \exp(2\pi i(\rho+J)) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \cdots\cdots 0 & 1 \end{pmatrix},$$

$$X(t;1) \xrightarrow{\ \circlearrowleft\, 1\ } X(t;1)E_1$$

$$E_1 = \begin{pmatrix} I_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \cdots\cdots 0 & e_1 \end{pmatrix}, \quad e_1 = \exp(2\pi i\rho_1).$$

Combining these with the connection formulas (4.3), i.e.,

$$\begin{cases} X(t) = X(t;0)L_0\ , \\ X(t) = X(t;1)L_1\ , \end{cases}$$

where

$$L_0 = \begin{pmatrix} I_{n-1} & \begin{matrix} c'_1 \\ c'_2 \\ \vdots \\ c'_{n-1} \end{matrix} \\ 0 \cdots 0 & 1 \end{pmatrix}, \qquad L_1 = \begin{pmatrix} I_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ c_1\ c_2 \cdots c_{n-1} & 1 \end{pmatrix},$$

we consequently obtain the following generators of the monodromy group with respect to X(t):

$$
(4.4)\quad \begin{cases} M_0 = L_0^{-1}E_0L_0 = \begin{pmatrix} & & \gamma_1' \\ & & \gamma_2' \\ \exp(2\pi i(\rho+J)) & & \vdots \\ & & \gamma_{n-1}' \\ 0 & \cdots\cdots & 0 & 1 \end{pmatrix}, \\ \\ M_1 = L_1^{-1}E_1L_1 = \begin{pmatrix} & & 0 \\ & I_{n-1} & \vdots \\ & & 0 \\ \gamma_1 & \gamma_2 \cdots\cdots \gamma_{n-1} & e_1 \end{pmatrix}, \end{cases}
$$

where $\gamma_j=(e_1-1)c_j$ $(j=1,2,\ldots,n-1)$, and denoting $e_0=\exp(2\pi i\rho)$,

$\delta_1=e_0(2\pi i)/1!$, $\delta_2=e_0(2\pi i)^2/2!,\ldots,\delta_{n-2}=e_0(2\pi i)^{n-2}/(n-2)!$,

we have

$$
\begin{cases} \gamma_1' = (e_0-1)c_1'+\delta_1c_2'+ \cdots + \delta_{n-2}c_{n-1}', \\ \gamma_2' = (e_0-1)c_2'+\delta_1c_3'+ \cdots + \delta_{n-3}c_{n-1}', \\ \vdots \\ \gamma_{n-1}' = (e_0-1)c_{n-1}'. \end{cases}
$$

Since we can give any values to (n-1) elements in (4.4), we put

$$
\gamma_1'=\delta_{n-1}\equiv e_0(2\pi i)^{n-1}/(n-1)!, \quad \gamma_2' =\delta_{n-2}, \ \ldots\ , \gamma_{n-1}'=\delta_1.
$$

So we have only to calculate $\gamma_1, \gamma_2,\ldots,\gamma_{n-1}$ by the relation (1.7), i.e.,

$$
(4.5)\qquad \det(M_0M_1-f) = \prod_{k=1}^{n} (f_k-f),
$$

where $f_k=\exp(2\pi i\nu_k)$ $(k=1,2,\dots,n)$. Since the determinant in the left hand side of (4.5) is written as

$$\det(M_0-M_1^{-1}f)\det M_1$$

$$=\begin{vmatrix} e_0-f & \delta_1 & \delta_2 & \cdots & \delta_{n-1} \\ & e_0-f & \delta_1 & \ddots & \vdots \\ & & \ddots & \ddots & \delta_2 \\ & \text{\Large 0} & & e_0-f & \delta_1 \\ \gamma_1 f & \gamma_2 f & \cdots & \gamma_{n-1}f & e_1-f \end{vmatrix},$$

putting $x=f-e_0$, $y=f-e_1$, we have

$$\begin{vmatrix} -x & \delta_1 & \delta_2 & \cdots & \delta_{n-1} \\ & -x & \delta_1 & \ddots & \vdots \\ & & \ddots & \ddots & \delta_2 \\ & \text{\Large 0} & & -x & \delta_1 \\ \gamma_1 f & \gamma_2 f & \cdots & \gamma_{n-1}f & -y \end{vmatrix}$$

$$=(-1)^{n+1}\gamma_1 f\begin{vmatrix} \delta_1 & \delta_2 & \cdots & \delta_{n-1} \\ -x & \delta_1 & \ddots & \vdots \\ & -x & \ddots & \delta_2 \\ \text{\Large 0} & & -x & \delta_1 \end{vmatrix}+(-x)\begin{vmatrix} -x & \delta_1 & \cdots & \delta_{n-2} \\ & -x & \delta_1 \ddots & \vdots \\ \text{\Large 0} & & -x & \delta_1 \\ \gamma_2 f & \cdots & \gamma_{n-1}f & -y \end{vmatrix}$$

$$=(-1)^{n+1}\sum_{j=1}^{n-1}\gamma_j f x^{j-1}A_{n-j}(x)+(-1)^n x^{n-1}y,$$

where

$$(4.6) \qquad A_j(x) = \begin{vmatrix} \delta_1 & \delta_2 & \cdots & \cdots & \delta_j \\ -x & \delta_1 & \ddots & & \delta_{j-1} \\ & -x & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \delta_2 \\ 0 & & & -x & \delta_1 \end{vmatrix} \qquad (j=1,2,\ldots,n-1).$$

Consequently, we can rewrite (4.5) in the form

$$(4.7) \qquad \sum_{k=1}^{n-1} \gamma_k \, x^{k-1} A_{n-k}(x) = \frac{1}{f}\{(f-e_0)^{n-1}(f-e_1) - \prod_{k=1}^{n}(f-f_k)\}$$

$$= \frac{1}{x+e_0}\{x^{n-1}(x+e_0-e_1) - \prod_{k=1}^{n}(x+e_0-f_k)\}$$

$$\equiv \psi(x),$$

where the member in the right hand side is easily seen to be a polynomial of degree (n-1) in f, i.e., in x, because of the Fuchs relation $\prod_{k=1}^{n} f_k = e_0^{n-1} e_1$.

We here make a remark on the properties of the determinant $A_j(x)$ $(j=1,2,\ldots,n-1)$:

(i) $A_j(x)$ is a polynomial of degree $(j-1)$ in x,

(ii) $A_j(0) = \delta_1^j$,

(iii) $A_j(x) = \sum_{k=1}^{j} \delta_k x^{k-1} A_{j-k}(x) \qquad (A_0(x) \equiv 1),$

e.g., $A_1(x) = \delta_1$, $A_2(x) = \delta_1 A_1(x) + \delta_2 x = \delta_1^2 + \delta_2 x, \ldots$.

Taking account of the above relations (i)(ii)(iii), we can calculate the explicit values of $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ by means of (4.7). To see this, we use the following notation

$$\begin{cases} u^{[p]}(x) \equiv \partial^p[u(x)] \equiv \frac{1}{p!}\frac{d^p}{dx^p}(u(x)), \\ (u(x)v(x))^{[p]} = \sum_{k=0}^{p} u^{[k]}(x)v^{[p-k]}(x) \qquad \text{(Leibniz rule)}, \end{cases}$$

and then obtain

$$[(x^q A(x))^{[p]}]_{x=0} = \begin{cases} A^{[p-q]}(0) & (p\geq q), \\ 0 & (p<q). \end{cases}$$

Now, differentiating both sides of the above relation (iii) in p times, and then putting x=0, we have for $0\leq p\leq j-1$

$$\text{(4.8)} \qquad A_j^{[p]}(0) = \sum_{k=1}^{j} \delta_k[(x^{k-1}A_{j-k}(x))^{[p]}]_{x=0}$$

$$= \sum_{k=1}^{p+1} \delta_k A_{j-k}^{[p-k+1]}(0) \qquad (j=1,2,\ldots,n-1)$$

and obviously, from (i), $A_j^{[p]}(0)\equiv 0$ for $p\geq j$.
Again, differentiating both sides of (4.7) in (j-1) times and then putting x=0, we have

$$\text{(4.9)} \qquad \sum_{k=1}^{j-1} \gamma_k A_{n-k}^{[j-k]}(0)+\gamma_j A_{n-j}(0) = \psi^{[j-1]}(0) \qquad (j=1,2,\ldots,n-1).$$

Combining (4.9) with (4.8), we can effectively evaluate $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$. For instance, from the relation (ii) we immediately obtain

$$\gamma_1 = \frac{\psi(0)}{A_{n-1}(0)} = -\frac{1}{\delta_1^{n-1} e_0} \prod_{k=1}^{n} (e_0 - f_k).$$

From (4.8) for p=1, i.e.,

$$\begin{cases} A_1^{[1]}(0) = 0, \\ A_j^{[1]}(0) = \delta_1 A_{j-1}^{[1]}(0) + \delta_2 A_{j-2}(0) \qquad (j=2,3,\dots,n-1), \end{cases}$$

we easily derive

$$A_j^{[1]}(0) = (j-1)\delta_1^{j-2}\delta_2 \qquad (j=1,2,\dots,n-1).$$

Then, from (4.9) for j=2, we obtain

$$\gamma_2 = (\psi^{[1]}(0) - \gamma_1 A_{n-1}^{[1]}(0))/A_{n-2}(0)$$

$$= \{ \frac{1}{e_0^2} - \frac{1}{e_0} (\sum_{k=1}^{n} \frac{1}{e_0 - f_k}) + \frac{(n-2)\delta_2}{e_0 \delta_1^2} \} \prod_{k=1}^{n} (e_0 - f_k)/\delta_1^{n-2}$$

$$= \frac{1}{\delta_1^{n-2} e_0} \{ \frac{n}{2e_0} - \sum_{k=1}^{n} \frac{1}{e_0 - f_k} \} \prod_{k=1}^{n} (e_0 - f_k),$$

and so on ...

We thus obtain the generators of the required monodromy group of (4.1) of the following form

$$M_0 = \begin{pmatrix} e_0 & \delta_1 & \delta_2 & \cdots & \delta_{n-1} \\ & e_0 & \delta_1 & & \delta_{n-2} \\ & & \ddots & \ddots & \vdots \\ & \text{\Large 0} & & e_0 & \delta_1 \\ & & & & 1 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 1 & & & \text{\Large 0} & 0 \\ & \ddots & & & \vdots \\ & \text{\Large 0} & & 1 & 0 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} & e_1 \end{pmatrix}.$$

The above example is a typical one among calculations of monodromy groups in more general cases. And, in practical calculations, we think that the computer works effectively by the algebraic manipulation based on the above algorithm.

References

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